Descriptive complexity vs. decidability for MSO

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Journées d'Informatique Fondamentale de Paris Diderot
25'th April 2013, Paris

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Regular languages of finite words

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Regular expressions

$$(aa)^* \; b \; (b \mid c)^*$$

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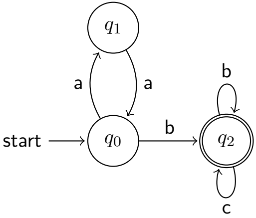

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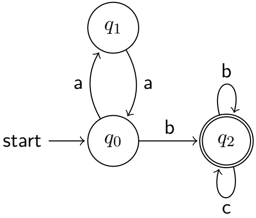

Monadic Second-Order logic

$$\exists_x\ b(x) \wedge \forall_{y<x}\ a(y)\ \wedge\ \ldots$$

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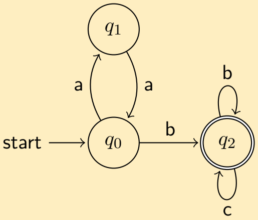

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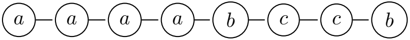
$w =$ 

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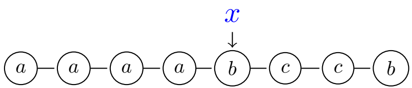

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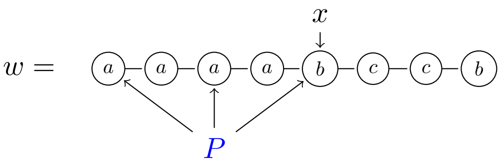

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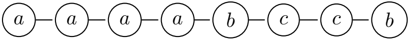
$w =$ 

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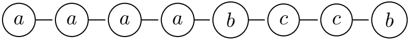
$w =$ 

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Satisfiability problem

Given an MSO formula ψ decide whether $\mathrm{L}(\psi) \neq \emptyset$?

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Formula is a declarative definition of the language.
Automata are more operational.

Finite automata

$$L = (a \mid b \mid c)^* \, abc \, (a \mid b \mid c)^*$$

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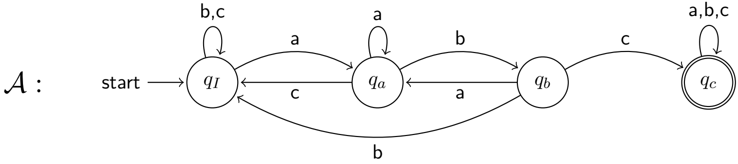
Deterministic automaton — only one transition to use



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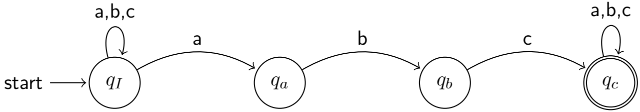

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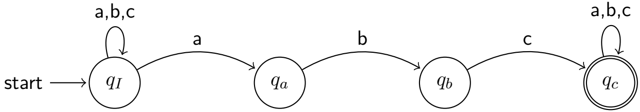

Run of an automaton

$$\begin{array}{lcl} w & = & \quad a \quad\; b \quad\; b \quad\; a \quad\; b \quad\; c \quad\; c \\ \rho & = & q_I \longrightarrow q_I \longrightarrow q_I \longrightarrow q_I \longrightarrow q_a \longrightarrow q_b \longrightarrow q_c \longrightarrow q_c \end{array}$$

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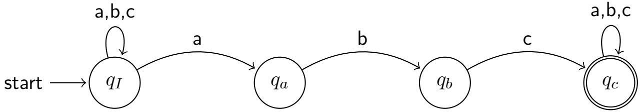

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A run ρ is accepting if it ends in a final state.

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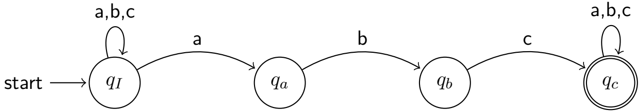

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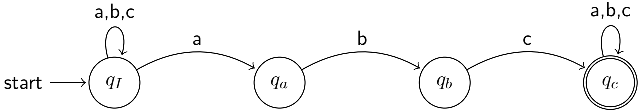

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deterministic : $\{w \in A^* :$ the run of $\mathcal{A}$ on w is accepting$\}$

non-det. : $\{w \in A^* :$ exists an accepting run of $\mathcal{A}$ on $w\}$

Equivalence

Theorem (Büchi [1960], Elgot [1961], Trakhtenbrot [1962])

For $L \subseteq A^*$ the following conditions are (effectively) equivalent:

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Such a language is called regular.

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Given a non-deterministic automaton compute an equivalent deterministic one (via powerset construction).

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Deciding satisfiability

Take ψ, compute an equivalent automaton $\mathcal{A}$, check if $\mathcal{A}$ accepts some word.

Simplifying the formulæ

ψ — MSO formula (many quantifiers inside)

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ψ' — simplified formula

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$\psi' = \exists_\rho \; (\rho \text{ is a run of } \mathcal{A}) \;\wedge\; (\rho \text{ is accepting})$

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$$\psi' = \exists_\rho \underbrace{(\rho \text{ is a run of } \mathcal{A}) \wedge (\rho \text{ is accepting})}_{\text{no set quantifiers here}}$$

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$$\psi'' = \underbrace{\forall_\rho \ (\rho \text{ is a run of } \mathcal{A}) \ \Rightarrow \ (\rho \text{ is accepting})}_{\forall_{\vec{X}} \ (\ldots) \text{ — universal formula}}$$

$$w = \quad a - a - a - a - b - c - c - b - \cdots \quad \in A^{\omega}$$

Infinite words

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ω-language defined by ψ — the same definition

$$\mathrm{L}(\psi) = \{w \in A^\omega :\ w \text{ satisfies } \psi\}\,.$$

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What about automata?

When an infinite run ρ is accepting?
There is no *last* state!

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A run of a Büchi automaton is accepting if it visits a final state infinitely many times.

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Parity automata

Each state q has priority $\Omega(q) \in \mathbb{N}$.
A run ρ of a parity automaton is accepting if:

the highest priority visited infinitely often is even.

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Theorem (Büchi [1962], McNaughton [1966], . . .)

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Original proof (Büchi)

Only Büchi automata, no determinization, direct complementation:

$$\text{given } \mathcal{A} \text{ compute } \mathcal{B} \text{ such that } \mathrm{L}(\mathcal{B}) = A^\omega \setminus \mathrm{L}(\mathcal{A}).$$

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Problems

Powerset construction is not enough to determinize!
Deterministic Büchi automata are too weak.
Determinization procedure is complicated.

Equivalence continued

Using non-deterministic Büchi automata and complementation

Decidability of $\mathrm{L}(\psi) \neq \emptyset$.

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Verification of interactive systems

- model a system as a finite automaton,
- write an MSO formula ψ specifying allowed behaviours,
- construct an automaton $\mathcal{B}$ recognising bad behaviours
- check for emptiness of $\mathcal{B}$.

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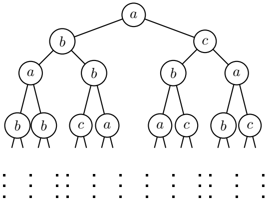
Using deterministic parity automata

Even more: memoryless winning strategies, Wagner hierarchy, . . .

Infinite trees — labellings of the full binary tree



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Trees are expressive

One infinite tree can encode:

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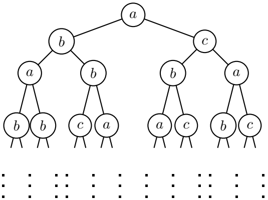

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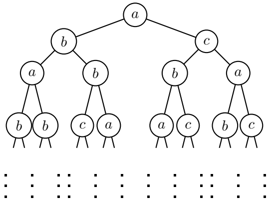

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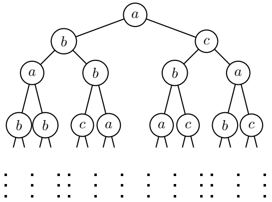

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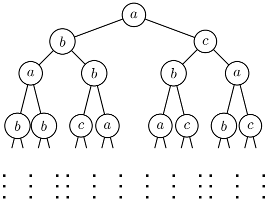

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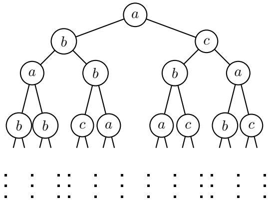

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What about decidability?

Given ψ check if there is a tree satisfying ψ?

Mother of all decidability results

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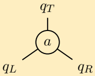
Branching transitions: 

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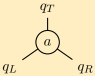
Branching transitions: 

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$$\mathrm{L}(\mathcal{A}) = \left\{ t : \underset{\text{run } \rho}{\exists} \ \underset{\text{branch } \pi}{\forall} \ \rho \text{ is accepting on } \pi \right\}$$

No simple automata for infinite trees

Theorem (Rabin [1969], Emerson&Jutla [1991], Mostowski [1991])

The following conditions are (effectively) equivalent for a language of trees L:

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Proof of 2)

Pumping argument **OR** descriptive complexity argument...

Descriptive complexity

Idea

How many set quantifiers need to appear in a definition of $L \subseteq X$:

$$L = \{x : \varphi\}?$$

(We don't restrict φ to MSO, it can be any formula of arithmetic.)

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Better than NP

The hierarchy is strict: for every n there is a set L that requires n set quantifiers: $\underbrace{\exists\forall\ldots\exists\forall}_{n}$.

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Intuition

Any machine recognising L gives an upper bound on the descriptive complexity of L.

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Note

All additional features allowed in $\mathcal{M}$: counters, stacks, tapes, ...

The other way round

Research plan

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Derive some undefinability / undecidability results.

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Conjecture (Shelah [1975])

The MSO theory of $(\mathbb{R}, <, \mathbf{Borel})$ *where set quantifiers range over Borel subsets of* $\mathbb{R}$ *is decidable.*

The other way round — a classical example

Theorem (Rabin)

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Find a regular tree language that requires a universal set quantifier. For example:

$$\left\{ t : \ \forall_{\text{branch } \pi} \ \text{almost all letters on } \pi \text{ are } a \right\}$$

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Theorem (Bojańczyk, Toruńczyk [2012])

Satisfiability problem of weak MSO+U is decidable over infinite trees.

Round 1 (joint with Hummel)

Theorem (Hummel, S. [2010])

There is an ω-word language definable in MSO+U that requires at least one universal set quantifier.
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Corollary

There is no model of alternating machines on ω-words with a fixed projective acceptance condition capturing MSO+U.

Round 2 (joint with Bojańczyk, Gogacz, and Michalewski)

Theorem (B., G., M., S. [2013] (unpublished))

Under the assumption that V=L, MSO+U logic is undecidable over infinite trees.

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Corollary

If there exists a proof that MSO+U is decidable over infinite trees

then

Set Theory is inconsistent.

Round 2 (joint with Bojańczyk, Gogacz, and Michalewski)

Intermediate statement

If $\mathbf{\Lambda}$ is any extension of MSO that defines some ω-word language requiring 6 alternations of set quantifiers

then (assuming V=L)

the $\mathbf{\Lambda}$-theory of the full binary tree is undecidable.

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MSO+U fits ideally in — it defines ω-word languages requiring arbitrarily many set quantifiers.

Summary & further work

Making long story short. . .

Decidability

$\updownarrow$

Definability

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Low descr. complexity

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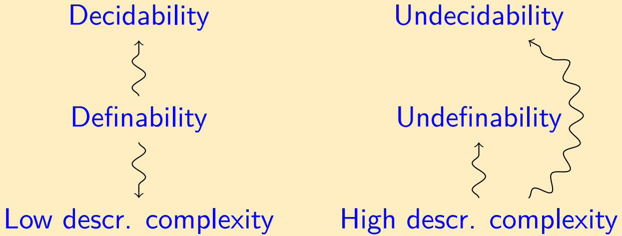

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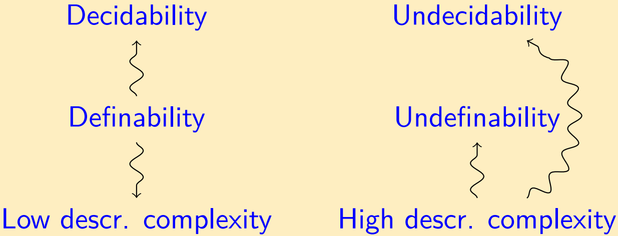

Getting rid of V=L

If there is no proof that MSO+U is decidable, there should be a direct proof of undecidability.

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Decidability ⇝ Definability ⇝ Low descr. complexity

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Rabin's theorem implies that $\mathrm{MSO}(\mathbb{R}, <, \mathbf{\Sigma}^0_2)$ is decidable.

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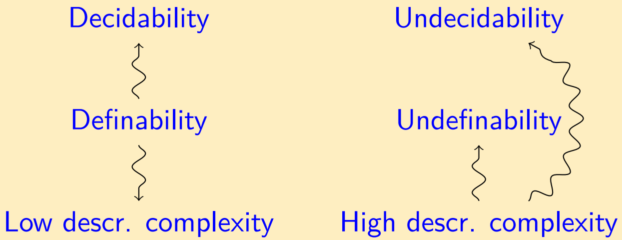

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What about $\mathrm{MSO}(\mathbb{R}, <, \mathbf{\Sigma}^0_3)$? Or $\mathrm{MSO}(\mathbb{R}, <, \mathbf{Borel})$?

Thank you for your attention!