Fractional differentiability results for elliptic partial differential equations with coefficients

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Motivation

Let us consider the following equation

$$-\mathrm{div}(a(x)Du) = 2\pi\delta_0 \quad \text{in } B_1 \subset \mathbb{R}^2,$$

where δ_0 is the Dirac measure at the origin.

For $a(x) = 1 + |x|^{1/2} \in C^{1/2}(B_1)$, we have a distributional solution

$$u(x) = 2\log(1 + |x|^{1/2}) - \log(|x|) \in W^{1,1}(B_1).$$

Known: $a \in C^{1/2}(B_1)$

$\Rightarrow \; Du \in W^{1/2-\varepsilon,1}(B_1)$

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Fact:

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Function spaces

For any $\alpha \in (0,1)$ and $\gamma \in [1,\infty)$, we consider the following function spaces of fractional order.

- Fractional Sobolev space

$$W^{\alpha,\gamma}(\Omega) = \left\{ f \in L^{\gamma}(\Omega) \middle| \int_{\Omega}\int_{\Omega} \frac{|f(x)-f(y)|^{\gamma}}{|x-y|^{n+\alpha\gamma}}\,dx\,dy < \infty \right\}.$$

- Nikolskii space

$$\mathcal{N}^{\alpha,\gamma}(\Omega) = \left\{ f \in L^{\gamma}(\Omega) \middle| \sup_{h \in \mathbb{R}^n \setminus \{0\}} \int_{\Omega_{|h|}} \frac{|\tau_h f|^{\gamma}}{|h|^{\alpha\gamma}}\,dx < \infty \right\},$$

where $\tau_h f(x) = f(x+h) - f(x)$ and $\Omega_{|h|} = \{x \in \Omega | \mathrm{dist}(\partial\Omega, x) > |h|\}$.

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- We define by $C^{\alpha}_{\gamma}(\Omega)$ the set of function $a \in L^1(\Omega)$ such that there exists $g \in L^{\gamma}(\Omega)$ satisfying

$$|a(x) - a(y)| \le |g(x) + g(y)||x-y|^{\alpha} \quad {}^{\forall} x, y \in \Omega.$$

Inclusions & Examples

- For any $\alpha \in (0,1), \gamma \in (1,\infty)$ and $\varepsilon \in (0,\alpha)$,

$$C^{\alpha}_{\gamma}(\Omega) \subsetneq W^{\alpha,\gamma}(\Omega) \subsetneq \mathcal{N}^{\alpha,\gamma}(\Omega) \subsetneq W^{\alpha-\varepsilon,\gamma}(\Omega) \subsetneq L^{\frac{n\gamma}{(\alpha-\varepsilon)\gamma-n}}(\Omega).$$

- $C^{\alpha}_{\infty}(\Omega) = C^{\alpha}(\Omega)$.

- For any $\alpha \in (0,1]$ and $\varepsilon \in (0,1)$, $\chi_{B_1} \in C^{\alpha}_{1/\alpha-\varepsilon}(B_2)$.

- For any $\varepsilon \in (0,1)$, $a(x) = 1 + |x|^{1/2} \in C^{1}_{4-\varepsilon}(B_1)$.

Heuristics for the above problem

Consider the following equation:

$$-\mathrm{div}(a(x)|Du|^{p-2}Du) = \mu, \quad \text{in } \Omega \subset \mathbb{R}^n,\, n \geq 2,\, p > 2 - \frac{1}{n}$$

where $\mu \in L^1(\Omega)$ and $a \in C^\alpha_\gamma$ for some $\alpha \in (0,1]$ and $\gamma \in [n,\infty)$.

For small $h \in \mathbb{R}^n$, we roughly have

$$\begin{aligned}|\tau_h(|Du|^{p-2}Du)| &\lesssim |\tau_h a||Du|^{p-1} + |h||\mu| \\ &\lesssim |h|^\alpha(|g||Du|^{p-1} + |h|^{1-\alpha}|\mu|).\end{aligned}$$

Then in view of scaling, we expect that if $\alpha\gamma \geq n$, then

$$|h|^{1-\alpha}|\mu| \in W^{1-\alpha,1}(\Omega) \hookrightarrow L^{n/(n-1+\alpha)}(\Omega) \;\Rightarrow\; |Du|^{p-2}Du \in W^{\alpha,n/(n-1+\alpha)}_{\mathrm{loc}}(\Omega).$$

% In the last inclusion, the exponents are irrelevant to γ.
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Selected references

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  Boundary Regularity in Variational Problems
  Arch. Rational Mech. Anal. (2010)

- A. Castro, G. Palatucci.
  Fractional Regularity For Nonlinear Elliptic Problems With Measure Data
  J. Convex Anal. (2013)

- A. L. Baisón, A. Clop, R. Giova, J. Orobitg, A. Passarelli di Napoli.
  Fractional Differentiability for Solutions of Nonlinear Elliptic Equations
  Potential Anal. (2017)

Fractional differentiability result for measure data problems

Theorem 1 (*S.-S. Byun, P. Shin, Y.* 2021 Calc. Var. PDE)

Let u be some distributional solution(so-called *SOLA*) to

$$-\mathrm{div}(a(x)|Du|^{p-2}Du) = \mu, \quad \text{in } \Omega \subset \mathbb{R}^n, \quad n \geq 2, \quad p \in [2, n]$$

where μ is a Radon measure with finite mass. Assume that $a \in C^{\alpha}_{\gamma}(\Omega)$ for some $\alpha \in (0, 1]$ and $\gamma \in [n, \infty]$ satisfying $\alpha\gamma \geq n$, and

$$0 < \nu \leq a(x) \leq L < \infty, \quad {}^{\forall}x \in \Omega.$$

Then for any $\sigma \in (0, \alpha)$ and $q_0 = n/(n - 1 + \alpha)$, we have

$$|Du|^{p-2}Du \in W^{\sigma, q_0}_{\mathrm{loc}}(\Omega).$$

What's new?

	Passarelli di Napoli et al.	Y. et al.
Equation	$-\mathrm{div}(a(x)\|Du\|^{p-2}Du) = -\mathrm{div}(\|F\|^{p-2}F)$	$-\mathrm{div}(a(x)\|Du\|^{p-2}Du) = \mu$
Concept of Solution	Weak solution	SOLA(Very weak solution)
Common Assumption	$\alpha\gamma \geq n$	
Result	$F \in \mathcal{N}^{\beta,p} \quad \Rightarrow$ $\|Du\|^{(p-2)/2}Du \in \mathcal{N}^{\alpha,2}$ $\quad (\alpha < \beta)$	$\|Du\|^{p-2}Du \in W^{\alpha-\varepsilon,\frac{n}{n-1+\alpha}}$
How?	Directly applying the difference quotient	Perturbation argument
Differences In Methods	Direct method is easier to apply but doesn't work for the very weak solutions.	

% See Theorem 1.4 in the paper "Higher differentiability for solutions to a class of obstacle problems" by M. Eleuteri and A. Passarelli di Napoli (2018) Calc. Var. PDEs.

Why $\alpha\gamma \geq n$?

Assume that the following rough inequality holds for some $f \in W^{\alpha,q}$ for any fixed $q \geq 1$ and $\alpha \in (0,1)$:

$$|\tau_h f| \lesssim |h|^{\alpha}|g||f|$$

where $g \in L^{\gamma}$.

To make the inequality have some meaning, the integrability of $|g||f|$ have to be higher than that of $|\tau_h f|/|h|^{\alpha}$.

$$\frac{1}{\gamma} + \frac{1}{q^*(n,\alpha)} \leq \frac{1}{q} \qquad \Rightarrow \qquad n \leq \alpha\gamma.$$

Here, $q^*(n,\alpha) = \frac{nq}{n-\alpha q}$ is the fractional critical exponent.

% From this one can expect further that the coefficient a does not oscillate too much in local like continuous functions.

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Key ingredient to prove main theorem

Let $a \in C^{\alpha}_{\gamma}(\Omega)$ for some $\alpha \in (0,1]$ and $\gamma \in [n,\infty)$ satisfying $\alpha\gamma \geq n$. Then

$$\begin{aligned}
\fint_{B_R} |a(x) - (a)_{B_R}|\, dx &= \fint_{B_R} \left| \fint_{B_R} (a(x) - a(y))\, dy \right| dx \\
&\leq R^{\alpha} \fint_{B_R} \fint_{B_R} (|g(x)| + |g(y)|)\, dy\, dx \\
&\leq c\, R^{\alpha} \left(\fint_{B_R} |g(x)|^{\gamma}\, dx \right)^{1/\gamma} \\
&\leq c\, R^{\alpha - n/\gamma} \|g\|_{L^{\gamma}(B_R)}.
\end{aligned}$$

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Hence, we have

- If $\alpha\gamma = n$, then $a \in \mathrm{VMO}(\Omega)$.
- If $\alpha\gamma > n$, then $a \in C^{\alpha - n/\gamma}(\Omega)$.

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Assuming $0 < \nu \leq a \leq L < \infty$, the regularity results below are well known for any weak solution $w \in W^{1,2}(\Omega)$ to

$$-\mathrm{div}(a(x)|Dw|^{p-2}Dw) = 0, \quad \text{in } \Omega.$$

- If $\alpha\gamma = n$, then $|Dw|^p \in L^q_{\mathrm{loc}}(\Omega)$ for every $q \in [1,\infty)$.
- If $\alpha\gamma > n$, then $|Dw|^{p-2}Dw \in C^{\beta}_{\mathrm{loc}}(\Omega)$ for some $\beta \in (0, \alpha - n/\gamma)$.

Outline of the proof of Theorem 1

- Compare *SOLA* to the weak solution $w \in u + W_0^{1,p}(B_{2R})$ to

$$-\mathrm{div}(a(x)|Dw|^{p-2}Dw) = 0 \quad \text{in } \ B_{2R}.$$

- Recall that $w \in W^{1,q}_{\mathrm{loc}}(B_{2R})$ for every $q \in [p, \infty)$.
- Then we can compare w to the weak solution $v \in w + W_0^{1,p}(B_R)$ to

$$-\mathrm{div}(a(0)|Dv|^{p-2}Dv) = 0 \quad \text{in } \ B_{2R}.$$

- Using $v \in W^{2,2}_{\mathrm{loc}}(B_R)$ and taking R relevant to h, we can employ bootstrap argument to deduce

$$|Du|^{p-2}Du \in W^{\sigma,q_0}_{\mathrm{loc}}(\Omega)$$

for any $\sigma \in (0, \alpha)$ and $q_0 = n/(n-1+\alpha)$.

What happens if we consider homogeneous equations?

We now consider

$$-\mathrm{div}(a(x)|Du|^{p-2}Du) = 0, \qquad \text{in} \quad \Omega \subset \mathbb{R}^n,\, n \geq 2,$$

where $a \in C^{\alpha}_{n/\alpha}$ for some $\alpha \in (0, 1]$.

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If $Du \in L^{\infty}_{\mathrm{loc}}(\Omega)$, then we expect

$$|Du|^{p-2}Du \in W^{\alpha,n/\alpha}_{loc}(\Omega)$$

What is known so far?

- Applying the known results obtained so far to

$$-\mathrm{div}(a(x)|Du|^{p-2}Du) = 0$$

with $a \in C^{\alpha,n/\alpha}$, one has only

$$|Du|^{(p-2)/2}Du \in \mathcal{N}^{\alpha-\varepsilon,2}.$$

Possible future works

1. For
$$-\mathrm{div}(a(x)|Du|^{p-2}Du) = 0$$
with $a \in C^{\alpha}_{\gamma}$ where $\alpha \in (0,1]$ and $\gamma \in (1, n/\alpha)$, the following may hold
$$|Du|^{\delta-1}Du \in W^{\alpha-\varepsilon,1}$$
for some $\delta = p/\gamma'$ and any $\varepsilon \in (0,\alpha)$.

2. Find an optimal assumptions on the variable exponent $p(x)$ to obtain fractional differentiablity results for the variable exponent problem:
$$-\mathrm{div}(|Du|^{p(x)-2}Du) = \mu.$$

Dziękuję Ci.

Thank you for your attention.