Projects Numerical Differential Equations 2019-20

Leszek Marcinkowski

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Abstract

Projects from the Numerical Differential Equations - for volunteers. As part of completing the project, one needs to know the basic properties of the methods implemented in the project. One has to prepare a short report describing the basic properties of the implemented methods and the results of the tests, I may also ask to show me the code and to run it.

Contents

1 Adaptive ODE schemes

(easy - the maximum grade is 4.5) Write two adaptive ODE schemes based on two Runge Kutta explicit methods: Heun and an appropriate third-order one. Use two approaches to chage the step.

1. Compute s such that $s * h$ is the right step giving the error below prescribed tolerance, then use the new step sh
2. if h is too large or two small, then take $h/2$ or $2 * h$ as new steps.

Find the details of the methods i.e., how to estimate the local (global) error. Implement the methods in octave. Test it for some differential equations with known solutions.

2 Nonconforming Crouzeix-Raviart FEM in 2D

Consider a BVP:

$$\begin{aligned} -a\triangle u^* + \vec{b}^T \nabla u^* + cu^* &= f \qquad in\ \Omega = (0,1)^2 \\ u &= g \qquad on\ \partial\Omega \end{aligned}$$

where $a > 0, \vec{b}^T = (b_1, b_2), c \geq 0$ constants.

Program in octave (or eg, C / C ++ but it will take more time) Crouzeix-Raviart finite element method on standard uniform triangulation of the unit square - i.e. we introduce fine squares with vertices: $(k * h, l * h)$ for $h = 1/N$ and divide them by a diagonal. That is, create in the appropriate nodal base a system of linear equations for the values of the approximate solution and solve it using the appropriate octave solver.

Then test it for different values of the a, b_1, b_2, c constants.

In particular, experimentally examine the order of convergence in the maximum norm, type L^2 and H^1 for known smooth and non-smooth solutions, we compare the FEM solution with the extension of the exact solution that we know. The extension is $I_h^{CR} u^*$ - a function from the CR FEM space taking values of u^* at the centers of the edges of the triangles (i.e., at the nodal points of the CR method).

As part of the project, one should familiarize himself with the method, implement the code, and conduct tests. As part of the course one needs to know the basic properties of the method e.g., what is the order of convergence in the H^1 standard, so-called broken or L^2?

3 Bilinear FEM in 2D

Consider a model problem:

$$\begin{aligned} -\triangle u^* + b * \partial_{x_1} u^* + cu^* &= f \qquad in\ \Omega = (0,1)^2 \\ u &= g \qquad on\ \partial\Omega \end{aligned}$$

where $b, c \geq 0$ constants.

Program in octave (or e.g. C / C ++ but it will take more time) Bilinear finite element method on standard uniform triangulation rectangle into small sub-rectangles

That is, create in the appropriate nodal base a system of linear equations for the values of the approximate solution and solve it using the appropriate octave solver. Then test it - for different values of the b, c constants.

In particular, experimentally examine the order of convergence in the maximum norm, type L^2 and H^1 for known smooth and non-smooth solutions to, we compare the FEM solution with the extension of the exact solution that we know. For extension, we take $I_h u^*$ - a function from the bilinear FEM space taking values of u^* at the vertices of rectangles, i.e., at the nodal points.

As part of the project, one should familiarize with the method, implement the code, and conduct tests. One needs to know the basic properties of the method e.g., what is the order of convergence in the maximum, H^1 or L^2 norm?

4 Quadratic element in 2D

A quadratic finite element method for the model problem:

$$\begin{aligned} -\triangle u^* + b * \partial_{x_2} u^* + cu^* &= f \qquad in\ \Omega = (0,1)^2 \\ u &= g \qquad on\ \partial\Omega \end{aligned}$$

where $b, c \geq 0$ constants.

Program in octave (or eg C / C ++ but it will take more time) quadratic finite element method on standard uniform triangulation square - i.e. we introduce squares with vertices: $(k * h, l * h)$ for $h = 1/N$ and divide them by a diagonal.

That is, create in the appropriate nodal base a system of linear equations for the values of the approximate solution and solve it using the appropriate octave solver.

Then test it - for different values of the b, c constants.

In particular, experimentally for known smooth and non-smooth solutions to examine the order of convergence in the maximum norm, type L^2 and H^1, we compare the MES solution with the extension of the exact solution that we know. For extension, we take $I_h u^*$ - a function from the square space of FEM taking values of u^* at the vertices and centers of the edges of the triangles, i.e. at the nodal points of the FEM quadratic method, cf.
`http://mst.mimuw.edu.pl/lecture.php?lecture=nrr&part=Ch12#S1.SS3`.

As part of the project, one should familiarize with the method, implement the code, and conduct tests. One needs to know the basic properties of the method, e.g., what is the order of convergence in the standard H^1 or L^2.

For calculating the right hand side, i.e. $\int_\Omega f\phi_x\,dx$ for ϕ_x of the base nodal function for node x use the triangle quadrature τ: $Q_\tau v = \int_\tau I_{\tau,1}(v)\,dx \approx \int_\tau v\,dx$, where $I_{\tau,1}(v)$ is a function linear such that $I_{\tau,1}(v)(x) = v(x)$ for x vertex of the triangle τ. Then the approximation $\int_\Omega f\phi_x\,dx$ (here $v = f\phi_x$) is the sum of approximations of integrals after the triangles into the support ϕ_x, i.e. after the triangles with x as the vertex.

5 FDM - Neumann condition - schemes with the higher order

Implement the finite difference method of order two in octave (or perhaps more labor-intensive in another language) for the model equation:

$$\begin{aligned} -\triangle u^* + cu^* &= f \qquad in\ \Omega = (0,1)^2 \\ \frac{\partial u^*}{\partial n} &= g_1 \qquad on\ \Gamma_1 \\ u^* &= g_2 \qquad on\ \partial\Omega \setminus \Gamma_1 \end{aligned}$$

where $c > 0$ constant, Γ_1 open right edge of the square edge.

Consider an even grid, i.e. $\overline{\Omega}_h = (k * h, l * h)$ for $h = 1/N$ Let us discret Laplacians by the standard difference of five points a normal derivative increasing the order - we assume that the equation is also met for Γ_1 and using this we increase the order because at the edge of the grid $x \in \Gamma_1$:

$$\frac{\partial u^*}{\partial n}(x) = \frac{\partial u^*}{\partial x_1}(x) = \overline{\partial}_{1,h} u^*(x) + 0.5\frac{\partial^2 u^*}{\partial x_1^2}(x)h + O(h^2)$$

now using the assumption that the output equation met in $x \in \Gamma_1$ we see

$$\frac{\partial^2 u^*}{\partial x_1^2}(x) = -\frac{\partial^2 u^*}{\partial x_2^2}(x) + c * u^*(x) - f(x) = -\partial\overline{\partial}_{2,h} u^*(x) + c * u^*(x) - f(x) + O(h^2)$$

From these two equations we construct a scheme of the order of two.

The task is to refine the details, then implement and test. I.e., create a system of linear equations on the values of the approximate solution of the grid points and solve it using the appropriate octave solver. Then test it for different values of the c constant and different values of f and g_k.

In particular, experimentally for known smooth and non-smooth solutions, examine the order of convergence in a discrete maximum norm and a L^2 norm. Compare the schema solution with the extension of the exact solution that we know. We take the mesh function as the extension taking values of u^* at grid points

As part of the project, one needs to familiarize onerself with the method, implement the code and conduct tests, in particular for known smooth solutions, test the convergence order in the discrete L^2_h standards and the discrete maximum standard.

As part of the course one also needs to know the basic properties of the method e.g. what is the order of convergence in the discrete maximum norm or L^2_h.

6 Finite difference method in 2 dimensions - non-rectangular area, i.e. sphere - Dirichlet condition - Collatz approximation

Implement the finite difference method of two on octave (or perhaps more labor-intensive in another language) for the model equation:

$$\begin{aligned} -\triangle u^* + cu^* &= f \qquad in\ \Omega = K(0,1) \\ u^* &= g \qquad on\ \partial\Omega \end{aligned}$$

where $c > 0$ constant.

Consider the grid equal to $[-1,1]^2$ cut from $\overline{\Omega}$, i.e. $\overline{\Omega}_h = (-1 + k * h, -1 + l * h) \cap \overline{\Omega}$ for $h = 2/N$. Let us discretize Laplacians by the standard difference of five points. On the mesh edge, we will use Collatz approximations of the boundary condition: for the edge point $x_{k,l}$, let $x_{k-1,l}$ be the inner point of the grid and $x_{k,l} + (h,0)$ will be outside Ω, i.e. outside the grid then there is a point $p = x_{k,l} + (\alpha h, 0) \in \partial\Omega$ for $0 < \alpha < 1$ - let $l(t)$ a linear interpolation polynomial such that $l(0) = u_{k-1,l}, l(h) = u_{k,l}$ and then the value of the grid operator for $x_{k,l}$ is equal to $l((1+\alpha)h) = u(p) = g(p)$ (we know the right side). Similarly, we proceed for all mesh border points, obtaining a scheme of the order of two (which should be justified).

The task is to refine the details, then implement and test. Ie. create a system of linear equations on the values of the approximate solution of the grid points and solve it using the appropriate octave solver. Then test it - for different values of the c constant and different values of f and g.

In particular, experimentally for known smooth and non-smooth solutions, examine the order of convergence in a discrete maximum norm and a L^2 norm. Compare the schema solution with the extension of the exact solution that we know. We take the mesh function as the extension taking values of u^* at grid points

As part of the project, one needs to familiarize with the method, implement the code and conduct tests, in particular for known smooth solutions, test the convergence order in the discrete L^2_h standards and the discrete maximum standard.

As part of the course one also need to know the basic properties of the method Â e.g. what is the order of convergence in the discrete maximum norm or L^2_h.

7 FEM - mixed boundary condition

Let $\Omega = (0,1)^2$ a unit square -the boundary comprise 4 edges: $\partial\Omega = \bigcup_{k,l=0,1} \overline{\Gamma_{k,l}}$. (let $\Gamma_{0,0}$ the left edge, $\Gamma_{1,0}$ the right one, $\Gamma_{0,1}$ the lower one, $\Gamma_{1,1}$ the upper one).

Consider a BVP with the mixed bnd conditions

$$\begin{aligned} -\triangle u^* + cu^* &= f \qquad in\ \Omega \\ l_{k,l}u &= g_{k,l} \qquad on\ \Gamma_{k,l} \end{aligned}$$

where a positive constant c

$$l_{k,l}u(s) = \begin{cases} u(s) & k = 0 \\ \frac{\partial u}{\partial n}(s) & k = 1 \end{cases}$$

Here $\frac{\partial u}{\partial n}$ normal derivative to the given edge. We set a A Dirichlet condition at the vertices, i.e. we assume that we know the solution values in the vertices.

Program in octave (or e.g. C / C ++ but it will take more time) linear finite element method on standard uniform triangulation square - i.e. we introduce squares with vertices: $(k*h, l*h)$ for $h = 1/N$ and divide them by a diagonal.

Ie. create in the appropriate nodal base a system of linear equations for the values of the approximate solution and solve it using the appropriate octave solver.

Then test it - for different values of the c constant and different values of f and $g_{k,l}$.

In particular, experimentally for known smooth and non-smooth solutions to examine the order of convergence in the maximum norm, type L^2 and H^1, compare the MES solution with the extension of the exact solution that we know. For extension we take $I_h u^*$ - a linear function from the MES space taking values of u^* at the vertices of the triangles (i.e. at the nodal points of the linear FEM method). Let us take the maximum from the module of a given function at nodal points as an approximation of the maximum norm.

One should familiarize himself with the method, implement the code and conduct tests, in particular for known smooth solutions, test the convergence order in the standards H^1, L^2, the discrete maximum standard.

As part of the course one also need to know the basic properties of the method e.g., what is the order of convergence in the standard H^1 or L^2?

8 Cubic element in 2 dimensions

Cubic finite element method for the model equation:

$$\begin{aligned} -\triangle u^* + b * \partial_{x_1} u^* + cu^* &= f \qquad in\ \Omega = (0.1)^2 \\ u &= g \qquad on\ \partial\Omega \end{aligned}$$

where $b, c \geq 0$ fixed.

Program in octave (or eg C / C ++ but it will take more time) cubic finite element method on standard uniform triangulation square - i.e. we introduce squares with vertices: $(k*h, l*h)$ for $h = 1/N$ and divide them by a diagonal.

That is, create in the appropriate nodal base a system of linear equations for the values of the approximate solution and solve it using the appropriate octave solver.

Then test it - for different values of the b, c constants.

In particular, experimentally for known smooth and non-smooth solutions to examine the order of convergence in the maximum norm, type L^2 and H^1, we compare the MES solution with the extension of the exact solution that we know. For extension we take $I_h u^*$ - a function from the MES cubic space taking values of u^* at the vertices, center of gravity, two points of the inner edges (at a distance of $1/3$ and $2/3$ of the edge length from the fixed end of this edge) of triangles, i.e. at the nodal points of the cubic FEM method, cf.
`http://mst.mimuw.edu.pl/lecture.php?lecture=nrr&part=Ch12#S1.SS3`.

As part of the project, one should familiarize himself with the method, implement the code and conduct tests.

As part of the course one needs to know the basic properties of the method e.g. what is the order of convergence in the standard H^1 or L^2.

For calculating the right hand side, i.e. $\int_\Omega f\phi_x\, dx$ for ϕ_x of the base nodal function for the node x, use either a procedure using the function `quad ()` in octave or quadrature on triangle τ: $Q_\tau v = \int_\tau I_{\tau,2}(v)\, dx \approx \int_\tau v dx$ where $I_{\tau,2}(v)$ is a quadratic function on a triangle τ such that $I_{\tau,2}(v)(x) = v(x)$ for x vertex or the middle of the edge of the triangle τ. Then the approximation

$\int_\Omega f\phi_x \, dx$ (here $v = f\phi_x$) is the sum of approximations of integrals after the triangles into the carriers ϕ_x, i.e. after the triangles with x as the vertex.

9 Cubic element in 1 dimension - different boundary conditions

(easy - the maximal grade 4.5)

Implement a continuous FEM cubic method on an uneven grid with Robin or Dirichlet boundary conditions or mixed for the equation

$$-u'' + cu = f \quad x \in (a, b)$$

Write an octave function solving this task, i.e. the parameters should be:

- Input:
  1. F - pointer (function handle) to the function f
  2. c - value of $c \geq 0$
  3. bca - as a scalar, we assume the Dirichlet boundary condition in a, i.e. $u(a) = bca$, as a two-element vector is $bca(1)$ value of Robin condition coefficient a $bca(2)$ right side in Robin condition i.e. $-u'(a) + bca(1)u(a) = bca(2)$
  4. bcb - by analogy but for b - for the Robin condition: $u'(b) + bcb(1)u(b) = bcb(2)$
  5. a, b - episode ends
  6. x - vector with nodes $[x_0, x_1, ..., x_n]$ flocks should be $a = x_0, b = x_n$ - if x is not given, the default value is 101 even nodes $[a, b]$
  7. QUAD - the option to calculate the right side - 0 - we count the Simpson square on each sub-dot, 1 with the octave function **quad** ()
- Output
  1. y - grid with nodal points of cubic FEM, i.e. we have points $x_0, x_0 + (1/3)h_0, x_0 + 2h_0/3, x_1, ..., x_k, x_k + h_k/3, x_k + (2/3)h_k, x_{k+1}, ..., x_n$ for $h_k = x_{k+1} - x_k$
  2. u - solution values in nodal points of cubic FEM, i.e. $u_k = u(y_k)$
  3. h - vector with subsection lengths $h_k = x_{k+1} - x_k$
  4. A stiffness matrix (corresponding to discretization $\int_a^b u'v'$ in the nodal database)
  5. M mass matrix (corresponding to discretization $\int_a^b uv$ in the nodal database)

The M and A matrices can then be used to compute discrete errors in the standard H^1 and L^2 norms, i.e. $|u_h|_{H^1(a,b)} = \sqrt{\vec{u}^T A \vec{u}}$ and $\|u_h\|_{L^2(a,b)} = \sqrt{\vec{u}^T M \vec{u}}$ for $\vec{u}$ - a vector with u_h values in FEM nodes. A discrte error is the error of $u_h - I_h u^*$. To compute a rela error (or its approimation - one has to use a finer mesh and some integration formulas.

The values of the right side, i.e. the integrals $\int_a^b f(x)\phi_k \, dx$ can be calculated using the appropriate octave function (**quad**()) or use, for example, Simpson's complex quadrature (i.e. Simpson's quadrature on each sub-dot included in the support of the corresponding nodal function). The function user will be able to choose which integration method to use.

Tests:

- The simplest test - please take the familiar smooth function e.g. $u = \sin(x)$ and check on the section $[-1, 3]$ will we get a good approximation of this function by choosing f and boundary values, respectively.
- Another simple test: u polynomial of various degrees $1, 2, 3, 4$ etc. (of course, the boundary conditions and f should be chosen accordingly)

- Convergence test in the standard L^∞, H^1 and L^2 for an even or slightly disturbed grid, e.g. $x_k = a + (k + eps_k)hh = (ba)/n$ z eps_k random value z $[-1, 1]/10$ for $k - 1, \dots, N - 1$.
- Test of convergence taking an uneven grid e.g. $h_k = 0.7h_{k-1}$ with a fixed h_1 (then one can compare the error ratios for the grids with half the h_1.)
- Error tests for the grid as above taking a solution strongly oscillating near the right end of the segment eg $u = \sin(x^2)$ to $[0.4]$ - one can draw an error graph, i.e. the $I_h u - u_h$ graph ($I_h u$ nodal interpolator - u_h discrete solution)
- Test of diffusive properties - we take the same right side e.g. $f(x)$ characteristic function of the segment $[0, 1]$ and we count solutions with zero boundary conditions $-u'' + cu = cf$ on $[-2, 2]$ for $c = 10^k$ for $k = -4, -3, -2, -$ 1,0,1,2,3,4.

For all convergence tests - compare the results when the right side is counted in both ways.