Descriptive set theoretic methods in automata theory

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prof. Igor Walukiewicz

Teoria automatów i weryfikacja

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$\mathcal{A}$:

Teoria automatów i weryfikacja

$\mathcal{A}$: 

np.: serwer, sterownik WiFi, czytnik kart, ...

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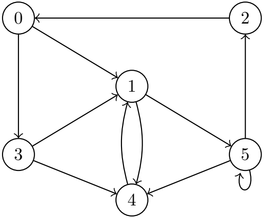

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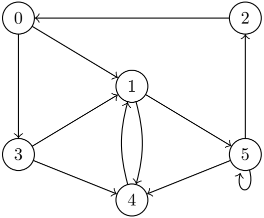

φ: "nieskończenie często odwiedzamy 1"

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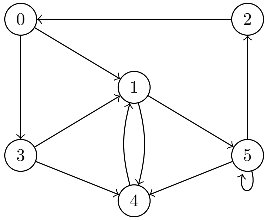

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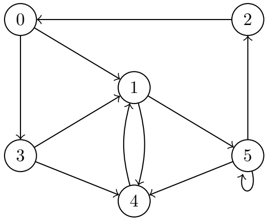

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$$\underbrace{3, 1, 4, 1, 5, \ldots}_{w}$$

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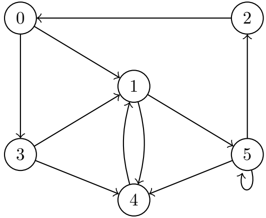

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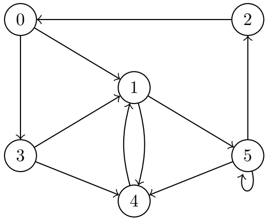

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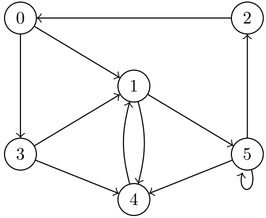

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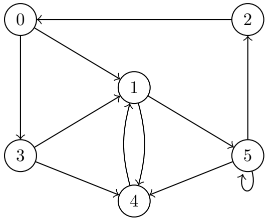

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liczba $3.1415\ldots$

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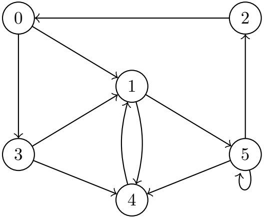

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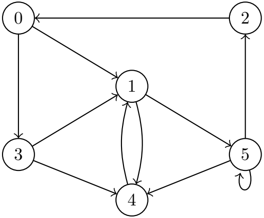

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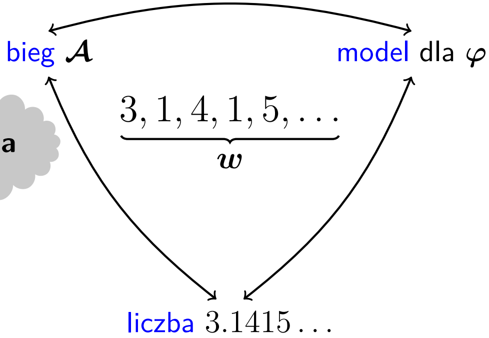
kombinatoryka



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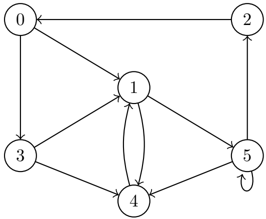

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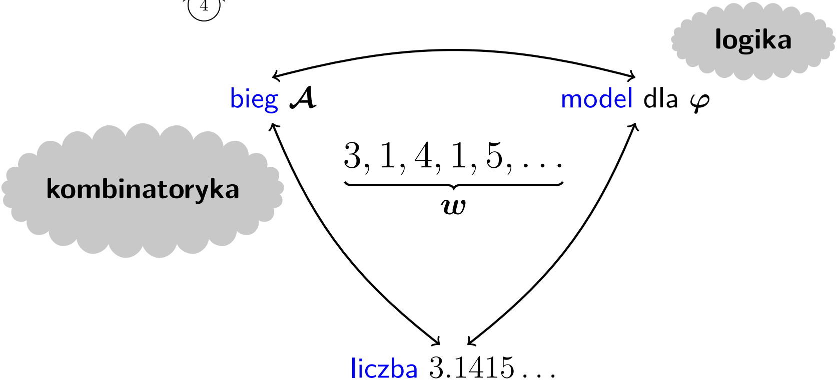

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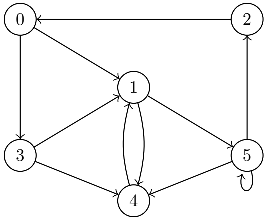

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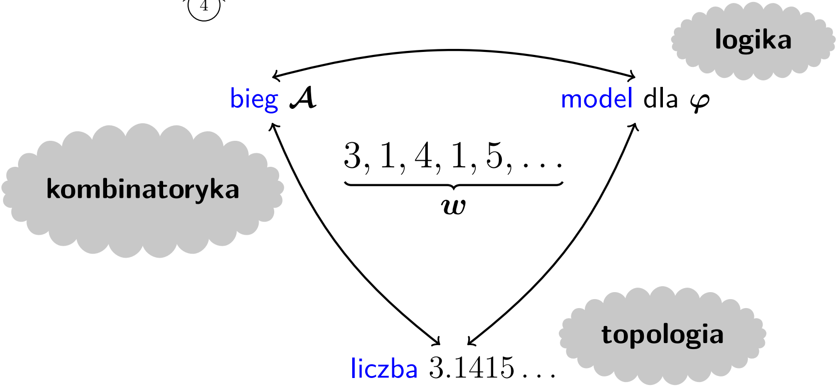

Infinite trees

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$t\colon \{\mathtt{L}, \mathtt{R}\}^* \to A$

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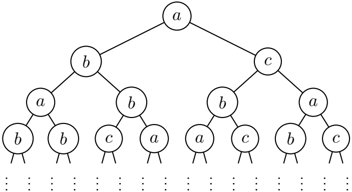

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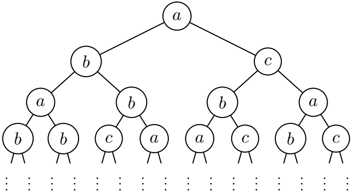

Monadic Second-Order logic (MSO)

Infinite trees

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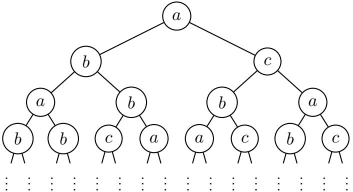

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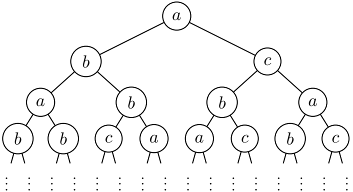

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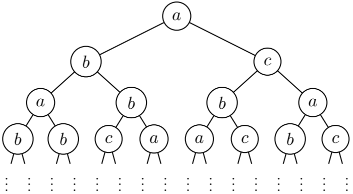

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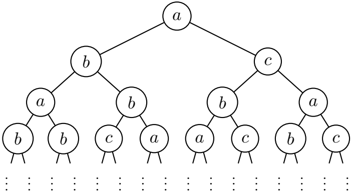

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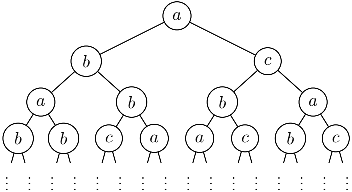

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Weak MSO

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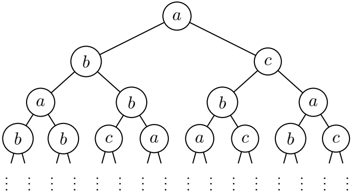

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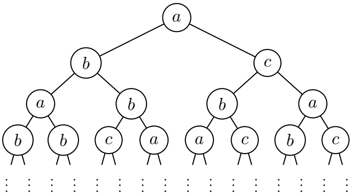

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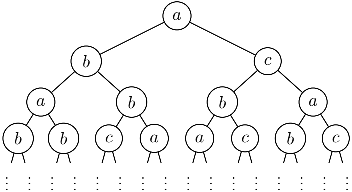

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Theorem (Rabin [1969])
MSO is decidable on infinite trees.

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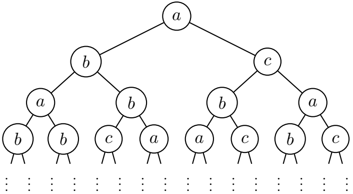

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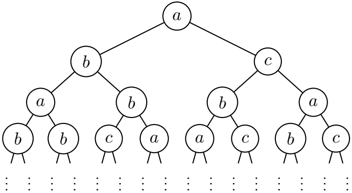

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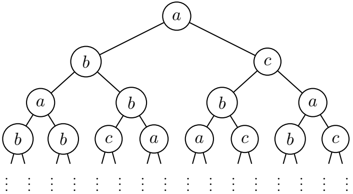

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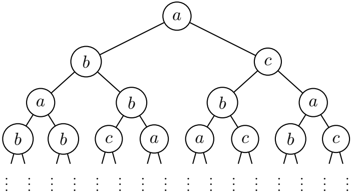

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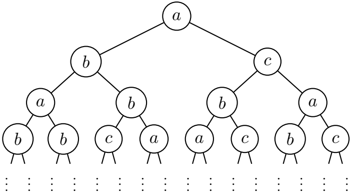

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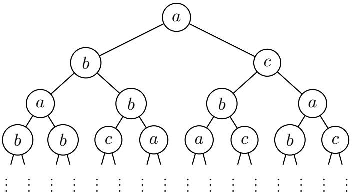

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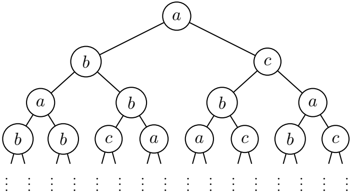

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$\rightsquigarrow$ the **Cantor** set

Regular languages

Regular languages

$\mathrm{L}(\mathcal{A}) = \left\{t : \ \mathcal{A} \text{ *accepts* } \mathrm{t}\right\}$

Regular languages

$\big[$Rabin [1969]: automaton $\mathcal{A} \longleftrightarrow$ formula $\varphi\big]$

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[Rabin [1969]: automaton $\mathcal{A} \longleftrightarrow$ formula φ]

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If L is regular then $L \in \mathbf{\Delta}^1_2$

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Decision problems

Input: Automaton $\mathcal{A}$

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$\mathrm{L}(\mathcal{A}) = \{t : \mathcal{A}$ *accepts* t$\} = \mathrm{L}(\varphi_{\mathcal{A}})$ } **regular** language of infinite trees

If L is regular then $L \in \mathbf{\Delta}^1_2$ There exist regular $L \in \mathbf{\Delta}^1_2 - \sigma(\mathbf{\Sigma}^1_1)$

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Conjecture (Skurczyński [1993])

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Chapter I : Subclasses of regular languages

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Rabin-Mostowski parity-index problem is decidable for **game automata**.

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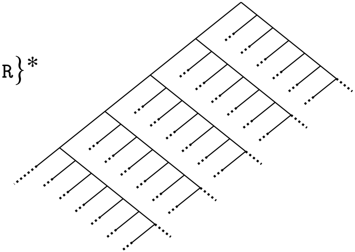

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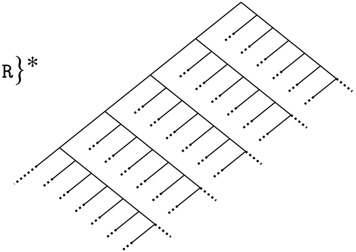

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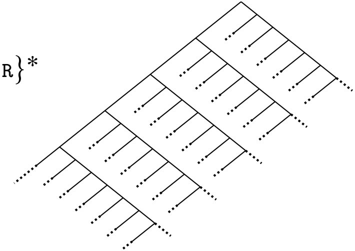

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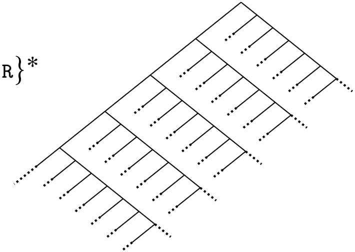

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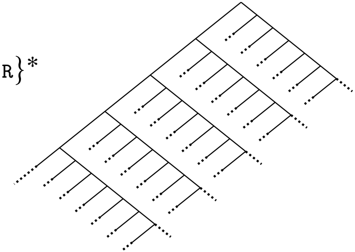

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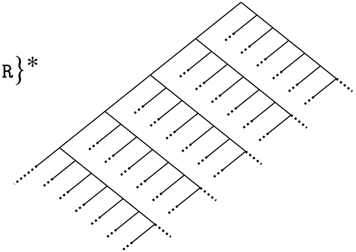

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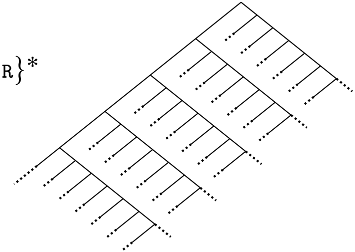

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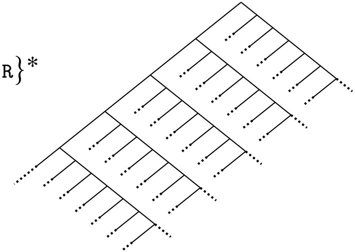

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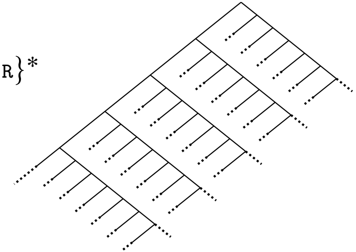

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Proof:

Continuous reductions + pumping + ranks à la Cantor-Bendixson

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$\rightsquigarrow$ **Theorem** (B.G.M.S [2013])

Assuming V=L the MSO+U theory of $\{\text{L}, \text{R}\}^*$ is undecidable.

"there is a universe of set theory where MSO+U is undecidable"

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Summary

Theorem 1 If $\mathcal{A}$ is an unambiguous $\min$-parity automaton of index $(0, j)$ then the language $\mathrm{L}(\mathcal{A})$ can be recognised by an alternating $\mathrm{Comp}(0, j-1)$-automaton of size polynomial in the size of $\mathcal{A}$.

Theorem 2 It is decidable if the language of infinite trees recognised by a non-deterministic Büchi tree automaton is weak MSO-definable.

Theorem 3 The non-deterministic and alternating index problems are decidable for game automata.

Theorem 4 A regular language of thin trees is either:
— $\mathbf{\Pi}^1_1$-complete among all infinite trees,
— WMSO-definable among all infinite trees (and thus Borel).

Moreover, it is decidable which of the cases holds.

Theorem 5 A language of infinite trees L is recognised by a homomorphism into a finite prophetic thin algebra if and only if L is bi-unambiguous.

Theorem 6 Non-existence of MSO-definable choice function on thin trees is equivalent to the fact that every finite thin algebra admits some consistent marking on every infinite tree.

The relation $\varphi(\sigma, t)$ stating that σ is a skeleton of t does not admit any MSO-definable uniformization of σ.

The language of all thin trees is ambiguous.

Theorem 7 There exist languages of ω-words that are definable in MSO+U logic and lie arbitrarily high in the projective hierarchy.

Theorem 8 The proj-MSO theory of $\{\mathrm{L}, \mathrm{R}\}^{\leqslant\omega}$ with prefix $\preceq$ and lexicographic $\leqslant_{\mathrm{lex}}$ orders effectively reduces to the MSO+U theory of the complete binary tree $\left(\{\mathrm{L}, \mathrm{R}\}^*, \preceq, \leqslant_{\mathrm{lex}}\right)$.

An algorithm deciding the proj-MSO theory of $\{\mathrm{L}, \mathrm{R}\}^{\leqslant\omega}$ (together with its proof of correctness) would imply that analytic determinacy fails.

Theorem 9 If L_1, L_2 are disjoint languages of ω-words both recognised by ωB- (respectively ωS)-automata then there exists an ω-regular language L_S such that

$$L_1 \subseteq L_S \quad \text{and} \quad L_2 \subseteq L_S^{\mathrm{c}}.$$

Additionally, the construction of L_S is effective.