Additive Schwarz method for quasilinear elliptic partial differential equations

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Abstract

We consider the solution of the algebraic system of equations which result from discretization of quasilinear elliptic problems of monotone type. We consider two methods one which is extension of multilevel additive Schwarz method for linear problems and second: the approximate Newton method with inner-outer iterations.

Keywords: nonlinear elliptic problems, preconditioner, finite elements, Schwartz algorithms, Newton's method, multigrid methods

AMS(MOS) subject classifications. 65H10, 65N30, 65N55

1 Introduction

In this paper we discuss parallel algorithms for solving systems of nonlinear algebraic equations which arise when quasilinear elliptic problems are discretized by finite elements. The multilevel iterative technique is a powerful technique for solving the systems of equations associated with discretized partial differential equations cf.[2],[11]. In this paper we present an extension of a multilevel iterative method for linear finite element equations discussed in [7], [8], [13], [3] and others, to nonlinear problems. We design two methods and then analyze their convergence. The first one combines multilevel additive Schwarz method (ASM) with Richardson method and second one arises from the approximated Newton method with outer-inner iterations. For inner iterations we use multilevel ASM. We note that in [11] for inner iterations multigrid methods are proposed. We show that the convergence of both methods is independent of number of unknowns and number of levels. Each iteration of the first method

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and every inner iterations in second one can be implemented on parallel computer. The first algorithm is an extension of algorithm derived from two level ASM discussed in [4].

In Section 2 we present the differential problem and specify some assumptions for this problem. In Section 3 we present the finite element discretization of such problem. For simplicity of presentation we consider only two dimensional case and we use piecewise continuous linear elements. In Section 4 we present multilevel additive Schwarz method for Poisson equation and an implementation of a preconditioner, which arises from this method. This section is auxiliary. In Section 5 we present Richardson method combined with preconditioner from Section 5 and analyze its convergence. Section 6 is devoted to Newton method with outer and inner iterations combined with preconditioner from Section 4. cf. [9]. In Section 7 some numerical results are reported for the considered algorithms.

2 Differential problem

Find $u \in H_0^1(\Omega)$ such that:

$$a(u,v) = f(v) \;\; \forall\, v \in \; H_0^1(\Omega), \tag{1}$$

where

$$a(u,v) = \int_\Omega (\sum_{i=1}^{2} a_i(x,u,\nabla u\;) \cdot D_i v + a_0(x,u,\nabla u) v)\, dx, \quad f(v) = \int_\Omega f v\, dx$$

Here $D_i = \frac{\partial}{\partial x_i}$, $\;f \in \; L^2(\Omega)$, $\nabla u = (\frac{\partial}{\partial x_1} u, \frac{\partial}{\partial x_2} u)^T$ and Ω is Lipschitz continuous, bounded region in R^2.

We assume that form (1) has bounded nonlinearity, cf.[12],

$$a_i \in C^1(\Omega \times R^3), \quad i = 0,1,2 \tag{2}$$

$$\max_{\overline{\Omega}\,,\, 0 \le i,j \le 2\,,\, 1 \le k \le 2} \{|a_i(x,0,0)|, |\frac{\partial a_i}{\partial x_k}|, \; |\frac{\partial a_i}{\partial p_j}|, \;\} \le \; M \tag{3}$$

We also assume that form $a(u,v) : H_0^1 \times H_0^1 \to R$ satisfies the condition of **strong ellipticity** i.e.

$$\exists\, \mu_0 > 0, \; \forall \xi = (\xi_0, \xi_1, \xi_2) \; \in \; R^3, \; \xi \neq 0, \; \forall p' = (p_0, p_1, p_2) \; \in \; R^3, \forall x \in \; \overline{\Omega}$$

$$\sum_{i,j=0}^{2} \frac{\partial}{\partial p_j} a_i(x,p') \xi_i \xi_j \ge \mu_0 \sum_{i=0}^{2} \xi_i^2 \tag{4}$$

Under the above assumptions the following properties of the form (1) can be shown:

Lemma 2.1 *Strong ellipticity of* $a(u,v) \ : H_0^1 \times H_0^1 \to R$ *implies its strong monotonicity i.e.*

$$\exists\, \gamma > 0 \ \ \forall\, u,v \in H_0^1 \ \ a(u,u-v) - a(v,u-v) \geq \gamma \|u-v\|^2_{H^1(\Omega)} \tag{5}$$

Lemma 2.2 *If* $a(u,v) \ : H_0^1 \times H_0^1 \to R$ *has the bounded nonlinearity then*

$$\exists\, \widehat{M} > 0 \ \ \forall\, u,v,w \in H_0^1 \quad |a(u,w) - a(v,w)| \leq \widehat{M} \|u-v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} \tag{6}$$

Proofs of these lemmas can be found for example in [12].

Remark 2.1 *Under the assumptions of strong monotonicity and (6) the problem (1) has an unique solution.*

For the proof see e.g. [12] □

3 The finite element approximation

We assume for simplicity of presentation that Ω is a polygonal region. We define a sequance of nested triangulations $\{\mathcal{T}^l\}_{l=1}^L$. We start with a coarse triangulation $\mathcal{T}^1 = \{\tau^1_{i=1}\}_{i=1}^{N_1}$ where τ^1_i represents an individual triangle. The successively finer triangulations $\mathcal{T}^l = \{\tau^l_{i=1}\}_{i=1}^{N_l}$ are defined by dividing individual triangles of the set $\mathcal{T}^{l-1}$ into a few triangles. We assume that all the triangulations are quasiuniform,(cf.[5]). Let $h^l_i = diameter(\tau^l_i)$ $h_l = \max_i h^l_i$ *and* $h = h_L$. We also assume that there exists a constant γ and constant C, such that if an element τ^{l+k}_i of level $l+k$ is contained in an element τ^l_j of level l, then $\frac{diam(\tau^{l+k}_i)}{diam(\tau^l_j)} \leq C\gamma^k$. For uniform refinement, with each triangle divided into four triangles $C = 1$ and $\gamma = 1/2$.

Let $V^l, \quad l = 1,\ldots,L$ be the space of continuous piecewise linear functions vanishing on $\partial\Omega$ associated with triangulation $\mathcal{T}^l$. Then each V^l posesses standard nodal basis associated with interior nodes of triangulation $\mathcal{T}^l$ and $V^l = span\{\varphi^l_i\}$ where $i = 1,\ldots,N_l$. In particulary $V^h = V^L = span\{\varphi^L_i\}$ for $i = 1,\ldots,N_h$. The spaces $V^l, \ \ i = 1,\ldots,L$ are auxiliary here and will be used in the next section.

We denote by $\underline{u}$ vector of coordinates of u_h in nodal basis of V^h, i.e. $\underline{u} = \{u_i\}_{i=1,\ldots,N_h}$ for $u_h = \sum_{i=1}^{N_h} u_i \varphi^L_i$, $u_i = u(x^L_i)$ for x^L_i - nodes.

The discrete problem is of the form:
Find $u_h \in V^L = V^h$ such that

$$\forall\, v_h \in V^h \ \ a(u_h, v_h) = f(v_h) \tag{7}$$

Using the standard nodal basis, it can be rewritten as a system of nonlinear equations:

$$A(\underline{u}) = \underline{f} \tag{8}$$

where $(A(\underline{u}), \underline{v})_{R^{N_h}} = a(u_h, v_h)$ and $(\underline{f}, \underline{v})_{R^{N_h}} = f(v_h)$

It is known that the problem (7) and, as consequence, the system (8), has an unique solution provided that $h < h_0$ is sufficiently small, see for example [9].

4 Additive multilevel Schwarz method for Poisson equation

In this section we recall a multilevel diagonal scaling (MDS) preconditioner for FE approximation of the Dirichlet problem for the Poisson equation, see [8] and [13]. It can be used as preconditioner for the problem (8).

Let us define the decomposition of V^l, $l = 1, \ldots, L$, as follows: $V_1^1 = V^1$, $V_i^l = span\{\varphi_i^l\}$, $i = 1, \ldots, N_l$, $l = 2, \ldots, L$. By the assumptions on the triangulations $V^l \subset V^L$, $l < L$. Thus the finite element space $V^h = V^L$ is represented as a sum:

$$V^h = \sum_{l=1}^{L} V^l = \sum_{l=1}^{L} \sum_{i=1}^{N_l} V_i^l$$

Let define operators $B : V^h \to V^h$ and $B_{V_i^l} : V_i^l \to V_i^l$ by:

$$(Bu_h, v_h)_{L^2} = b(u_h, v_h) = (\nabla u_h, \nabla v_h)_{L^2} \ \ \forall v_h \in V^h,$$

$$(B_{V_i^l} u_h, v_h)_{L^2} = (u_h, v_h)_B \ \ \forall\, v_h \in V_i^l$$

and $B_1 = B_{V_1^1} : V^1 \to V^1$:

$$(B_1 u_h, v_h)_{L^2} = (u_h, v_h)_B \quad \forall v_h \in V^1$$

We also define projections $P_{V_i^l} : V^h \to V_i^l$ and $Q_{V_i^l} : V^h \to V_i^l$ by:

$$(P_{V_i^l} u_h, v_h)_B = (u_h, v_h)_B \quad \forall v_h \in V_i^l$$

$$(Q_{V_i^l} u_h, v_h)_{L^2} = (u_h, v_h)_{L^2} \quad \forall v_h \in V_i^l$$

Since V_i^l for $l > 1$ are one dimensional spaces then the projections $P_{V_i^l}$ take the form:

$$P_{V_i^l} u_h = \frac{1}{b(\varphi_i^l, \varphi_i^l)} b(u_h, \varphi_i^l)\varphi_i^l = B_{V_i^l}^{-1} Q_{V_i^l} B u_h \quad l = 2, \ldots, L \ \ i = 1, \ldots, N_l \tag{9}$$

and

$$P_{V^1} u_h = B_1^{-1} Q_{V^1} B u_h \ \ for \ \ l = 1$$

Let us define P_{MAS} and $(\widehat{B}^{-1})$ by:

$$P_{MAS} = \sum_{l=1}^{L}\sum_{i=1}^{N_l} P_{V_i^l} = (\sum_{l=1}^{L}\sum_{i=1}^{N_l} B_{V_i^l}^{-1} Q_{V_i^l})B = (\widehat{B})^{-1}B \tag{10}$$

In [13] the following Lemma is proved:

Lemma 4.1 *For any $u_h \in V^h$*

$$c_1(u_h, u_h)_B \leq (P_{MAS}u_h, u_h)_B \leq c_2(u_h, u_h)_B$$

where c_1 and c_2 are constants independent of $\{h_l\}$ and L.

$A \sim B$ denotes that SPD (symmetric, positive definite) operators A and B are spectrally equivalent, i.e. exist constants $c_1, c_2 > 0$ independent of h such that for any $u_h \in V^h$

$$c_1(u_h, u_h)_A \leq (u_h, u_h)_B \leq c_2(u_h, u_h)_A$$

The Lemma 4.1 implies the following corollary

Corollary 1 *For $\widehat{B}$, see (10), we have that $B \sim B(\widehat{B})^{-1}B$ and thus $(\widehat{B})^{-1} \sim B^{-1}$*

This implies

Corollary 2 *For $\widehat{B}$, see (10), we have that $\widehat{B} \sim B$.*

It yields that the norms generated by $\widehat{B}$ and B are equivalent to $\|\cdot\|_{H^1}$ in V^h, i.e. $\|u_h\|_{\widehat{B}} \sim \|u_h\|_B = b^{1/2}(u_h, u_h) \sim \|u_h\|_{H^1} \;\; \forall\, u_h \in V^h$.

We now write down the preconditioner $\widehat{B}$ in the matrix form. Let K_l be the stiffness matrix associated with $b(u, v)$ in $V^l = span\{\varphi_i^l\}_{i=1,\ldots,N_l}$ and $D_l = diag(K_l)$, i.e.:

$$K_l = \{b(\varphi_i^l, \varphi_j^l)\}_{j,i=1,\ldots,N_l}$$

Let $\Pi_l : R^{N_l} \to R^{N_h}$, $(l \leq L)$ be the standard interpolation (prolongation) operator and $\Pi_l^t : R^{N_h} \to R^{N_l}$ be the adjoint operator to Π_l. Then a matrix representation of $\widehat{B}^{-1}$ denoted also by $\widehat{B}^{-1}$ is of the form:

$$\widehat{B}^{-1} = \Pi_1 K_1^{-1}\Pi_1^t \; + \; \ldots \; + \Pi_l D_l^{-1}\Pi_l^t \; + \; \ldots \; + \Pi_{L-1}D_{L-1}^{-1}\Pi_{L-1}^t \; + \; D_L^{-1} \tag{11}$$

Note that to compute $\widehat{B}^{-1}\underline{u}$ for the nodal vector $\underline{u}$ of $u_h \in V^h$ we need to solve L independent problems and we can do it in parallel. The cost of computation of $\Pi_l D_l^{-1}\Pi_l^t$ is of $O(N_h)$. If we compute $\widehat{B}^{-1}$ sequentialy, then we can interpolate from neighbouring levels and the cost of computing $\widehat{B}^{-1}$ is of $O(N_h)$.

5 Richardson iterative method

In this section we discuss the Richardson iterative method for the problem (8) with $\widehat{B}^{-1}$ as preconditioner. That method is an extension of method discussed in [4] for two levels only. The Richardson method is well known for linear systems, see [10].

We solve the problem (8) by:

$$u^{n+1} = u^n - \tau \widehat{B}^{-1}(A(u^n) - f), \tag{12}$$

where $\widehat{B}^{-1}$ is defined in (11), and τ is a parameter which will be chosen later. Here and below we drop an underlying for vectors.

Lemma 5.1 *The operator A, see (8) satisfies the inequalities:*

$$\exists\, \delta_0 \geq 0\ \forall\, u, v \in R^{N_h}\ \ (A(u) - A(v), u - v)_{R^{N_h}} \geq \delta_0 \|u - v\|^2_{\widehat{B}}$$

$$\exists\, \delta_1 \geq 0\ \forall\, u, v \in R^{N_h}\ \ \|A(u) - A(v)\|^2_{\widehat{B}^{-1}} \leq \delta_1\, \|u - v\|^2_{\widehat{B}}$$

Proof: We will use the fact that $\|\cdot\|_{H^1} \sim \|\cdot\|_{\widehat{B}}$ in V^h. Thus the first inequality follows from the strong monotonicity of A, see Lemma 2.1.
For the second one Lemma 2.2 yields that $\exists\, \delta_1 > 0\ \forall\, u, v, w \in R^{N_h}$

$$|(A(u) - A(v), w)_{R^{N_h}}| \leq \delta_1^{1/2} \|u - v\|_{\widehat{B}} \|w\|_{\widehat{B}}$$

Indeed let $g = A(u) - A(v)$ and

$$\|A(u) - A(v)\|_{\widehat{B}^{-1}} = \|\widehat{B}^{-1} g\|_{\widehat{B}} = \sup_{w \in R^{N_h}} |(\widehat{B}^{-1} g, \frac{w}{\|w\|_{\widehat{B}}})_{\widehat{B}}|$$

$$\leq \sup_{w \in R^{N_h}} |(A(u) - A(v), \frac{w}{\|w\|_{\widehat{B}}})_{R^{N_h}}| \leq \delta_1^{1/2}\, \|u - v\|_{\widehat{B}}\ \square$$

The previous lemma yields the following theorem (cf. [9]):

Theorem 5.1 *The method (12) for the problem (8) is convergent for τ: $0 < \tau < 2\delta_0\, \delta_1^{-1}$ with estimate*

$$\|u^n - u\|_{\widehat{B}} \leq \rho(\tau)^n \|u^0 - u\|_{\widehat{B}}$$

where $\rho(\tau) = (1 - 2 * \tau\delta_0 + \tau^2\delta_1)^{1/2} < 1$ and is independent of h and L. The optimal parameter $\tau^* = \delta_0\, \delta_1^{-1}$ and then $\rho(\tau^*) = (1 - \delta_0^2\, \delta_1^{-1})^{1/2}$

Proof: Let $z^{n+1} = u^{n+1} - u$. Then from (12) we obtain

$$\|z^{n+1}\|^2_{\widehat{B}} = \|z^n\|^2_{\widehat{B}} - 2\tau(A(u^n) - A(u), z^n)_{R^{N_h}} + \tau^2 \|(A(u^n) - A(u))\|^2_{\widehat{B}^{-1}} \leq$$

$$\leq (1 - 2\tau\delta_0 + \tau^2\delta_1)\|z^n\|^2_{\widehat{B}} = \rho(\tau)^2 \|z^n\|^2_{\widehat{B}}$$

Certainly $\rho(\tau^*) \leq \rho(\tau) < 1\ for\ 0 < \tau < 2\delta_0\delta_1^{-1}\ \ and\ \ \tau^* = \delta_0\delta_1^{-1}\ \ \square$

6 The Newton method with outer-inner iterations

In this section we present Newton method applied for solving (8).
As in the previous Section 5 we drop an underlying for vectors. Operator A, see Under the assumption (3) the operator A has a Gateaux derivative A' :

$$\forall\, u, v, w \in R^{N_h}\ (A'(u)v, w)_{R^{N_h}} = s(u_h; v_h, w_h) =$$

$$\int_\Omega \sum_{i,j=1}^{2} \frac{\partial}{\partial p_j} a_i(x, u_h, \nabla u_h) \frac{\partial}{\partial x_j} v_h \frac{\partial}{\partial x_i} w_h + \sum_{j=1}^{2} \frac{\partial}{\partial p_j} a_0(x, u_h, \nabla u_h) \frac{\partial}{\partial x_j} v_h w_h \quad (13)$$

$$+ \sum_{i=1}^{2} \frac{\partial}{\partial p_0} a_i(x, u_h, \nabla u_h) v_h \frac{\partial}{\partial x_i} w_h + \frac{\partial}{\partial p_0} a_0(x, u_h, \nabla u_h) v_h w_h \, dx$$

Note that the form $s(u_h; v_h, w_h)$ is bilinear for variables v_h and w_h. Since $\|u_h\|_{H^1} \sim \|u_h\|_B \sim \|u\|_{\widehat{B}}$ then under the condition of strong ellipticity we obtain that:

$$\exists\, \gamma > 0\ \ \forall\, u, v \in R^{N_h}\ \ (A'(u)v, v) \geq \gamma \, \|v\|^2_{\widehat{B}} \quad (14)$$

and respectively using (3) that:

$$\exists\ M > 0\ \ \forall u, v, w \in R^{N_h}\ \ |(A'(u)v, w)| \leq M \, \|v\|_{\widehat{B}} \|w\|_{\widehat{B}} \quad (15)$$

We additionally assume that in some ball $S_{\widehat{B}}$ with center as the solution u of (8) the derivative A' is Lipschitz continuous i.e.:

$$\exists\, b > 0\, \forall\, u, v \in S_{\widehat{B}}\ w \in R^{N_h}\ \|(A'(u) - A'(v))w\|_{\widehat{B}^{-1}} \leq\ b\|u - v\|_{\widehat{B}} \|w\|_{\widehat{B}} \quad (16)$$

In order to solve (8) we apply Newton method :

$$A'(u^n)(u^{n+1} - u^n) = -A(u^n) + f \quad (17)$$

where u^0 should be chosen as in Theorem 6.1, i.e. near u the solution of (8). To compute u^{n+1} we have to solve a linear system:

$$A'(u^n)v = g \quad (18)$$

where $u^{n+1} = u^n + v, \quad g = -A(u^n) + f$. This system is solved by an iterative method, for example Richardson method from Section 5 which converged under our assumptions for A', see Theorem 5.1, (14) and (15). For the detailed discussion of Richardson method and other methods of solving large linear systems we refer to [10]. Note, that $A'(u^n)$ is not symmetric in general. The Richardson method for (18) takes the form (cf. (12))

$$v^{m+1} = v^m - \tau \widehat{B}^{-1}(A'(u^n)v^m - g) \quad (19)$$

We denote v^m as an inner iteration, unlike u^n - an iteration of the Newton method, which we denote as an outer iteration. When we solve (18) with (19), we take an initial guess as $v^0 = 0$.

We now analyse a convergence of the method (19). cf. for example [9].

Let R_n be an operator of decreasing error after k_n inner iterations of iterative method for solving (18), i.e. if v is solution of (18) and v^{k_n} inner iteration then

$$v^{k_n} - v = R_n(v^0 - v)$$

Using Richardson method (i.e. (19)) we obtain that:

$$R_n = (I - \tau\, \widehat{B}^{-1} \cdot A'(u^n))^{k_n} \;\; \textit{and} \;\; \|R_n\| \le q_n = \rho(\tau)^{k_n} < 1$$

where $\rho(\tau) = (1 - 2\,\tau\gamma + \tau^2\, M)^{1/2}$

Lemma 6.1 *If $v^0 = 0$, then the k_n-inner iteration is equal to the solution of*

$$A'(u^n)(I - R_n)^{-1} w = g$$

Proof: We have

$$v^{k_n} - v = R_n(v^0 - v) = -R_n v$$

and

$$w := v^{k_n} = -R_n v + v = (I - R_n)v$$

Substituting this into (18) ends the proof. □

We now take $v + u^n$ as approximation of u^{n+1}, see (17), and also denote it by u^{n+1}. It easy to see that it satisfies

$$A'(u^n)(I - R_n)^{-1}(u^{n+1} - u^n) = -A(u^n) + f \tag{20}$$

Note that (20) can be interpreted as Richardson method for (18) with preconditioner $C^{-1} = A'(u^n)(I - R_n)^{-1}$.

We now present several Lemmas, that are used later in the proof of convergence. The most important is Lemma 6.4, which yields a convergence of the method.

Lemma 6.2 *Under the assumption (16) holds*

$$\forall\, u, v \in S_{\widehat{B}} \|A(v) - A(u) - A'(u)\,(v - u)\|_{\widehat{B}^{-1}} \le b\, 2^{-1}\, \|v - u\|^2_{\widehat{B}}$$

Proof: Let $z = v - u$. Then

$$A(u + z) - A(u) = \int_0^1 A'(u + tz)\, z\, dt$$

and

$$\|A(u+z) - A(u) - A'(u)\, z\|_{\widehat{B}^{-1}} \le \int_0^1 \|(A'(u+tz) - A'(u))\, z\|_{\widehat{B}^{-1}}\, dt \le \frac{b}{2}\|z\|^2_{\widehat{B}} \quad \square$$

Lemma 6.3 *Under the asssumptions (15) and (14) holds*

$$\forall\, u, v \,\in\, V^h \;\; \|A'(u)v\|_{\widehat{B}^{-1}} \leq M\, \|v\|_{\widehat{B}} \tag{21}$$

and

$$\forall\, u, v \,\in\, R^{N_h} \;\; \|(A'(u))^{-1}v\|_{\widehat{B}} \leq \frac{1}{\gamma}\, \|v\|_{\widehat{B}^{-1}} \tag{22}$$

Proof: By (15)

$$\|A'(u)v\|_{\widehat{B}^{-1}} = \|\widehat{B}^{-1}A'(u)v\|_{\widehat{B}} = \sup_{w\in R^{N_h}} \frac{(\widehat{B}^{-1}A'(u)v, w)_{\widehat{B}}}{\|w\|_{\widehat{B}}}$$

$$= \sup_{w\in R^{N_h}} \frac{(A'(u)v, w)_{R^{N_h}}}{\|w\|_{\widehat{B}}} \leq M\, \|v\|_{\widehat{B}}$$

For $v = (A'(u))^{-1}w \neq 0$ using (14) we get

$$\gamma\, \|v\|_{\widehat{B}} \leq \frac{(A'(u)v, v)_{R^{N_h}}}{\|v\|_{\widehat{B}}} = \frac{(\widehat{B}^{-1}A'(u)v, v)_{\widehat{B}}}{\|v\|_{\widehat{B}}} \leq$$

$$\sup_{w\in R^{N_h}} \frac{(\widehat{B}^{-1}A'(u)v, w)_{\widehat{B}}}{\|w\|_{\widehat{B}}} = \|\widehat{B}^{-1}A'(u)v\|_{\widehat{B}} = \|A'(u)v\|_{\widehat{B}^{-1}} \;\square$$

Next lemma implies a convergence of the method (20) cf. [9]:
Let $B_{\widehat{B}}(u, r) = \{v \in R^{N_h} : \|u - v\|_{\widehat{B}} \leq r\}$

Lemma 6.4 *Let u and u^{n+1} be solutions of (8) and (20) respectively, and $S = B_{\widehat{B}}(u, r)$. Under the assumptions (15), (14), (16), and for R_n with $\|R_n\| \leq q_n < 1$, and $u^n \in S$ holds*

$$\|z^{n+1}\|_{\widehat{B}} \leq \frac{1}{\gamma}\,(2^{-1}\, l\, \|z^n\|^2_{\widehat{B}} + \xi^n) \tag{23}$$

$$\textit{where} \quad z^n = u^n - u, \quad \textit{and} \quad \xi^n = q_n(1 - q_n)^{-1}M\|u^{n+1} - u^n\|_{\widehat{B}}$$

Proof: By (20) $A'(u^n)z^{n+1} = g_0 + g_1$
where

$$g_0 = A(u) - A(u^n) + A'(u^n)z^n,$$

and

$$g_1 = A'(u^n)[I - (I - R_n)^{-1}](u^{n+1} - u^n)$$

Thus (22) implies that

$$\|z^{n+1}\|_{\widehat{B}} = \|A'(u^n)^{-1}(g_0 + g_1)\|_{\widehat{B}} \leq \gamma^{-1}(\|g_0\|_{\widehat{B}^{-1}} + \|g_1\|_{\widehat{B}^{-1}})$$

By Lemma 6.2

$$\|g_0\|_{\widehat{B}^{-1}} = \|A(u^n) - A(u) - A'(u)\, z^n\|_{\widehat{B}^{-1}} \leq 2^{-1} b \|z^n\|^2_{\widehat{B}}$$

Using (21) and $\|R_n\| < q_n < 1$ we obtain that:

$$\|g_1\|_{\widehat{B}^{-1}} = \|A'(u^n)[I - (I - R_n)^{-1}](u^{n+1} - u^n)\|_{\widehat{B}^{-1}} \leq$$

$$\leq M\|R_n(I - R_n)^{-1}(u^{n+1} - u^n)\|_{\widehat{B}} \leq q_n(1 - q_n)^{-1}\|u^{n+1} - u^n\|_{\widehat{B}} \ \square$$

Theorem 6.1 *Let the assumptions of Lemma 6.4 be satisfied and a radius of ball S and q_n be such that*

$$q_n(1 - q_n)^{-1}\, M\gamma^{-1} \leq q < 1 \ \ \textit{and} \ \ (1 - q)^{-1}(2^{-1}\gamma^{-1}\, b\, r + q) \leq \alpha < 1$$

Then for $u^0 \in S$ the method (20) for (8) converges with estimate

$$\|z^k\|_{\widehat{B}} \leq \alpha^k\, \|z^0\|_{\widehat{B}}$$

Proof: Let $\|z^k\|_{\widehat{B}} \leq r$. By the assumptions of the theorem it follows that

$$\|z^{k+1}\|_{\widehat{B}} \leq \gamma^{-1}\, b\, 2^{-1}\, r\|z^k\|_{\widehat{B}} + q(\|z^{k+1}\|_{\widehat{B}} + \|z^k\|_{\widehat{B}})$$

Thus

$$\|z^{k+1}\|_{\widehat{B}} \leq \alpha\|z^k\|_{\widehat{B}}$$

what completes the proof $\square$

7 Numerical experiments

In this section we report on some numerical experiments with algorithms presented in Sections 5 and 6. These experiments were carried on a unit square for the following differential problem:

$$\left\{ \begin{array}{rcll} -\triangle u \ + u\,(u_x + u_y) & = & f(x,y) & in \quad \Omega \\ u & = & 0 & on \quad \partial\Omega \end{array} \right.$$

The triangulations are obtained by dividing Ω into squares and then each square into two triangles. The problem was chosen rather for Newton method as it is the same as in [2], where experiments for Newton method are reported. In first Table we report on results for algorithm presented in Section 5 i.e. Richardson method, The parameter τ was choosen experimentaly to obtain minimal number

of iterations. The last column of the table gives the number of iterations required to decrease the norm $\|\cdot\|_{\widehat{B}^{-1}}$ of the residual by a factor 10^{-4}.

Level	*Unknows*	τ	*Iter. no.*
2	$(4-1)^2$	1.7	32
3	$(8-1)^2$	1.73	32
4	$(16-1)^2$	1.74	33
5	$(32-1)^2$	1.735	34
6	$(64-1)^2$	1.715	38
7	$(128-1)^2$	1.7	41
8	$(256-1)^2$	1.69	44

The experiments show that the method is independant of number of levels and the number of unknows. The next table shows dependance of the Richarson method of the parameter τ. Experiments were carried for 5th levels, i.e. for $(32-1)^2$ unknows.

parameter τ	*Iteration number.*
1.5	140
1.6	76
1.7	41
1.73	35
1.74	33
1.75	52
1.76	*method does not converge*

The next table presents the results of experiments for Newton method with outer-inner iterations presented in Section 6. The last column of the table gives the number of outer iterations required to decrease the norm $\|\cdot\|_{\widehat{B}^{-1}}$ of the residual by factor 10^{-4}. The number of inner iteration depends on how an exact approximation we need and on parameter τ. We stop the inner iterations if we decrease the norm of the inner residual by a factor 10^{-4}. We note that 'too' good approximation of inner iteration does not decrease the number of outer iterations, but of course if the inner approximation is too bad it increase the number of outer iterations.

Level	*Unknowns*	*Outer iter. no.*
2	$(4-1)^2$	4
3	$(8-1)^2$	6
4	$(16-1)^2$	6
5	$(32-1)^2$	6
6	$(64-1)^2$	5
7	$(128-1)^2$	5
8	$(256-1)^2$	5

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