HABILITATION THESIS

NUMERICAL METHODS OF SOLVING ELLIPTIC EQUATIONS ON NONMATCHING MESHES

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The list of papers comprising the thesis

[H1] Leszek Marcinkowski. A mortar finite element method for fourth order problems in two dimensions with Lagrange multipliers. *SIAM J. Numer. Anal.*, 42 (5): 1998–2019, 2005.

[H2] Leszek Marcinkowski. Additive Schwarz Method for mortar discretization of elliptic problems with P1 nonconforming finite element. *BIT*, 45 (2): 375–394, 2005.

[H3] Leszek Marcinkowski. An Additive Schwarz Method for mortar Morley finite element discretizations of 4th order elliptic problem in 2d. *Electron. Trans. Numer. Anal.*, 26: 34–54, 2007.

[H4] Leszek Marcinkowski, Talal Rahman. Neumann-Neumann algorithms for a mortar Crouzeix-Raviart element for 2nd order elliptic problems. *BIT*, 48 (3): 607–626, 2008.

[H5] Leszek Marcinkowski. A Neumann-Neumann algorithm for a mortar finite element discretization of fourth-order elliptic problems in 2d. *Numer. Methods Partial Differential Equations*, 25 (6): 1425–1442, 2009.

The journals, in which the papers are published, are included in the ISI Master Journal List.

Introduction

The thesis consists of the papers which are included in the research area which concerns the numerical solving of boundary value problems for elliptic partial differential equations of the second and fourth order on parallel computers. To be more precise, the papers belong to the part of this area which is called Domain Decomposition (DD) or Domain Decomposition Methods of solving differential equation problems discretized on nonmatching meshes in parallel.

Here the nonconformity of the mesh means that the discretization is constructed on local independent grids in subdomains, contrary to the case of a conforming mesh when we have to deal with one global mesh defined over the whole domain. Nonmatching grids are utilized, e.g. in adaptive refinements of the meshes. The discretizations considered in the thesis are

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constructed by the Finite Element Method (FEM), cf. [8], where discrete spaces are constructed by imposing special conditions on the restrictions of discrete functions on an element (e.g. triangle) and by introducing a respective class of smoothness of the functions from this space.

The paper [H1] is devoted to the construction and analysis of the discretization of a fourth order differential equation problem on nonmatching meshes. The inf-sup (Ladyzhenskaya - Babuška - Brezzi) condition is proven for a mixed type formulation of mortar method discretization of a fourth order problem with constants independent of the parameters of local meshes. This condition is important for proving convergence of the method. In [H2], a new domain decomposition algorithm is developed for a mortar Crouzeix-Raviart discretization on nonmatching meshes of a second order elliptic equation problem with jumps of the coefficients. The proven convergence estimates are almost optimal, i.e. they depend logarithmically only on the number of degrees in subdomains, and are independent of the number of subdomains or jumps of the differential problem's coefficients. In [H3], the problem of constructing a domain decomposition method is solved for a mortar nonconforming Morley finite element discretization of a fourth order elliptic problem with discontinuous coefficients. A fully parallel substructuring algorithm, which is based on the abstract Schwarz framework, is developed. Almost optimal convergence estimates have also been proven in the paper. The paper [H4] is devoted to solving the problem of the construction of a fully parallel method of the Neumann-Neumann type for a discretization of a second order elliptic problem with discontinuous coefficients on nonmatching grids with a Crouzeix-Raviart nonconforming finite element. Almost optimal convergence estimates were also established in the paper. In [H5], a method of the Neumann-Neumann type is discussed for the discretization of a fourth order problem with discontinuous coefficients on nonmatching meshes with the reduced Hsieh-Clough-Tocher macro finite element. New results were obtained, i.e. almost optimal convergence estimates were proven.

A detailed discussion of the results of the papers, which this thesis consist of, is preceded by an introduction, in which the subject of the research area of the thesis is briefly presented.

Domain decomposition methods

Discretizations of differential equation problems usually are constructed on the basis of one global mesh (triangulation) of the domain, in which this differential problem is defined. However, in many applications there is a need to construct discretizations on independent nonmatching meshes in subdomains or to use different discretizations methods in the given subdomains (substructures) of the original domain. It may be caused by physical or mathematical properties of the model or it may be due to parallel implementation issues. Then, for example, adaptive mesh refinement (which is a construction of a new discretization) takes place locally in processors.

One of the methods of constructing discretizations of differential equation problems on nonmatching meshes is the mortar method proposed by Ch. Bernardi, Y. Madey, and T. Patera in the early 1990s [3]. Here spaces of discrete functions are constructed by imposing integral continuity conditions of the L^2 type on traces of solutions onto the common part of the boundary of two neighboring subdomains. Since then this part of the Domain Decomposition Methods is a domain of intensive ongoing research. There are at least a couple of hundreds of research papers related to the mortar method. The monograph of B. Wohlmuth [27] contains an extensive bibliography on this method (see also [12], [25]).

By discretizing differential problems on matching or nonmatching grids, we get large systems of linear or nonlinear algebraic equations which have to to be solved. The number of unknowns can exceed hundreds of thousands or even a couple of millions. Solving such systems requires parallel computers and nonstandard numerical methods. The convergence speed of standard iterative methods depends on the condition number of the system, which grows polynomially with the number of unknowns in the finite element discretization case. (In the case of linear symmetric positive definite systems of equations, the condition number equals the ratio of the largest and the smallest eigenvalues.)

The iterative methods, which are utilized in practice, are based on the parallel preconditioning technique, where we replace the original problem with a new equivalent one with the lower condition number, that is algebraically equivalent to multiplying the matrix of the system by another matrix - a preconditioner - in the case of linear systems. Multiplication of this matrix - preconditioner - on a vector is implemented as an efficient parallel algorithm. The condition number estimates of the new problem are optimal or almost optimal if the constants in the bounds are independent or logarithmically dependent on local parameters of discretizations and are independent of the jumps of the coefficients of the original differential problem which can be arbitrarily large. In practical constructions of parallel preconditioners the almost optimal bounds are considered as being completely sufficient.

An effective parallel preconditioner cannot be constructed for just any system of linear equation. There are efficient parallel preconditioners for some specific systems arising from discretizations of differential problems, where information from the original differential problems is utilized.

Domain Decomposition Methods or Domain Decomposition is a research area which is a part of Numerical Methods, and it offers one of the best techniques of constructing effective parallel preconditioners. The term Domain Decomposition refers to the splitting of an original differential equation problem on a given domain or an approximation thereof into a collection of coupled subproblems on subdomains forming decomposition of the original domain. In numerical methods this decomposition may take place on the discretization level - when in subdomains we may utilize different discretization methods, or on the level of solving the systems of equations (linear or nonlinear) arising from the discretization of the original differential problem. Then, instead of solving the system of equations by the standard iterative methods, that is usually completely ineffective, we solve independent smaller subproblems which are defined on respective subdomains, in order to get a faster convergence speed.

Decomposition may also enter at the differential equation level, where different mathematical models describing physical phenomena are used in subdomains. In practice these three approaches may be intertwined.

Despite the fact that Domain Decomposition Methods is a research field that historically emerged from the paper of H. A. Schwarz, published in 1869 [19] (S. L. Sobolev [21], I. Babuška [1], and J. von Neumann [24], participated in Domain Decomposition in the context of the partial differential equations or the functional analysis, respectively), in the last few decades only we observe a rapid development of these methods, which started with the paper of P.-L. Lions [14], published in 1988, on the alternating Schwarz method. The results of this paper initiated a development the abstract theory of the Alternating (Multiplicative) and Additive Schwarz Methods, cf. [20], [23].

The Additive Schwarz Method is a fully parallel extension of the Alternating (Multiplicative) Schwarz Method. The abstract scheme of the Additive Schwarz Method enables the construction and analysis of many types of domain decomposition methods by defining a decomposition of the discrete space into a sum of subspaces and by introducing special projection operators onto these subspaces.

A very important class of the Domain Decomposition Methods is the so-called substructuring methods, i.e. first interior variables are eliminated in subdomains and the discrete problem is reduced to finding a solution on the skeleton of a decomposition (i.e. the sum of the boundaries of the subdomains).

In the class of the substructuring methods, it is worth distinguishing a very important class of methods of the Neumann-Neumann type, cf. [20], [23], [13]. The class of the Neumann-Neumann methods takes its name from the fact that local subproblems, which are solved in subdomains, are boundary value problems with the Neumann boundary conditions. While constructing a domain decomposition method a crucial key role is played by a so-called *coarse* space. If there is no *coarse* space in the case of constructing an algorithm of the Additive Schwarz Method type, then worse convergence estimates are established, i.e. the convergence estimates are dependent on the number of subdomains. The construction of a *coarse* space should be relatively simple due to implementation, but at the same time the functions from this subspace should satisfy respective estimates. The *coarse* space plays an important role for communication between local problems and its construction is specially important in the construction of all domain decomposition algorithms based on the scheme of the Additive Schwarz Method, especially, in the construction of the Neumann-Neumann methods, cf. [23], [13]. It is well known that for the substructuring methods the almost optimal estimates of the condition number cannot be improved (cf. [6]).

From the end of the 1980s Domain Decomposition Methods is an area of intensive research by groups of scientists from many research centers, cf. seven monographs on Domain Decomposition Methods [20], [18], [27], [23], [22], [11], [16], proceedings of conferences on Domain Decomposition Methods, which have been organized regularly every eighteen months for more than twenty years (cf. e.g. [12], [25]), or even text books (cf. e.g. [5], [17]).

Results of the thesis

The papers comprising the thesis are devoted to the first aspect of Domain Decomposition, i.e. to the solving systems of equations (cf. [H2] , [H3], [H4], [H5]), and to a lesser extent to the second aspect, i.e. to the construction and analysis of discretizations on nonmatching meshes (cf. [H1]).

Mortar discretizations of fourth order elliptic equations

In [H1], a mortar discretization of a model fourth order problem is developed and analyzed for a saddle problem formulation and a partition of the original domain in many subdomains.

There were no results of this type before [H1] was published for fourth order partial differential elliptic equations. Our results are extensions of the results for second order elliptic equations obtained by F. Ben Belgacem [2], D. Bräss, W. Dahmen, and Ch. Wieners [4], and B. Wolhmuth [26], [27].

As was mentioned above in mortar discretizations, special L^2 type conditions are imposed on the traces of discrete functions onto the common parts of boundaries of two neighboring subdomains. These conditions are equivalent to the equality of orthogonal projections onto special test spaces which are defined over these common parts of the boundaries. In [H1], new test spaces were introduced. These test spaces are less complicated than the standard test ones, that significantly reduces the cost of the implementation of the method while the optimal convergence is preserved (cf. e.g. [10], [15]).

The second important result, which was obtained in [H1], is the proof of the fact that for the saddle point formulation of the mortar discrete problem, the Ladyzhenskaya - Babuška - Brezzi (LBB) condition is satisfied. This condition is necessary for proving the existence and uniqueness of a discrete solution and it is essential for proving convergence of the method. Here the LBB condition, which is also called an inf-sup condition (cf. [7]) is satisfied with constants which are independent of the mesh parameters. The constants are independent of the number of subdomains, which is also very important.

The estimates established in the paper are proven in two types of norms: in the dual trace norms, which appears naturally in the context of the LBB condition for this differential equation problem, and in the discrete norms which depend on the discretization parameters.

These results enable us to apply many well known methods for solving systems of equations arising from discretizations of saddle point problems, like some methods for solving the Stokes problem. In the case of these methods it is required that the LBB condition is satisfied with the constants independent of the discretization parameters. In the paper we also additionally obtained some estimates on the Lagrange multiplier approximation error in both types of norms.

The results of the paper required proving a number of auxiliary lemmas from the area of functional analysis of discrete function spaces.

Domain Decomposition Methods for a mortar Crouzeix-Raviart discretization of second order problems

In [H2] and [H4], domain decomposition algorithms are developed and analyzed for the mortar discretization of second order elliptic problems with discontinuous coefficients on nonmatching meshes which utilize the locally nonconforming linear finite element of the Crouzeix-Raviart type (cf e.g. [8]).

The algorithms of [H2] and [H4] are substantially different, despite the fact that they are based on the abstract Additive Schwarz Method (ASM) framework but with completely different discrete space decompositions. In the case of the element of the Crouzeix-Raviart type the construction and implementation of Domain Decomposition algorithms are usually simpler, *inter alia* there are no problems with so-called crosspoints, but as it is a nonconforming element (local finite element spaces contain discontinuous functions), proofs of respective estimates are usually more complex. Large jumps of the coefficients of the original differential problems yield low regularity of the solution of the differential problem and this makes the construction and analysis more difficult for algorithms for solving the discrete problem.

The estimates of the condition number proven in both papers are almost optimal, i.e. they depend logarithmically only on the number of degrees in subdomains, and are independent of the number of subdomains or the jumps of the coefficients of the differential problem which can be arbitrarily large. The convergence rate of the iterative methods for solving the discrete problems is entirely dependent on the condition number. In the context of the substructuring methods, it is well known that these estimates cannot be improved even for discretizations on one global mesh (cf. [6]).

In [H2] we develop and analyze a first domain decomposition algorithm for a mortar Crouzeix-Raviart discretization on nonmatching meshes of a second order elliptic differential problem with discontinuous coefficients. The papers, which were published before, were concerned with problems with continuous coefficients, whose analysis is much simpler. The domain decomposition method from this paper is based on the abstract scheme of the Additive Schwarz Method framework and is of the substructuring type.

In [H4], (a joint work with Prof. Talal Rahman from The University of Bergen, Norway) we construct and analyze two additive versions of the Neumann-Neumann methods: the first one is of the substructuring type and the second one is another version of the first one. The construction of a fully additive (i.e. fully parallel) version of a Neumann-Neumann domain decomposition method for a mortar discretization of a second order elliptic differential problem was an unsolved open problem for a long time, which was finally solved in [9] in 2005 for the case of standard conforming linear elements in subdomains. Both versions of algorithms in [H4] are fully parallel and are extensions of the results from [9] for the case of the Crouzeix-Raviart mortar discretizations.

The Domain Decomposition Method for mortar discretizations of fourth order problems on nonmatching meshes

A Schwarz method for locally nonconforming discretizations of the Morley type

In [H3], a domain decomposition method was presented and analyzed for the mortar discretization of a model fourth order elliptic problem with discontinuous coefficients with the nonconforming Morley element in subdomains (cf. [8]). The nonconforming Morley element is the most popular element used in practical physical and technical applications, which are modeled by fourth order differential equations.

The presented algorithm is based on the abstract scheme of the Additive Schwarz Method framework and it is fully parallel. Almost optimal convergence estimates were proven, and they depend only on the numbers of local degrees of freedom in the subdomain and are independent of the parameters of local meshes, the number of subdomains, or the discontinuities of the coefficients of the original differential problem.

Before paper [H3] was published, there were no other domain decomposition algorithms for mortar Morley discretizations or for a mortar discretization of a fourth order problem with discontinuous coefficients, even for the case of discretization with locally conforming elements, e.g. the Hsieh-Clough-Tocher element (cf. e.g. [8]).

The Morley finite element for fourth order problems is a nonconforming finite element method, and its nonconformity is very high, i.e. local finite element spaces contain functions which are not even continuous. As a consequence, we have a relatively small number of degrees of freedom (a small number of unknowns) in the cases of this type of discretization. But the analysis of an algorithm is much more complex due to the high degree of nonconformity of this type of finite element. In proofs, a complicated mathematical apparatus has to be applied in the case of such nonconforming discretization.

It is worth adding that the proofs in this paper are very complex and they required introducing a number of special discrete operators defined over the common part of the boundaries of two neighboring substructures and we had to to prove their stability in special trace norms defined over the boundaries of the subdomains.

A Neumann-Neumann method for locally conforming mortar discretizations of a fourth order differential problem

The paper [H5] is devoted to the construction and analysis of a Neumann-Neumann type method for a mortar discretization of a fourth order elliptic equation problem with discontinuous coefficients with the locally conforming reduced Hsieh-Clough-Tocher macro element (cf. [8]). The problem of the construction of a Neumann-Neumann method for a mortar discretization of a fourth order elliptic problem, even for the case of differential problems with continuous coefficients, was unsolved for a long time.

The construction and analysis of the algorithm are extensions of the results from [9] and [H4], where Neumann-Neumann methods were presented for mortar discretizations of second order elliptic problems, to the case of fourth order elliptic differential problems.

The reduced Hsieh-Clough-Tocher element is a so-called composite finite element or a macro finite element, and it is more complicated on each triangle, but it has a lower number of degrees of freedom than the finite element methods, where functions restricted to a triangle are polynomials, e.g. the Argyris triangle finite element (cf. [8]).

In the paper a algorithm of the Neumann-Neumann type was developed and we proved almost optimal convergence estimates dependent logarithmically on the numbers of local unknowns and independent of the discontinuities of the parameters of the original differential problems. In the context of the Neumann-Neumann class of methods these results cannot be improved (cf. [6]). The key problem which had to be solved was the question how to construct a coarse subspace, i.e. a special subspace of the discrete space with a possible low dimension which is necessary in the construction of domain decomposition methods. In the case of this discretization, it was completely unclear how to do it.

Summary

The thesis consists of the papers in which a number of fully parallel Domain Decomposition Methods were developed for solving a system of equations arising from mortar method discretizations of differential elliptic equations of second and fourth orders on nonmatching grids. The almost optimal condition estimates were proven for these methods. The almost optimal condition estimates mean that the condition number bounds depend only logarithmically on the number of unknowns in subdomains and are independent of the number of subdomains, jumps of the coefficients of the original differential problem, or the global number of unknowns. For the substructuring methods it is well known that the almost optimal estimates cannot be improved. In the papers there are also results on the discretization of elliptic differential equations of the fourth order by the mortar method. In particular, the inf-sup (Ladyzhenskaya - Babuška - Brezzi) condition for the saddle form of mortar discretizations for fourth order elliptic differential problems were proven with constants independent of discretizations parameters, or the number of subdomains. This condition is important e.g. for proving the convergence of a discretization, or for a possible application of well-known general algorithms for solving discrete problems in such a formulation. All the results of the papers comprising the thesis are new.

References

[1] Ivo Babuška. Über Schwarzsche Algorithmen in partiellen Differentialgleichungen der mathematischen Physik. *Czechoslovak Math. J.*, 8 (83):328–343, 1958.

[2] Faker Ben Belgacem. The mortar finite element method with Lagrange multipliers. *Numer. Math.*, 84(2):173–197, 1999. First published as a technical report in 1994.

[3] Christine Bernardi, Yvon Maday, and Anthony T. Patera. A new nonconforming approach to domain decomposition: the mortar element method. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991)*, volume 299 of *Pitman Res. Notes Math. Ser.*, pages 13–51. Longman Sci. Tech., Harlow, 1994.

[4] Dietrich Braess, Wolfgang Dahmen, and Christian Wieners. A multigrid algorithm for the mortar finite element method. *SIAM J. Numer. Anal.*, 37(1):48–69, 1999.

[5] Susanne C. Brenner and L. Ridgway Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[6] Susanne C. Brenner and Li-Yeng Sung. Lower bounds for nonoverlapping domain decomposition preconditioners in two dimensions. *Math. Comp.*, 69(232):1319–1339, 2000.

[7] Franco Brezzi and Michel Fortin. *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, New York, 1991.

[8] P. G. Ciarlet. Basic error estimates for elliptic problems. In *Handbook of Numerical Analysis, Vol. II*, pages 17–351. North-Holland, Amsterdam, 1991.

[9] Maksymilian Dryja. A Neumann-Neumann algorithm for a mortar discetization of elliptic problems with discontinuous coefficients. *Numer. Math.*, 99:645–656, 2005.

[10] Jianguo Huang, Likang Li, and Jinru Chen. On mortar-type Morley element method for plate bending problem. *Appl. Numer. Math.*, 37(4):519–533, 2001.

[11] John E. Lagnese and Günter Leugering. *Domain decomposition methods in optimal control of partial differential equations*, volume 148 of *International Series of Numerical Mathematics*. Birkhäuser Verlag, Basel, 2004.

[12] Ulrich Langer, Marco Discacciati, David E. Keyes, Olof B. Widlund, and Walter Zulehner, editors. *Domain decomposition methods in science and engineering XVII*, volume 60 of *Lecture Notes in Computational Science and Engineering*. Springer-Verlag, Berlin, 2008. Selected papers from the 17th International Conference on Domain Decomposition Methods (DD17) held in Strobl, July 3–7, 2006.

[13] Patrick Le Tallec. Domain decomposition methods in computational mechanics. *Comput. Mech. Adv.*, 1(2):121–220, 1994.

[14] P.-L. Lions. On the Schwarz alternating method. I. In *First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Paris, 1987)*, pages 1–42. SIAM, Philadelphia, PA, 1988.

[15] Leszek Marcinkowski. A mortar element method for some discretizations of a plate problem. *Numer. Math.*, 93(2):361–386, 2002.

[16] Tarek P. A. Mathew. *Domain decomposition methods for the numerical solution of partial differential equations*, volume 61 of *Lecture Notes in Computational Science and Engineering*. Springer-Verlag, Berlin, 2008.

[17] Alfio Quarteroni and Alberto Valli. *Numerical Approximation of Partial Differential Equations*. Springer-Verlag, Berlin, 1994.

[18] Alfio Quarteroni and Alberto Valli. *Domain Decomposition Methods for Partial Differential Equations*. Oxford Science Publications, Oxford, 1999.

[19] H. A. Schwarz. *Gesammelte Mathematisches Abhandlungen. Vol. II.* Springer-Verlag, Berlin, 1890.

[20] Barry F. Smith, Petter E. Bjørstad, and William D. Gropp. *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations*. Cambridge University Press, Cambridge, 1996.

[21] Serguei L. Sobolev. L'Algorithme de Schwarz dans la Theórie de l'Elasticité. *Comptes Rendus (Dokładý) de l'Académie des Sciences de l'URSS*, IV((XIII)6):243–246, 1936.

[22] Olaf Steinbach. *Stability Estimates for Hybrid Coupled Domain Decomposition Methods*. Lecture Notes in Mathematics. Springer, Berlin, Heidelberg, New York, 2003.

[23] Andrea Toselli and Olof Widlund. *Domain decomposition methods—algorithms and theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2005.

[24] John von Neumann. *Functional Operators. II. The Geometry of Orthogonal Spaces*. Annals of Mathematics Studies, no. 22. Princeton University Press, Princeton, N. J., 1950.

[25] Olof B. Widlund and David E. Keyes, editors. *Domain decomposition methods in science and engineering XVI*, volume 55 of *Lecture Notes in Computational Science and Engineering*. Springer, Berlin, 2007. Papers from the 16th International Conference on Domain Decomposition Methods held in New York, January 11–15, 2005.

[26] Barbara I. Wohlmuth. A mortar finite element method using dual spaces for the Lagrange multiplier. *SIAM J. Numer. Anal.*, 38(3):989–1012, 2000.

[27] Barbara I. Wohlmuth. *Discretization Methods and Iterative Solvers Based on Domain Decomposition*, volume 17 of *Lectures Notes in Computational Science and Engineering*. Springer-Verlag, Berlin, 2001.