Regular languages of thin trees

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The setting — regular languages

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Motivating problem

Except decidability, little is known about regular languages of infinite trees.
No *simple* algebras for infinite trees.

Outline

- Thin trees: structures in-between words and trees.
- Thin forest algebra: Wilke algebra $\oplus$ forest algebra.
- Effective characterisations.
- Topological properties: thin trees are much poorer than all trees.

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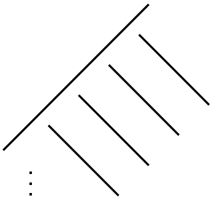
- Topological properties: thin trees are much poorer than all trees.

- Finite alphabet A.
- Infinite, finitely branching, labelled trees t (leafs allowed).
- Regular languages L (those definable in MSO logic).
- Also, weak regular languages (definable in weak MSO logic).

Definition

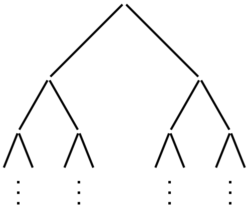
A tree is *thin* if it has only countably many infinite branches.

THIN



A *comb*

THICK



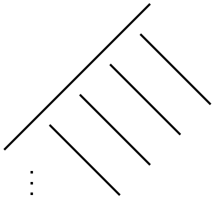
The full binary tree

Lemma (Cantor, Bendixson [1882])

A tree is either:

- **thin** — has countably many infinite branches,
- **thick** — contains a full binary tree as a minor.

THIN



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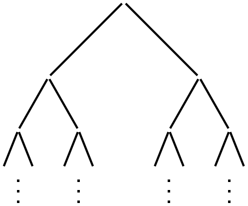

Lemma (Cantor, Bendixson [1882])

A tree is either:

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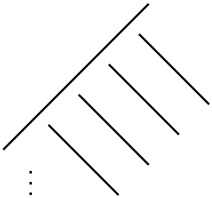
Corollary

Being a thin tree is MSO-definable.

THIN



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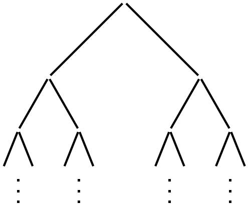
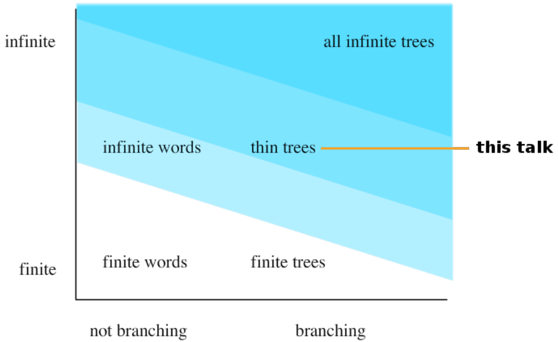
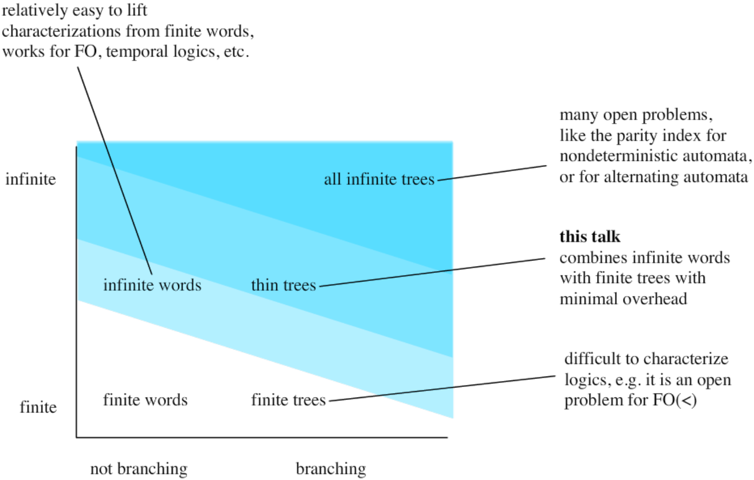


Diagram



infinite
all infinite trees
infinite words
thin trees
this talk
finite
finite words
finite trees
not branching
branching

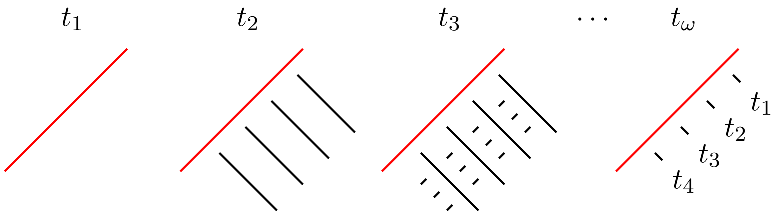
Diagram



relatively easy to lift characterizations from finite words, works for FO, temporal logics, etc.
many open problems, like the parity index for nondeterministic automata, or for alternating automata
infinite
all infinite trees
this talk
combines infinite words with finite trees with minimal overhead
infinite words
thin trees
difficult to characterize logics, e.g. it is an open problem for FO(<)
finite
finite words
finite trees
not branching
branching

Structural induction

$\mathrm{rank}(t)$ — a measure of the complexity of t.



$\mathrm{rank}(t_1) = 1$ $\mathrm{rank}(t_2) = 2$ $\mathrm{rank}(t_3) = 3$ $\mathrm{rank}(t_\omega) = \omega$

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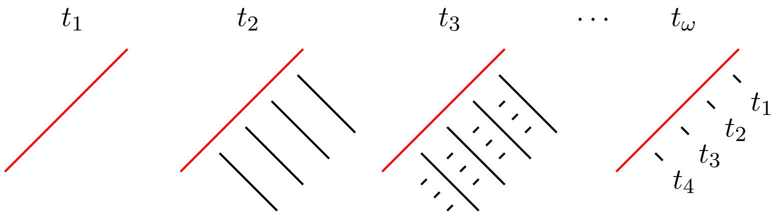

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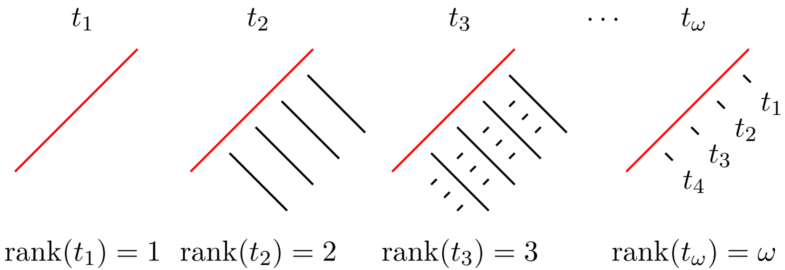


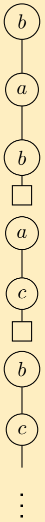
A thin tree consists of a *spine* and subtrees of smaller rank along it.

Cannot assign rank to a thick tree.
Every thin tree t has $\mathrm{rank}(t) < \omega_1$.
The spine need not be the leftmost branch!

Thin forest algebra

Thin forest algebra

Wilke algebra



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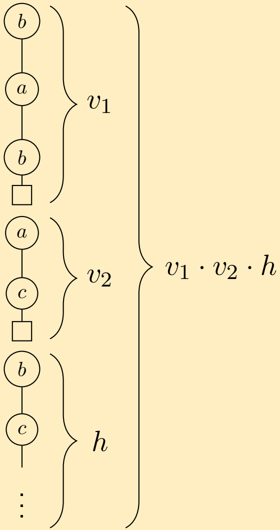

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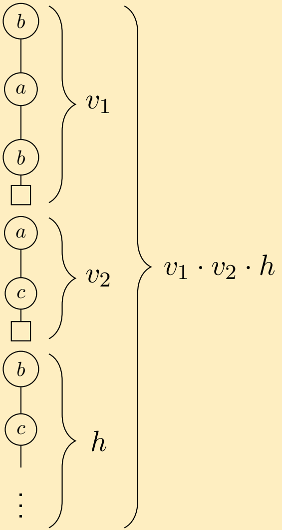

⊕

For the sake of algebra

Use forests instead of trees!

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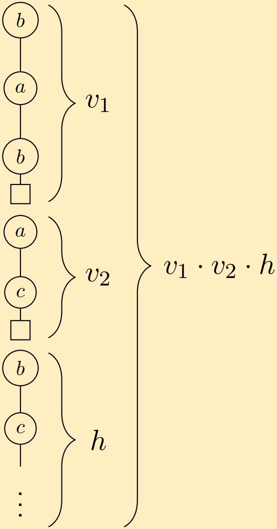

$\oplus$

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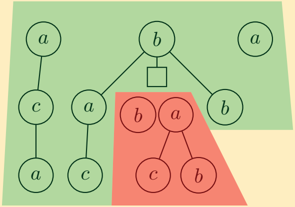
Forest algebra



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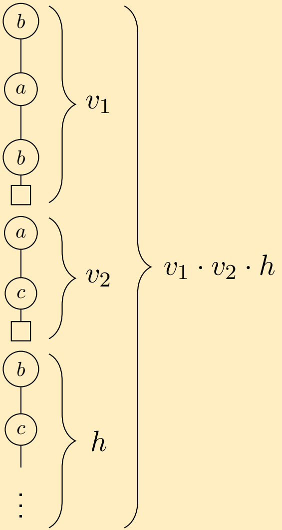

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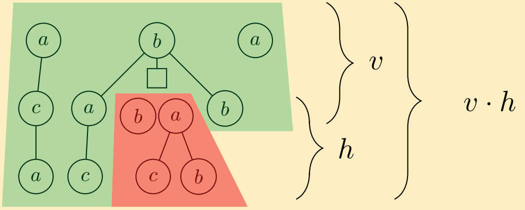
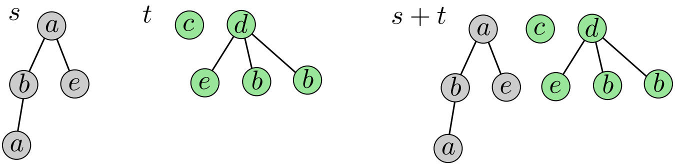
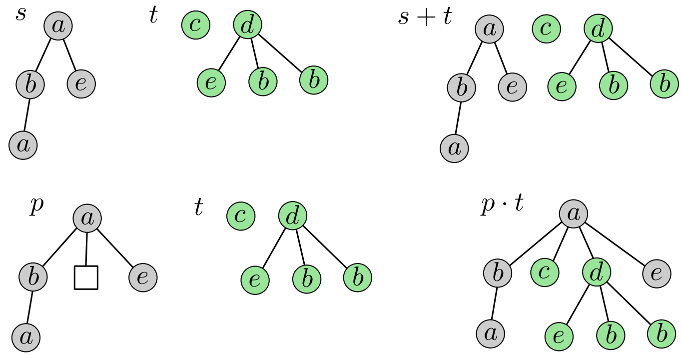
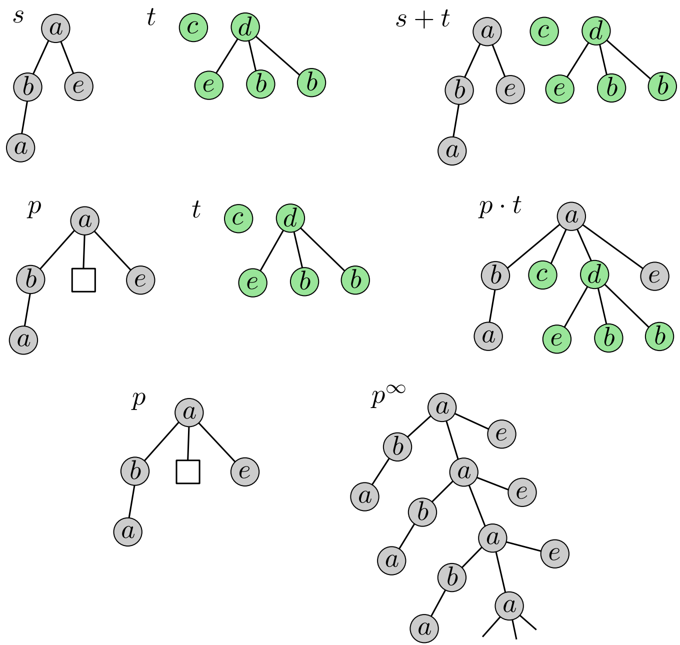



s
t
s + t


s
t
s + t
p
t
p · t


s
t
s + t
p
t
p · t
p
p^∞

Effective characterisations

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[1] Only finitely many changes on every path.

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Intermediate step

For every forests s and t check:

$$s + t \ \sim_L \ t + s \quad \text{(Myhill-Nerode style equivalence)}$$

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Theorem

A regular language of thin trees is closed under bisimulational equivalence iff its syntactic algebra satisfies identities

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Remark

For thick trees no such equational characterisation is known!

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Theorem

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1. *L is weak MSO-definable among all trees,*
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1. L *is weak MSO-definable among all trees,*
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3. L *is* ***not*** *co-analytic* $(\mathbf{\Pi}^1_1)$*-hard among all trees,*
4. *the syntactic morphism for* L *satisfies condition*

$$\text{if} \quad h = v(w+h)^\infty \quad \text{or} \quad h = v(h+w)^\infty \quad \text{then} \quad h = \perp.$$

Descriptive complexity

Theorem

Every regular language of thin trees is co-analytic $(\mathbf{\Pi}^1_1)$ *among all trees.*

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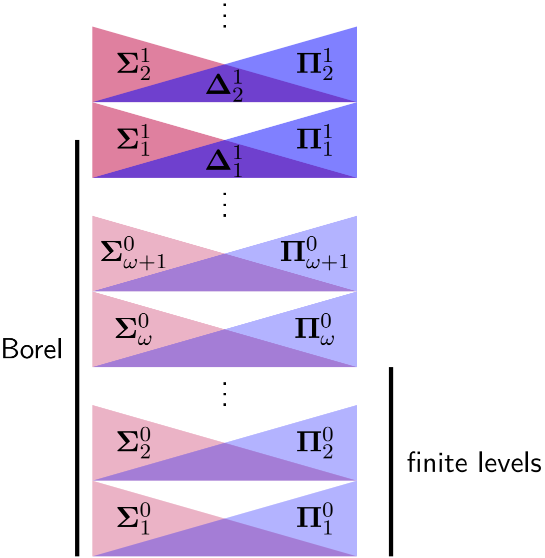
Conjecture in all trees (*a gap property*):

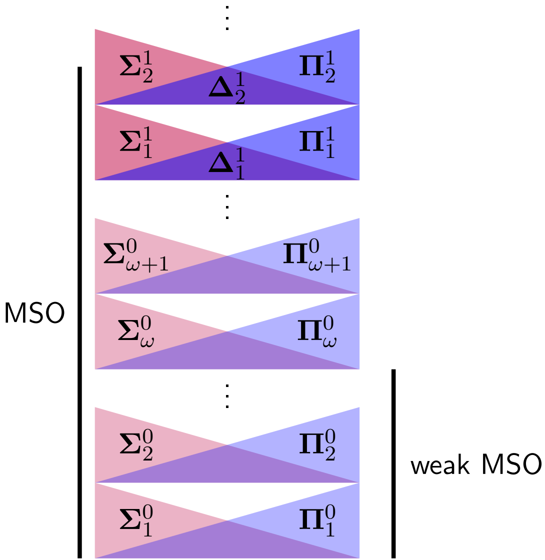
$$\text{Borel and regular} \implies \text{weak MSO-definable}$$

Corollary

A regular language of thin trees is either:

- *definable in weak MSO among all trees,*
- $\mathbf{\Pi}^1_1$*-complete among all trees.*


Σ^1_2
Δ^1_2
Π^1_2
Σ^1_1
Δ^1_1
Π^1_1
$\Sigma^0_{\omega+1}$
$\Pi^0_{\omega+1}$
Σ^0_ω
Π^0_ω
Σ^0_2
Π^0_2
Σ^0_1
Π^0_1
Borel
finite levels



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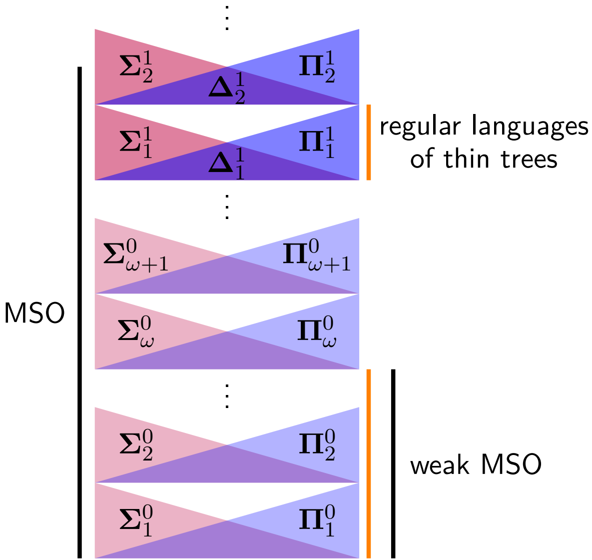
Π^0_1
MSO
weak MSO


Σ^1_2
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regular languages of thin trees
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MSO
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weak MSO
Σ^0_1
Π^0_1

Summary

- Structures in-between words and trees.
- Nice (simple) algebras.
- Equational characterisations of various properties.
- Collapse of the complexity comparing to all trees.

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Open problems

- Decidability of the weak MSO-definability **among thin trees**?
- Is it possible to extend these techniques/results to all trees?

Effective characterisations

Effective (equational) characterisations of regular languages of thin trees that are:

- open in the standard topology,
- commutative (in two flavours),
- invariant under bisimulation (in two flavours),
- weak MSO-definable among all trees.

Descriptive complexity

Every regular language of thin trees is:

- co-analytic ($\mathbf{\Pi}^1_1$) among all trees,
- recognisable by a non-det. (1,3)-automaton among all trees,
- recognisable by an unambiguous automaton among thin trees,
- not harder than Borel sets (as a subset of thin trees).

Thank you for your attention!