On the decidability of MSO+U on infinite trees

M. Bojańczyk [1] T. Gogacz [2]
H. Michalewski [1] M. Skrzypczak [1]

[1] University of Warsaw

[2] University of Wrocław

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MSO+U logic (Bojańczyk '04)

$$\mathsf{U}X.\ \varphi(X) \quad \equiv \quad \forall n.\ \exists X.\ \varphi(X)\ \wedge\ n < |X| < \infty.$$

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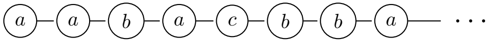
Example

The delays between request and response are uniformly bounded.

Satisfiability problem is φ true in some model

for ω-words — open



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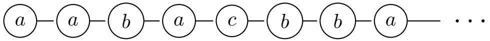

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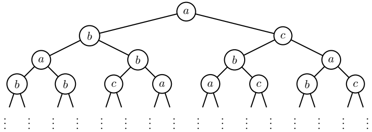

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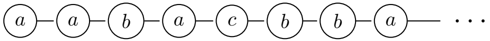

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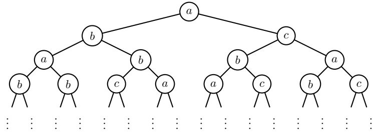

Compositionality

MSO+U admits finite index for natural Myhill–Nerode equivalences

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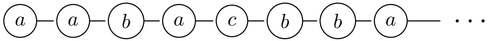

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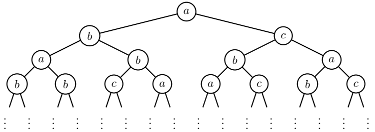

Compositionality

MSO+U admits finite index for natural Myhill–Nerode equivalences

$\rightsquigarrow$ "finitarily: $\mathrm{MSO{+}U} \equiv \mathrm{MSO}$"

Main result

under a set-theoretic assumption **($\star$)**

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$(\star)$: there exists a well-order $\preceq$ on 2^ω s.t.

$$\preceq \text{ is projective } \text{ i.e. } \preceq \in \boldsymbol{\Sigma}^1_n(2^\omega \times 2^\omega)$$

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Independent of axioms of Zermelo–Fraenkel set theory (ZFC)

$\leadsto$ uncoditional undecidability remains **open**

Relative consistency

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If ZFC has a model **then** $\mathrm{ZFC}+(\star)$ has a model

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($\mathrm{MSO+U}$ is undecidable there)

$\rightsquigarrow$ no algorithm for $\mathrm{MSO+U}$ over infinite trees
with correctness proof in ZFC

Main tool: **topology**

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$$
\begin{array}{ccccccccc}
 & & \mathbf{\Sigma}^1_1 & & \mathbf{\Sigma}^1_2 & & \mathbf{\Sigma}^1_3 & & \mathbf{\Sigma}^1_4 & & \\
\mathcal{B} & \begin{array}{c}\diagup \\ \diagdown\end{array} & & \times & & \times & & \times & & & \cdots \\
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Projective hierarchy

$\boldsymbol{\Sigma}^1_1$ = projections of Borel sets

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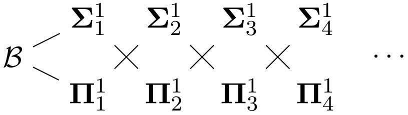

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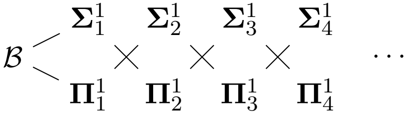

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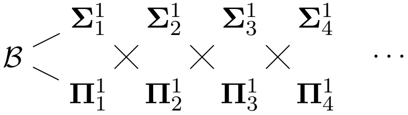

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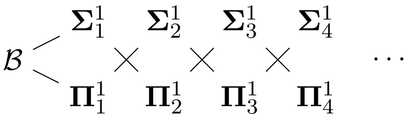

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$\exists_{X_1} \forall_{X_2} \exists_{X_3}$ (Borel condition) $\rightsquigarrow$ $\mathbf{\Sigma}^1_3$ set

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Decidability of projective MSO $\Rightarrow$ **Projective Determinacy fails**

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Proof:
Shelah's proof (reformulated and rewritten).

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Theorem (Rabin 1969)

The MSO *theory of* $(2^\omega, \leq)$ *with quantifiers ranging over* F_σ *sets is decidable.*

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Open problem

Is $\mathrm{MSO{+}U}$ *decidable over* ω*-words?*

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[relation with $\mathrm{MSO{+}inf}$ over profinite words (Toruńczyk 2012)]