On uniformisation in MSO over infinite trees

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Paris

Uniformisation

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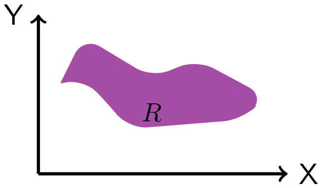
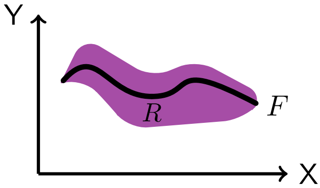


Relation $R \subseteq X \times Y$

Uniformisation



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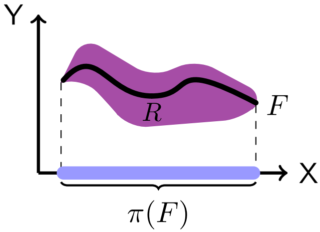

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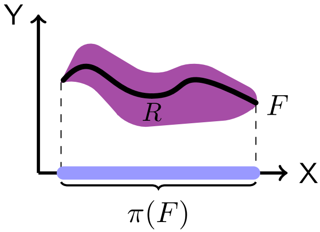

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Every relation admits a uniformisation.

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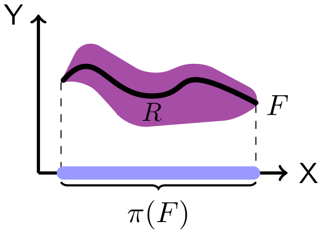

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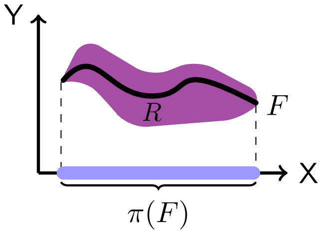

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Theorem (Novikov, Kondô [1938])

Every co-analytic ($\mathbf{\Pi}^1_1$) relation admits a co-analytic uniformisation.

In monadic second-order (MSO) logic?

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[a model-theoretic argument with some subtleties]

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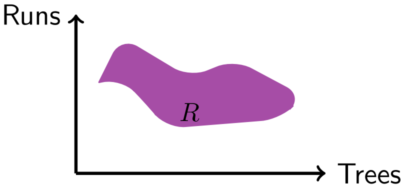

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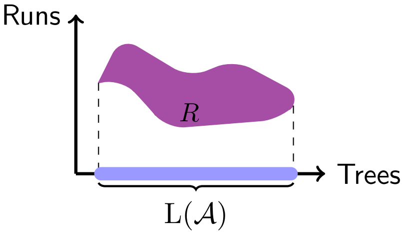

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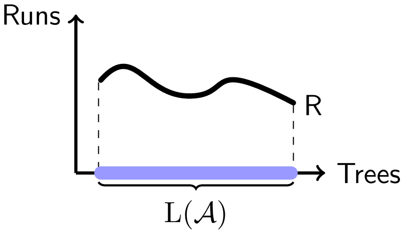

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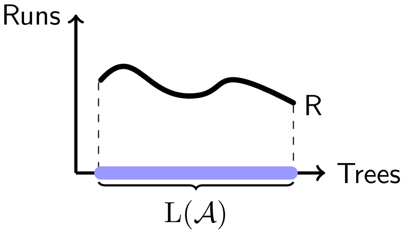

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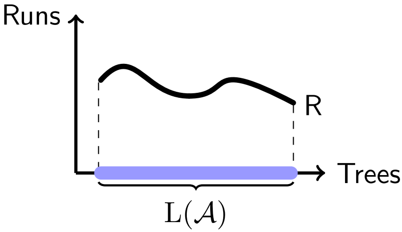

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Any unambiguous automaton for $\exists_y\ a(y)$ induces an MSO-definable choice function.

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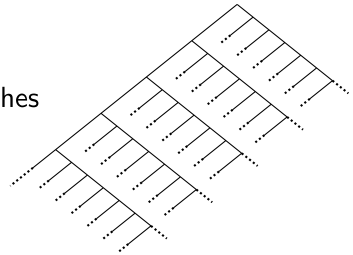

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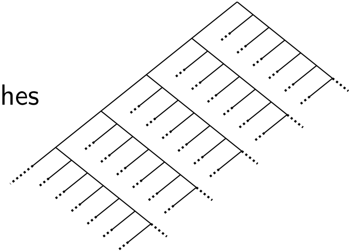

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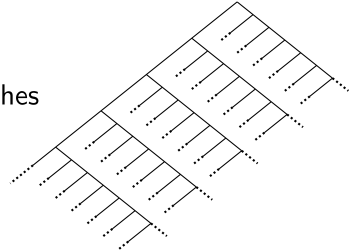

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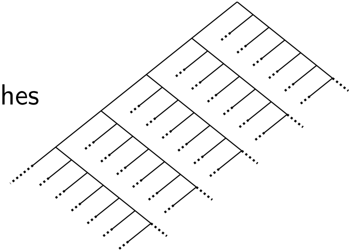

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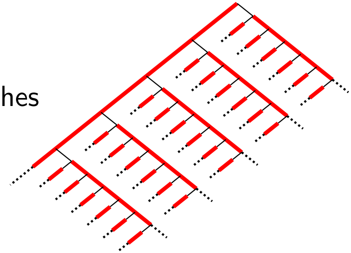

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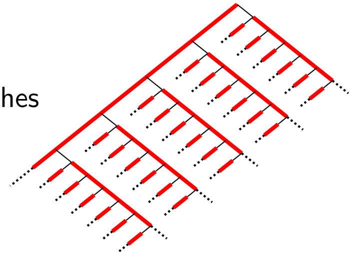

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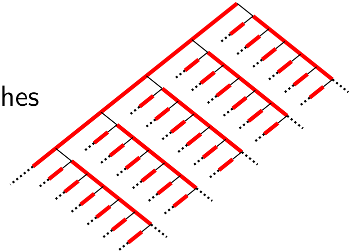

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t has a skeleton ⇔ t is thin

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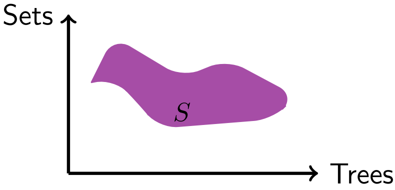

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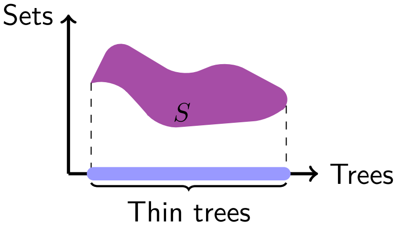

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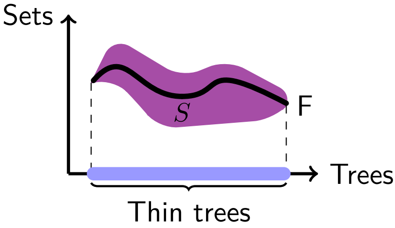

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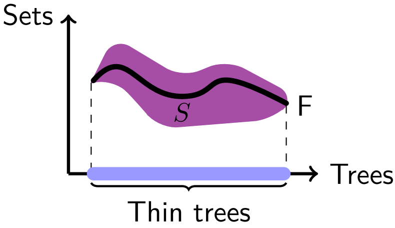

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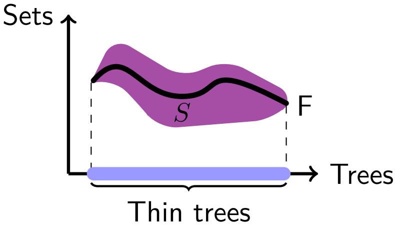

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⇝ new examples of non-uniformisable / ambiguous languages

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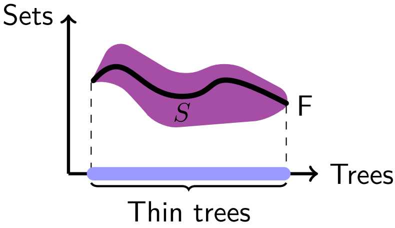

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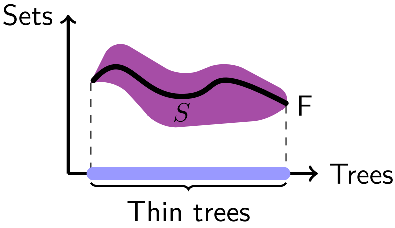

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Conjecture (S. [2013])

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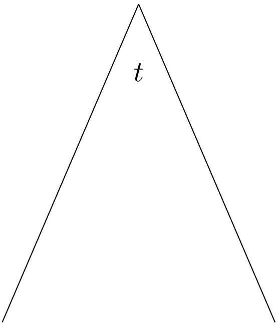

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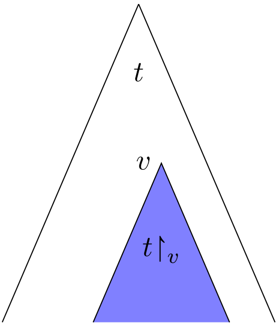

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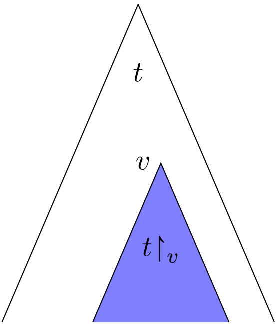

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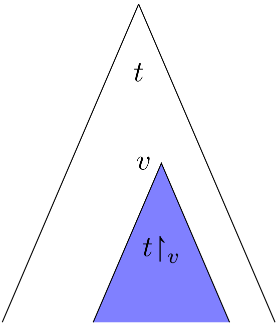
if there exists $\alpha\colon \mathrm{Trees} \to H$...



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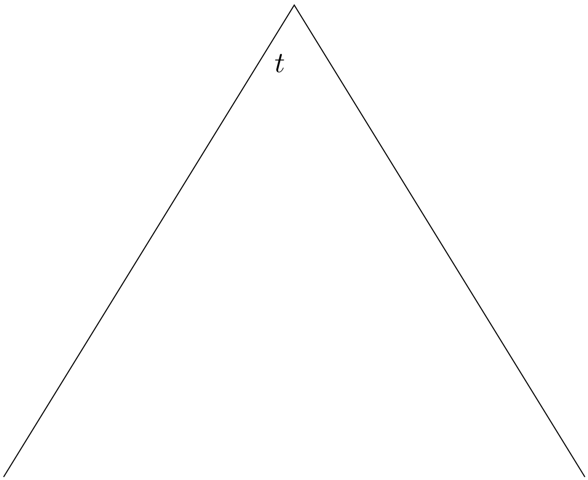

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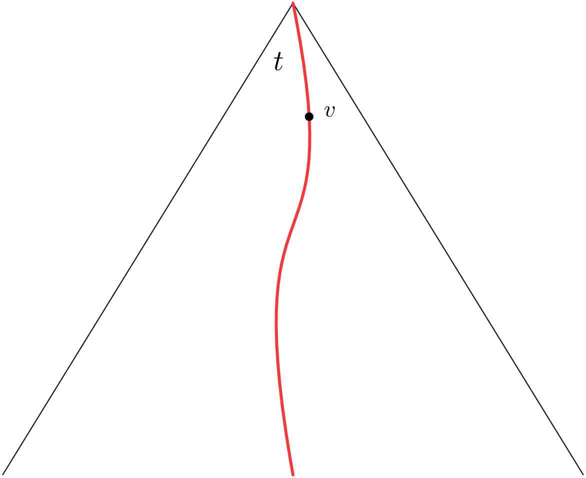

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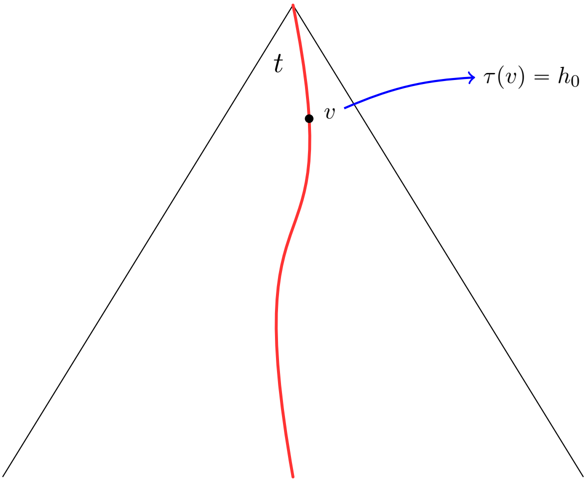

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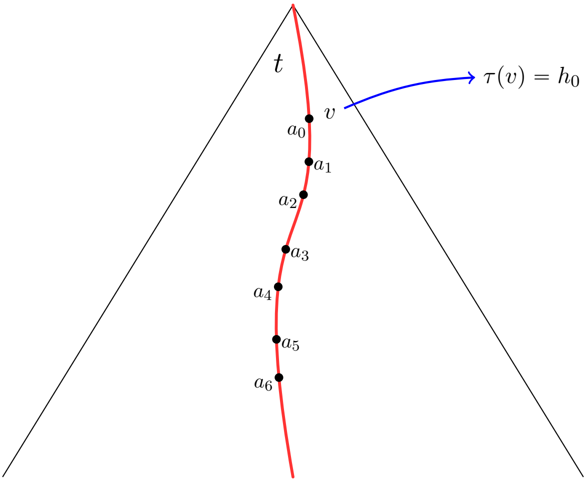

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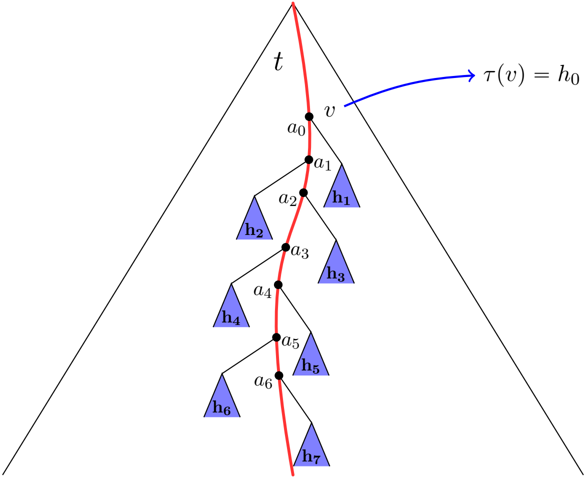

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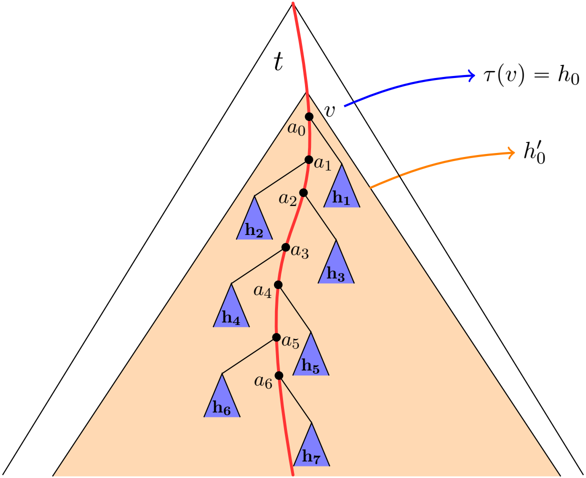

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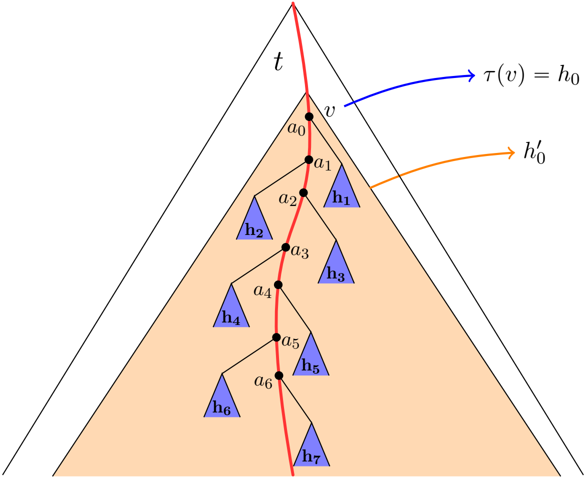

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[no actual marking because $\alpha\colon \mathrm{Thin} \to H$ (not $\alpha\colon \mathrm{Trees} \to H$)]

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