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CLASSROOM NOTES

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This section contains brief notes which are essentially self-contained applications of mathematics that can be used in the classroom. New applications are preferred, but examplary applications not well known or readily available are accepted.

Both "modern" and "classical" applications are welcome, especially modern applications to current real world problems.

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A MOTIVATIONAL EXAMPLE FOR THE NUMERICAL SOLUTION OF TWO-POINT BOUNDARY-VALUE PROBLEMS*

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Abstract. This paper presents an example of a two-point boundary-value problem which can be used to motivate the study of numerical techniques for solving such problems by undergraduates. The problem is referred to as the *putting problem* and illustrates the shooting method for two-point boundary-value problems. The differential equations are developed and numerical results are given.

Key words. shooting method, two-point boundary-value problem, golf, putting problem

AMS subject classifications. 65L10, 34B15

1. Introduction. Many times in a course in numerical analysis, numerical methods are introduced without any real world application for the technique. Usually an equation is given and we are asked to solve it not knowing what the equation represents. This leaves students with the feeling that they must memorize the material of the course so that they can fulfill a requirement. The best way to teach a course in numerical analysis is to motivate each technique by using a real world example which produces the type of problem to which the numerical method can be applied. This approach is sometimes difficult since many good examples require a background in mathematics and physics that is too advanced for undergraduates. The following example has been used to motivate the study of the numerical solution of two-point boundary-value problems and, in particular, is a very good way to visualize the shooting method for such problems. In addition, the example only requires a basic understanding of physics, differential equations, and calculus with which the student should already be familiar by the time a course in numerical analysis is taken.

2. The example. This problem will be referred to as the *putting problem*. It is presented in the follow manner.

Suppose that Arnold Palmer is on the 18th green at Pebble Beach. He needs to sink this putt to beat Jack Nicklaus and walk away with the $1,000,000 grand prize. What should he do? Solve a BVP! By modeling the surface of the green, Arnie sets up the equations of motion of his golf ball. Letting $\boldsymbol{x}(t)$ be the position of the ball at time t, he gets the BVP

$$\ddot{\boldsymbol{x}} = \boldsymbol{F}(t, \boldsymbol{x}, \dot{\boldsymbol{x}}), \quad \boldsymbol{x}(0) = \boldsymbol{0}, \quad \boldsymbol{x}(T) = \boldsymbol{x}_H,$$

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where T is unknown, the initial position of the ball is at the origin, and $\boldsymbol{x}_H$ is the position of the hole relative to the origin. Solving this BVP, Arnie finds out how hard and in what direction to putt the ball. Of course, he can solve it in his head; we need a computer!

3. The mathematical model. At this point we need to formulate $\boldsymbol{F}(t, \boldsymbol{x}, \dot{\boldsymbol{x}})$. It is sufficient for our purpose to assume that the ball is a point mass with mass m. Let

$$m\ddot{\boldsymbol{x}} = \boldsymbol{f}_G + \boldsymbol{f}_N + \boldsymbol{f}_F, \quad \boldsymbol{x}(0) = \boldsymbol{0}, \quad \boldsymbol{x}(T) = \boldsymbol{x}_H,$$

where $\boldsymbol{f}_G$ is the force due to gravity, $\boldsymbol{f}_N$ is the normal force, and $\boldsymbol{f}_F$ is the frictional force. The gravitational force is given by $\boldsymbol{f}_G = -mg\,\boldsymbol{k}$, where g is the acceleration due to gravity and $\boldsymbol{k} =$ (0,0,1), the unit vector in the x_3, or vertical, direction. The normal force is in the direction of the unit outward surface normal $\boldsymbol{n}$ and is the projection of $-\boldsymbol{f}_G$ onto $\boldsymbol{n}$. Thus, $\boldsymbol{f}_N = m\,g\,(\boldsymbol{n}\cdot\boldsymbol{k})\,\boldsymbol{n}$. Assuming the surface is given explicitly as $x_3 = S(x_1, x_2)$ such that $S(0, 0) = 0$, we have

$$\boldsymbol{n} = \frac{\boldsymbol{N}}{\|\boldsymbol{N}\|}, \quad \text{where} \quad \boldsymbol{N} = \left(-\frac{\partial S}{\partial x_1}, -\frac{\partial S}{\partial x_2}, 1\right).$$

Therefore,

$$\boldsymbol{n}\cdot\boldsymbol{k} = n_3 = \frac{1}{\|\boldsymbol{N}\|} \quad \text{and} \quad \boldsymbol{f}_N = m\,g\,n_3\,\boldsymbol{n}.$$

The frictional force is in the direction opposite the velocity of the ball $\dot{\boldsymbol{x}}$ and its magnitude is $\|\boldsymbol{f}_F\| = \mu_K\|\boldsymbol{f}_N\|$, where μ_K is the *kinetic coefficient of friction*. Thus,

$$\boldsymbol{f}_F = -\mu_K\|\boldsymbol{f}_N\|\frac{\dot{\boldsymbol{x}}}{\|\dot{\boldsymbol{x}}\|}.$$

Combining all of these forces and noting that the mass of the ball m can be eliminated, we have

$$\ddot{\boldsymbol{x}} = \boldsymbol{F}(t, \boldsymbol{x}, \dot{\boldsymbol{x}}) = -g\boldsymbol{k} + g\,n_3\,\boldsymbol{n} - \mu_K g\,n_3\,\frac{\dot{\boldsymbol{x}}}{\|\dot{\boldsymbol{x}}\|}.$$

This second-order system can now be transformed into a first-order system in the usual way [4]. Letting $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3$, $y_4 = \dot{x}_1$, $y_5 = \dot{x}_2$, and $y_6 = \dot{x}_3$, we get the first-order system $\dot{\boldsymbol{y}} = \boldsymbol{G}(t, \boldsymbol{y})$, where

$$\boldsymbol{G}(t, \boldsymbol{y}) = \left(y_4, y_5, y_6, gn_1\,n_3 - \mu_K g\,n_3\frac{y_4}{s}, gn_2\,n_3 - \mu_K g\,n_3\frac{y_5}{s}, gn_3\,n_3 - \mu_K g\,n_3\frac{y_6}{s} - g\right)$$

and $s = \sqrt{y_4^2 + y_5^2 + y_6^2}$ is the speed of the ball.

This BVP is an example of a *free boundary-value problem* since the final time T must also be determined [3], [4]. To handle this, we let $y_7 = T$, $\dot{y}_7 = 0$, and define a new independent variable τ by setting $t = \tau y_7$ for $0 \le \tau \le 1$. Thus,

$$\frac{dt}{d\tau} = T = y_7 \quad \text{and} \quad \frac{dy}{dt} = \boldsymbol{G}(t, \boldsymbol{y}) \quad \text{becomes} \quad \frac{dy}{d\tau} = \frac{dy}{dt}\frac{dt}{d\tau} = \boldsymbol{G}(\tau, \boldsymbol{y})y_7.$$

We then get a new system of seven first-order differential equations which requires another boundary condition. For simplicity, we choose to force the ball to have zero speed at the final time T; that is, $\|\dot{\boldsymbol{x}}(T)\| = 0$.

Remarks. The boundary condition $\boldsymbol{x}(T) = \boldsymbol{x}_H$ is somewhat artificial. In reality the hole is not a point but is a circle of radius R_H. Thus, a more appropriate test is $\|\boldsymbol{x}(T) - \boldsymbol{x}_H\| \le R_H$.

The boundary condition $\|\dot{x}(T)\| = 0$ is also not realistic. We should actually test $\|\dot{x}(T)\| \leq s_T$, where s_T is the maximum final speed. See [2] for more details. In this simple model, we are not modeling the possibility of the ball leaving the surface of the green when rolling over a steep hill. In this case, the ball becomes a projectile with $\boldsymbol{f}_F = \boldsymbol{f}_N = \mathbf{0}$ and we must also model the bouncing of the ball as it falls and collides with the green.

4. Numerical results. We now present numerical results of three examples based on a C implementation of the above model using the shooting method as described in [3]. All floating point computations where performed using the `double` type. For completeness, we briefly summarize the shooting method for nonlinear problems. Given the n-dimensional boundary-value problem

$$\dot{y} = G(\tau, y) \quad \text{for } 0 \leq \tau \leq 1, \quad A(y(0)) = y_0, \quad B(y(1)) = y_1,$$

where y_0 is a p-vector, $A(y)$ is a p-vector-valued function, y_1 is a q-vector, $B(y)$ is a q-vector-valued function, and $p + q = n$, we consider the initial-value problem

$$\dot{y} = G(\tau, y) \quad \text{for } 0 \leq \tau \leq 1, \quad C(y(0)) = x,$$

where x is a q-vector and $C(y)$ is a q-vector-valued function. We express the solution of such a problem as $y(\tau; x)$. Using this and the above boundary condition at $\tau = 1$, we get $f(x) = B(y(1; x)) - y_1 = \mathbf{0}$. To evaluate this function we need to integrate the equations of motion to $\tau = 1$. Thus, we see that the boundary-value problem is reduced to a multidimensional root-finding problem. In [3] a multidimensional Newton–Raphson method is employed which uses the finite difference approximation

$$\frac{\partial f_i}{\partial x_j} \approx \frac{f(x_1, \ldots, x_j + \Delta x_j, \ldots, x_n) - f(x_1, \ldots, x_n)}{\Delta x_j}$$

to compute the Jacobian matrix. The initial conditions for our first-order system are $y(0) = (x(0), \dot{x}(0), T_0) = (\mathbf{0}, v_0, T_0)$, where v_0 is the guess for the initial velocity of the ball and T_0 is the initial guess for T, which we take to be one second. The shooting method then tries to improve the values for v_0 and T_0.

In all of the following examples values of $R_H = 0.005$ feet, $s_T = 0.005$ feet per second, $\Delta x_j = 10^{-10}$, $\mu_K = 0.2$, and $g = 32$ feet per second squared have been used to produce the data in Tables 4.1–4.3. The Attempt column represents each iteration of the shooting method with the first attempt representing the initial guess and the final attempt representing the solution. The Speed and Direction are the magnitude and direction of the velocity vector at the beginning of each iteration where a direction of zero degrees is shooting directly at the hole with positive angles to the left of the hole and negative angles to the right. Finally, the Time column is the value of the final time T used for solving the initial-value problem during that particular iteration. The hole is positioned at $x_H = (20, 0, 0)$ in all examples and all distances are in feet.

Example 1 (Existence of a unique solution). Here the green is described by the function $S(x_1, x_2) = 0$, which is a perfectly flat green. Clearly the only solution this green can have is to putt directly at the hole. The equations of motion become

$$\ddot{x} = -\mu_K g \frac{\dot{x}}{\|\dot{x}\|}.$$

For the initial conditions $x(0) = \mathbf{0}, \dot{x}(0) = \dot{x}_0$, we have the solution

$$x(t) = \frac{\dot{x}_0}{2\|\dot{x}_0\|} t(2\|\dot{x}_0\| - \mu_K g t) \quad \text{with} \quad \dot{x}(t) = \frac{\dot{x}_0}{\|\dot{x}_0\|}(\|\dot{x}_0\| - \mu_K g t),$$

TABLE 4.1
Putting problem numerical results. Example 1.

Attempt	Speed	Direction	Time
1	10.00	10.00	1.00
2	17.80	-6.71	2.65
3	16.16	-0.69	2.50
4	16.05	0.00	2.49

TABLE 4.2
Putting problem numerical results. Example 2.

Attempt	Speed	Direction	Time	Attempt	Speed	Direction	Time
1	10.00	0.00	1.00	1	10.00	-90.00	1.00
2	13.83	−21.54	2.45	2	23.58	−2.80	3.04
3	16.48	−12.15	2.66	3	16.79	−12.43	2.98
4	16.85	−13.71	3.78	4	17.28	−14.36	4.57
5	16.57	−13.21	3.51	5	17.36	−14.62	4.76
6	16.64	−13.35	3.64				

TABLE 4.3
Putting problem numerical results. Example 3.

α	Speed	Direction	Time
0.0	16.05	0.00	2.49
0.1	16.35	−15.47	2.95
0.15	17.19	−26.58	4.08
0.175	18.50	−35.72	6.02
0.19	20.81	−45.87	10.53
0.1925	21.63	−48.69	12.61
0.195*	22.87	−52.40	16.28

for $0 \le t \le \|\dot{\boldsymbol{x}}_0\|/(\mu_K g)$. Solving $\boldsymbol{x}(T) = \boldsymbol{x}_H = (20, 0, 0)$ and $\|\dot{\boldsymbol{x}}(T)\| = 0$ for $\dot{\boldsymbol{x}}_0$ and T, we get $\dot{\boldsymbol{x}}_0 = (16, 0, 0)$ and $T = 2.5$. Table 4.1 shows the results of initially putting 10 degrees to the left of the hole.

Example 2 (Nonuniqueness). A green that has a least two solutions is given by

$$S(x_1, x_2) = \frac{(x_1 - 10)^2}{125} + \frac{(x_2 - 5)^2}{125} - 1,$$

which is a paraboloid with a minimum at (10,5). Because the green has a minimum to the left of the hole at (20,0,0), one would expect to have a solution with a negative direction. We might also expect that there may be multiple solutions, since if we putt slightly more to the right with an increase in initial speed, the ball could roll uphill a little more and then roll back to the hole. In fact, there are at least two such solutions which are illustrated in Table 4.2. To obtain the first, we initially putt directly at the hole and to obtain the second we initially putt 90 degrees to the right of the hole.

Example 3 (Nonexistence). A very simple green for which a solution can fail to exist is given by $S(x_1, x_2) = \alpha x_2$, where $\alpha \ge 0$ is the slope of the plane tilted along the x_1-axis. We have already seen that the case $\alpha = 0$ has a unique solution. The equations of motion are now

$$\ddot{\boldsymbol{x}} = -g\left(0, \frac{\alpha}{1+\alpha^2}, \frac{\alpha^2}{1+\alpha^2}\right) - \frac{\mu_K g}{(1+\alpha^2)^{1/2}} \frac{\dot{\boldsymbol{x}}}{\|\dot{\boldsymbol{x}}\|}.$$

Since $\dot{\boldsymbol{x}}$ must lie in the tilted plane, we have $\dot{x}_3 = \alpha \dot{x}_2$. Now the only way a solution can exist with $\dot{\boldsymbol{x}}(T) = \boldsymbol{0}$ is if the frictional force is large enough to overcome the force causing the ball to roll down the plane. Comparing the x_2 components of the acceleration, we see that we must have

$$\frac{\alpha}{1+\alpha^2} \leq \frac{\mu_K \dot{x}_2}{(1+\alpha^2)^{1/2}(\dot{x}_1^2+(1+\alpha^2)\dot{x}_2^2)^{1/2}} \to \frac{\mu_K}{1+\alpha^2} \quad \text{as} \quad \dot{x}_1 \to 0.$$

From this inequality, we can see that in order for the frictional force to slow the ball down in the x_2 and x_3 directions, we must have $\alpha \leq \mu_K$. If $\alpha > \mu_K$ then no solution can exist. In fact, even if $\alpha < \mu_K$ but is very close to μ_K the numerical solution may not converge. Table 4.3 shows the results of slowly increasing the slope from 0 toward 0.2. The inital guess for a given α is the solution of the previous α. The hole is again at (20,0,0). For $\alpha = 0.195$ the initial speed, direction, and T converge after five iterations but the final speed remains slightly above the tolerance of 0.005 preventing convergence even after 100 iterations due to the finite precision of the computations. For the case $\alpha = 1.0$ which has no solution, the iteration process diverged before reaching 50 iterations for every initial guess used.

5. Conclusion. The above simple model has been used successfully in the classroom to motivate the study of numerical techniques for solving two-point boundary-value problems and for dynamically illustrating the shooting method using computer graphics to display each iteration as it is computed. Further study for students could include modeling the sliding (the ball most likely slides initially) and rolling motion of the ball, modeling the surface by a more realistic function as in [2], modeling the ball leaving the surface and bouncing, and trying to determine realistic values for μ_K. Such a study could be performed by students working in groups with each student assigned a specific task. This gives students some experience working on a project such as they might do in industry.

Note. At the time of this writing, the author has become aware of a paper by Lorensen and Yamrom [2] where the *putting problem* is examined for the purpose of visualizing a putt using computer graphics. The authors applied the problem to the real world by surveying golf greens and replaying putts during golf tournaments to help viewers better understand how the golf green topography affects putting. This application reinforces the use of this example in the classroom to motivate the shooting method. The same $\boldsymbol{F}(t, \boldsymbol{x}, \dot{\boldsymbol{x}})$ is used except that the authors model the surface of the green as a mesh of polygons with a surface normal at each vertex computed by averaging as in the Gouraud shading technique [1] and compute the normal in the interior by averaging the vertex normals to obtain a continuous right-hand side to the above differential equations.

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