$\mathrm{MSO} + \mathrm{U}$ describes sets at arbitrarily high levels of the projective hierarchy

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2. Topology, projective hierarchy.
3. Trees, combinatorics and the main construction.

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The setting

Finite alphabet A, infinite words $w \in A^{\omega}$.

Quantifier U

Introduced by Bojańczyk and Colcombet [Boj04], [BC06].

$$\mathsf{U}X.\ \varphi(X)$$

iff

there are finite sets X satisfying $\varphi(X)$ of arbitrarily big size.

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Definition

$\mathrm{MSO}+\mathrm{U}$ = Monadic Second Order logic extended by U.

Example

infinitely many $b \ \wedge \ \mathsf{U}X.\ X$ is a block of a's.

defines words $a^{n_1}ba^{n_2}b\dots$ such that (n_i) is unbounded.

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$\exists_B\ B$ is an infinite set of blocks $ba^nb\ \wedge$

$\neg\mathsf{U}_{X\subseteq B}\ X$ is a block of a's.

defines words $a^{n_1}ba^{n_2}b\ldots$ such that $\liminf_{i\to\infty}\ n_i < \infty$.

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Problem: MSO + U

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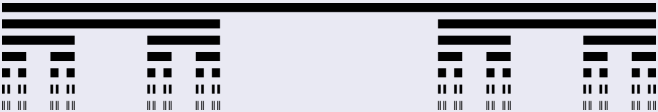
Problem: $\mathrm{MSO} + \mathrm{U}$

- Decidability?
- Automata model?
- Topological complexity?

Topology

Cantor space

A^ω is the Cantor space.



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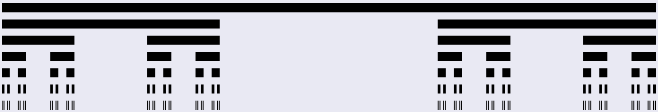

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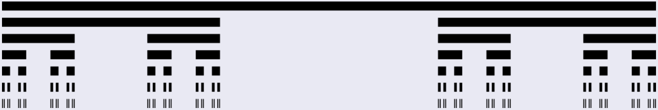

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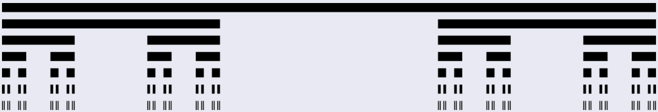

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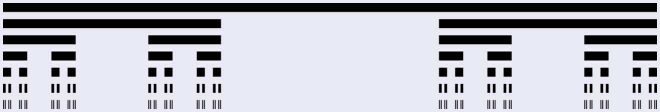

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Borel sets $\mathcal{B}$

- Closure of the family of open sets by countable unions and countable intersections.
- Well behaved (constructive) sets — e.g. they satisfy Continuum Hypothesis and determinacy.
- Many *natural* sets are Borel: properties like boundedness, liveness, safety, $\limsup$, $\liminf$, ...

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- $\liminf < \infty$ — union of

  „there are infinitely many values smaller then n".

Projection

Take a set $L \subseteq A^\omega \times B^\omega$. Consider

$$\pi_1(L) \;=\; \{u \in A^\omega : \exists_{v \in B^\omega}\,(u, v) \in L\}\,.$$

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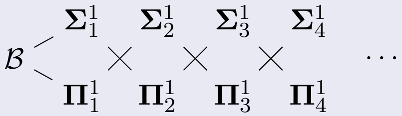
Projective hierarchy

$\mathbf{\Sigma}^1_1$ — the family of projections of Borel sets,
$\mathbf{\Pi}^1_1$ — complements of $\mathbf{\Sigma}^1_1$,
$\mathbf{\Sigma}^1_2$ — projections of $\mathbf{\Pi}^1_1$,
. . .



Estimations

Example: alternating Borel automata

Let $\mathcal{A}$ be an alternating automaton with configuration space C.
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In particular alternating ω-BS automata are in $\mathbf{\Sigma}^1_2$.

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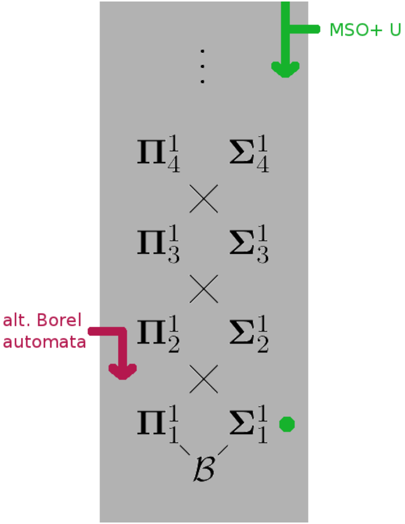
Similarly

Take φ an $\mathrm{MSO}+\mathrm{U}$ formula. Assume that φ contains k quantifiers. Then $L(\varphi) \in \mathbf{\Sigma}^1_{k+1}$.

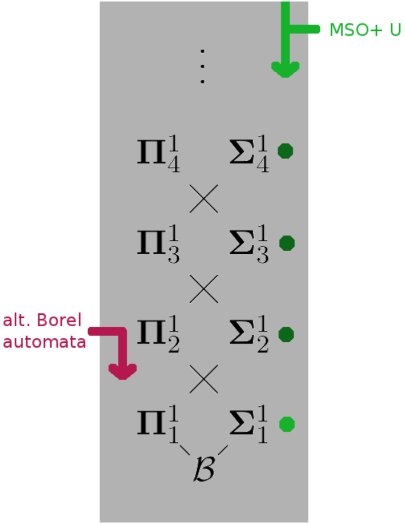
Theorem (Hummel, S., Toruńczyk 2010)

There is a $\boldsymbol{\Sigma}^1_1$-complete set definable in $\mathrm{MSO} + \mathrm{U}$.



New theorem

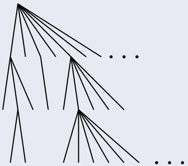
For every i there is a $\boldsymbol{\Sigma}^1_i$-hard set definable in $\mathrm{MSO} + \mathrm{U}$.



Trees

Definition

A tree over $\mathbb{N}$ is a prefix-closed subset $t \subseteq \mathbb{N}^*$.



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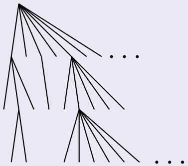

Example

The set of trees that contain an infinite branch

$$\{t \subseteq \mathbb{N}^* : \exists_{\eta \in \mathbb{N}^\omega}\ \eta \text{ is an infinite branch of } t\}$$

is $\mathbf{\Sigma}^1_1$-complete.

Idea of the proof

1. Take $\mathbf{\Sigma}^1_i$-hard set of *multidimensional* trees.
2. Iteratively encode trees into infinite words. Do it in a way *convenient* for $\mathrm{MSO}+\mathrm{U}$.
3. Write an $\mathrm{MSO}+\mathrm{U}$ formula expressing this hard property.

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Encoding: a basic ingredient

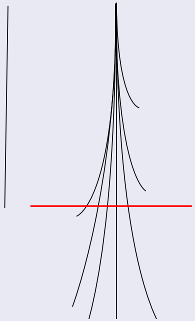
Enumerate all vertices of a given tree encoding $v = (v_1, v_2, \ldots, v_m) \in \mathbb{N}^*$ as

$$a^{v_1} b a^{v_2} b \ldots a^{v_m}$$

A witness of a branch (last year's result)

A set of vertices of a tree $G \subseteq \mathbb{N}^*$ is:

- deep contains arbitrarily deep vertices,
- thin on any finite depth there are only finitely many paths to elements of G.



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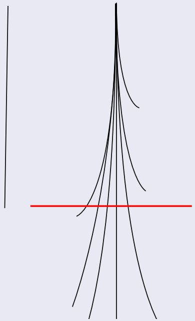
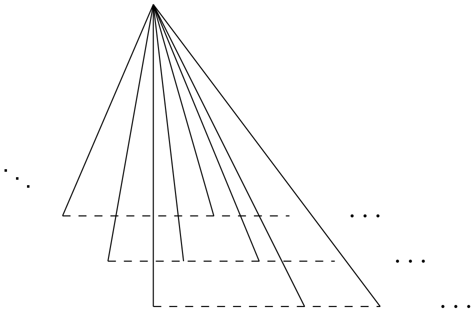


Hint

König's lemma

Trees on $\mathbb{N}^i$

A tree on $\mathbb{N}^i$ is a prefix closed subset $t \subseteq (\mathbb{N}^i)^*$. Let Tr^i be the set of all such trees.



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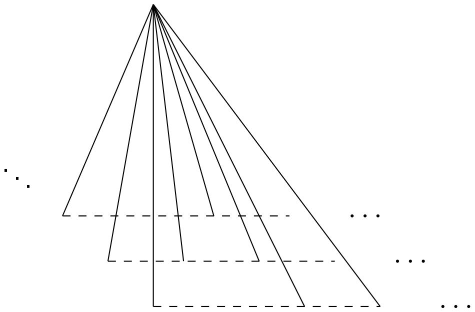
For a sequence $\eta \in \mathbb{N}^\omega$ and a tree $t \in \mathrm{Tr}^i$ let $t\restriction_\eta \in \mathrm{Tr}^{i-1}$ be:

the subtree of t where i'th coordinate of vertices correspond to η.



Hard property

Inductive definition $\mathrm{IF}^i \subseteq \mathrm{Tr}^i$:

- IF^1 are trees in Tr^1 with an infinite branch,
- IF^{i+1} are trees $t \in \mathrm{Tr}^{i+1}$ for which there exists a sequence $\eta \in \mathbb{N}^\omega$ such that

$$t\restriction_\eta \notin \mathrm{IF}^i.$$

Fact: IF^i is $\mathbf{\Sigma}^1_i$-complete.

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Notice

We cannot express in $\mathrm{MSO}+\mathrm{U}$ that a given word $u \in A^{\omega}$ encodes a tree $t \in \mathrm{Tr}^{i}$. But we don't need to! It's enough to build formulas φ_i such that

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Summary

- $L(\varphi_i)$ is $\mathrm{MSO}+\mathrm{U}$ definable and $\mathbf{\Sigma}^1_i$-hard.
- $\mathrm{MSO}+\mathrm{U}$ defines languages as complicated as possible.
- There is no alternating automata model with Borel (or even fixed projective) accepting condition that captures whole $\mathrm{MSO}+\mathrm{U}$.

Thank you for your attention!

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