Topological properties of infinite computations

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Task

- *Define a semantics of programs,*
- *express some expected properties,*
- *proof or better automatically verify them.*

Solution

For usual (terminating) programs:

- *Write programs in declarative languages,*
- *(and/or) define small steps/big steps/continuation semantics,*
- *(and/or) use Hoare logic,*
- *(and/or) . . .*
- *neither way: **treat a program as a partial function** in $\rightarrow$ out.*

Sometimes one doesn't want to terminate!

Example

Mail server has three states: receiving, reordering, sending. We want to ensure, that infinitely often server will receive:

$$\forall_{n \in \mathbb{N}} \exists_{k \geq n} \ S(k) = \text{receiving}.$$

Problems

- *We cannot even simulate the whole computation — its infinite,*
- *first order logic over* $(\mathbb{N}, +, \cdot)$ *is undecidable,*
- *servers are parallel, so testing may not expose possible errors.*

Idea

Model infinite computation as a sequence $\alpha \in \Sigma^\omega$, for a given finite set Σ of states of a server.
Use Monadic Second Order (MSO) logic over $(\omega, <, \Sigma)$ to express properties of such computations.

Example

For $\Sigma = \{A, B\}$ consider a formula

$$\varphi = \exists_{P \subseteq \omega} \forall_{n \in \omega} \ (n \in P \Leftrightarrow s(n) \notin P) \wedge 0 \in P \wedge \exists_{k \in \omega} \, k \in P \wedge A(k).$$

$\alpha \in \Sigma^\omega$ has a property $\alpha \models \varphi$ iff there exists even k such that $\alpha_k = A$.

More formally

Definition

MSO is SO logic with additional restriction, that second order quantification may bind only sets of elements. For φ over $<, \Sigma$, let

$$L(\varphi) = \{\alpha \in \Sigma^\omega : \alpha \models \varphi\}.$$

Theorem (Büchy 1960)

MSO logic over $(\omega, <, \Sigma)$ is decidable — there exists an algorithm that reads a formula and outputs whether $L(\varphi) = \emptyset$.

1. Define a model of automaton reading infinite word and accepting or rejecting it.
2. Show that each formula can be translated into an automaton.
3. Construct a program as above, operating on the graph of the automaton.

Definition

Deterministic parity automaton is a tuple $\mathcal{A} = \langle q_0, Q, \delta, \Omega \rangle$ where

- q_0 is an element of Q,
- Q is finite set of *states*,
- $\delta\colon Q \times \Sigma \to Q$ is a *transition function*,
- $\Omega\colon Q \to \mathbb{N}$ maps a state $q \in Q$ into its *rank* $\Omega(q) \in \mathbb{N}$.

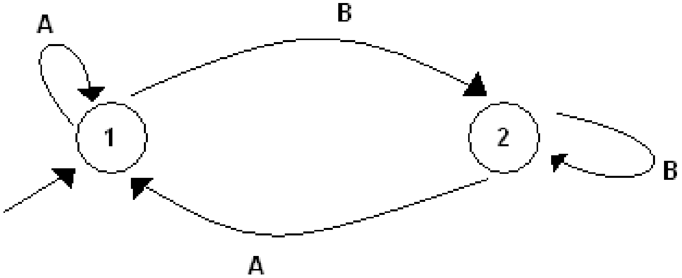
For a given word $\alpha \in \Sigma^\omega$, we define a *run* $\tau \in Q^\omega$ inductively $\tau_0 = q_0$ and $\tau_{n+1} = \delta(\tau_n, \alpha_n)$. We say that $\mathcal{A}$ accepts a word α iff

$$\liminf_{n\to\infty} \Omega(\tau_n) \equiv 1 \mod 2.$$

By $L(\mathcal{A})$ (language recognised by automaton $\mathcal{A}$) denote the set of all words $\alpha \in \Sigma^\omega$ accepted by $\mathcal{A}$.

Example

Consider automaton $\mathcal{A}$: $Q = \{1, 2\}$, $q_0 = 1$, $\Omega(q) = q$ and $\delta(q, A) = 1$ and $\delta(q, B) = 2$. Then $\mathcal{A}$ accepts α iff α contains infinitely many A's.



Definition

WMSO *(Weak MSO) is* MSO *logic with restriction that quantifiers bind only finite sets.*

Definition

ω-regular languages are (equivalently):

- *defined by* MSO *formulas,*
- *defined by* WMSO *formulas,*
- *recognised by deterministic parity automata,*
- *recognised by nondeterministic Büchi automata.*

Fact

The family of ω-regular languages is closed under boolean operations and projections: $\pi : \Sigma^\omega \times \Gamma^\omega \to \Sigma^\omega$.

Idea

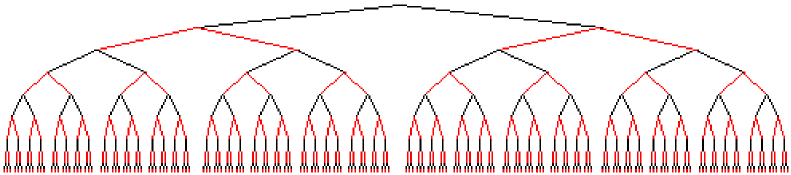
Σ^ω is homeomorphic to Cantor's discontinuum. Why not use methods known from descriptive set theory to investigate the complexity of languages $L(\varphi) \subseteq \Sigma^\omega$?

Definition

Consider a topology on Σ^ω generated by sets

$$[s] := \left\{\alpha \in \Sigma^\omega : \alpha|_{|s|} = s\right\},$$

for $s \in \Sigma^*$.

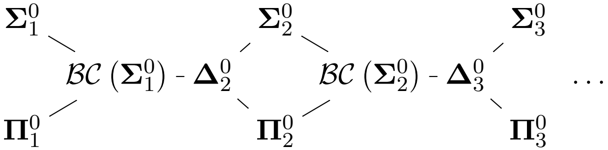


Borel hierarchy

Definition

Inductive definition for $\eta < \omega_1$*.*

- $\mathbf{\Sigma}^0_1$ — *open sets,*
- $\mathbf{\Pi}^0_1$ — *closed sets,*
- $\mathbf{\Sigma}^0_\eta$ — *countable unions of sets from* $\bigcup_{\tau<\eta} \mathbf{\Pi}^0_\tau$*,*
- $\mathbf{\Pi}^0_\eta$ — *completions of sets from* $\mathbf{\Sigma}^0_\eta$*,*
- $\mathbf{\Delta}^0_\eta = \mathbf{\Sigma}^0_\eta \cap \mathbf{\Pi}^0_\eta$.



Fact

Borel hierarchy is strict — each inclusion on the following schema is not an equality.

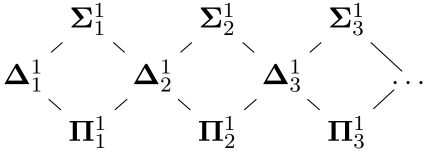
$$
\begin{array}{ccccccccc}
\boldsymbol{\Sigma}^0_1 & & & & \boldsymbol{\Sigma}^0_2 & & & & \boldsymbol{\Sigma}^0_3 \\
& & \mathcal{BC}\left(\boldsymbol{\Sigma}^0_1\right) \text{ - } \boldsymbol{\Delta}^0_2 & & & & \mathcal{BC}\left(\boldsymbol{\Sigma}^0_2\right) \text{ - } \boldsymbol{\Delta}^0_3 & & & \ldots \\
\boldsymbol{\Pi}^0_1 & & & & \boldsymbol{\Pi}^0_2 & & & & \boldsymbol{\Pi}^0_3
\end{array}
$$

Projective hierarchy

Definition

Set $A \subseteq \Sigma^\omega$ is called analytic (den. $\mathbf{\Sigma}^1_1$) iff it is a projection of some Borel set.

- $\mathbf{\Pi}^1_n$ are completions of sets from $\mathbf{\Sigma}^1_n$,
- $\mathbf{\Sigma}^1_{n+1}$ are projections of sets from $\mathbf{\Pi}^1_n$,
- $\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$.



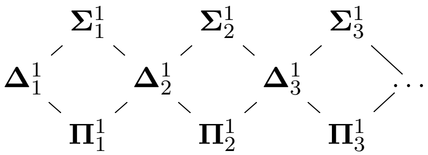
Fact

The projective hierarchy is strict.

Theorem (Souslin)

Borel sets are exactly $\mathbf{\Delta}^1_1$ sets.

$$\mathcal{B} = \mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1.$$



Definition

A continuous reduction of $A \subseteq X$ to $B \subseteq Y$ is a continuous function $f\colon X \to Y$ such that

$$f^{-1}(B) = A.$$

Fact

If A reduce to B, then topological complexity of A is at most equal to these of B.

Definition

A set $B \subseteq X$ is called complete for topological complexity class $\mathcal{C}$ iff $B \in \mathcal{C}$ and for each $A \in \mathcal{C}$, A continuously reduce to B.

Fact

For each $\eta < \omega_1$ and $n < \omega$, there exist complete sets for $\mathbf{\Sigma}^0_\eta, \mathbf{\Pi}^0_\eta, \mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$.

Example

Set $\{\alpha \in \{0,1\}^\omega : \exists_{i \in \mathbb{N}}\, \alpha_i = 1\}$ is $\mathbf{\Sigma}^0_1$-complete.

$$\left\{\alpha \in 2^{\mathbb{N}^i} : \exists_{m_i} \forall_{m_{i-1}} \exists_{m_{i-2}} \ldots \forall_{m_1}\ \alpha(m_i, m_{i-1}, \ldots, m_1) = 1\right\}$$

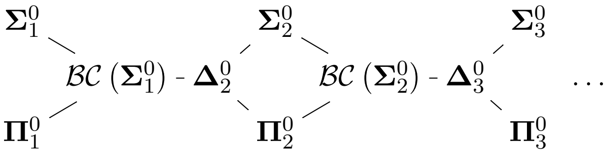
is $\mathbf{\Sigma}^0_i$-complete.

Fact

If $\mathcal{C}$-complete set reduce to $B \subseteq X$ then topological complexity of B is at least $\mathcal{C}$.

Fact

For each MSO formula φ, there holds $L(\varphi) \in \mathcal{BC}\left(\mathbf{\Sigma}^0_2\right)$.
Language „exists infinitely many 1's" is $\mathbf{\Sigma}^0_2$-complete and ω-regular.



Fact

Language $\{a^{n_0} b a^{n_1} b \dots : \lim_{i \to \infty} n_i = \infty\}$ is $\mathbf{\Pi}^0_3$-complete, so not ω-regular.

Remark

There are countably many ω-regular sets. So there exist open sets that are not ω-regular.

Remark

Consider a set $P \subseteq \Sigma^\omega$ that is $\mathbf{\Sigma}^1_{1024}$-complete. Consider boolean calculus over one atom „Φ": set of formulas like $\Phi \wedge (\neg\Phi \vee \bot)$. Let

$$\alpha \models \Phi \Leftrightarrow \alpha \in P$$

The emptiness problem for this calculus is decidable (trivially).

Extensions of the MSO logic

Idea

Extend a concept of ω-regular language, to express asymptotic properties like "blocks of a^n are bounded in length".

Definition

Formula $\mathrm{U}X.\varphi(X)$ holds iff

$$\forall_{n\in\mathbb{N}}\exists_{X\subset\omega}\ n \leq |X| < \infty \wedge \varphi(X).$$

- $\mathrm{MSO}+\mathrm{U} = \mathrm{MSO}$ with U,
- $\mathrm{WMSO}+\mathrm{U} = \mathrm{WMSO}$ with U.

$\mathrm{WMSO+U}$, results by Mikołaj Bojańczyk

Fact

$\mathrm{WMSO+U}$ *is strictly stronger than* MSO.

Theorem

$\mathrm{WMSO+U}$ *is decidable.*

Construct adequate automata model — $\max$-automata. Show that emptiness problem for that automata is decidable.

Fact

Languages definable in $\mathrm{WMSO+U}$ *lie in* $\mathcal{BC}\left(\mathbf{\Sigma}^0_2\right)$.

$\mathrm{MSO} + \mathrm{U}$

Question

Is $\mathrm{MSO} + \mathrm{U}$ *decidable?*
Is there any appropriate automata model for $\mathrm{MSO} + \mathrm{U}$*?*

Theorem

There exists non Borel set definable in $\mathrm{MSO} + \mathrm{U}$.

Corollary

There is no nondeterministic Borel automata model catching full $\mathrm{MSO} + \mathrm{U}$.

Thank you for your attention!

- W.Thomas, *Languages Automata and Logics*, Technical Report 9607.
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- S.Hummel, M.S., S.Toruńczyk, *On the Topological Complexity of* $\mathrm{MSO} + \mathrm{U}$ *and Related Automata Models*, MFCS 2010.