# Nonstandard phenomena in the study of double-phase problems

Vicenţiu D. Rădulescu

Department of Mathematics, University of Craiova, Romania
Institute of Mathematics of the Romanian Academy, Bucharest
`radulescu@inf.ucv.ro`
`http://math.ucv.ro/∼radulescu`

Monday's Nonstandard Seminar
Seminar of Section of Differential Equations at MIMUW
Warsaw, 22 February 2021

# Outline of the talk

1. Motivation and pioneers of the field

# Outline of the talk

1. Motivation and pioneers of the field

2. Double phase versus a discontinuity property of the spectrum

# Outline of the talk

1. Motivation and pioneers of the field
2. Double phase versus a discontinuity property of the spectrum
3. Double phase problems with mixed regime

# Outline of the talk

1. Motivation and pioneers of the field
2. Double phase versus a discontinuity property of the spectrum
3. Double phase problems with mixed regime
4. Open problems

# 1. Motivation and pioneers of the field

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ with smooth boundary. If $u : \Omega \to \mathbb{R}^N$ is the displacement and if $Du$ is the $N \times N$ matrix of the deformation gradient, John Ball proved that the total energy can be represented by an integral of the type

$$I(u) = \int_\Omega f(x, Du(x))dx. \tag{1}$$

One of the simplest examples considered by Ball is given by

$$f(\xi) = g(\xi) + h(\det \xi),$$

where $\det \xi$ is the determinant of the $N \times N$ matrix $\xi$, and $g$, $h$ are nonnegative convex functions, which satisfy the growth conditions

$$g(\xi) \geqslant c_1 \, |\xi|^p; \qquad \lim_{t \to +\infty} h(t) = +\infty,$$

where $c_1 > 0$ and $1 < p < N$. The condition $p \leqslant N$ is necessary to study the existence of equilibrium solutions with cavities, that is, minima of the integral (1) that are discontinuous at one point where a cavity forms; in fact, every $u$ with finite energy belongs to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^N)$, hence it is a continuous if $p > N$.

Next, Zhikov intended to provide models for strongly anisotropic materials in the context of homogenisation. In particular, Zhikov considered three different model functionals for this situation in relation to the Lavrentiev phenomenon. These are

$$
\begin{aligned}
\mathcal{M}(u) \quad &:= \int_\Omega c(x)|Du|^2 dx, \quad 0 < 1/c(\cdot) \in L^t(\Omega),\ t > 1\\
\mathcal{V}(u) \quad &:= \int_\Omega |Du|^{p(x)} dx, \quad 1 < p(x) < \infty\\
\mathcal{P}_{p,q}(u) \quad &:= \int_\Omega (|Du|^p + a(x)|Du|^q) dx, \quad 0 \leqslant a(x) \leqslant L,\ 1 < p < q.
\end{aligned}
$$

Next, Zhikov intended to provide models for strongly anisotropic materials in the context of homogenisation. In particular, Zhikov considered three different model functionals for this situation in relation to the Lavrentiev phenomenon. These are

$$\begin{aligned}
\mathcal{M}(u) &:= \int_\Omega c(x)|Du|^2 dx, \quad 0 < 1/c(\cdot) \in L^t(\Omega),\ t > 1\\
\mathcal{V}(u) &:= \int_\Omega |Du|^{p(x)} dx, \quad 1 < p(x) < \infty\\
\mathcal{P}_{p,q}(u) &:= \int_\Omega (|Du|^p + a(x)|Du|^q) dx, \quad 0 \leqslant a(x) \leqslant L,\ 1 < p < q.
\end{aligned}$$

These functionals fall in the realm of the functionals of $(p,q)$–type, according to Marcellini's terminology. These are functionals of the type in (1), where the energy density satisfies

$$|\xi|^p \leqslant f(x,\xi) \leqslant |\xi|^q + 1, \quad 1 \leqslant p \leqslant q.$$

Another model studied by Mingione *et al.* is given by

$$u \mapsto \int_\Omega |Du|^p \log(1 + |Du|) dx, \quad p \geqslant 1,$$

which is a logarithmic perturbation of the $p$-Dirichlet energy.

As shown by Zhikov (1987), the smooth functions are in general **not** dense in $W^{1,p(x)}(\Omega)$.

As shown by Zhikov (1987), the smooth functions are in general **not** dense in $W^{1,p(x)}(\Omega)$. If $p$ is logarithmic Hölder continuous (**notation:** $p \in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$), that is,

$$|p(x) - p(y)| \leqslant \frac{C}{|\log|x-y||} \quad \forall\, x, y \in \overline{\Omega},\ |x-y| \leqslant 1/2,$$

then the smooth functions are dense in $W^{1,p(x)}(\Omega)$ and so the Sobolev space $W_0^{1,p(x)}(\Omega)$ with zero boundary values is the closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|$.

As shown by Zhikov (1987), the smooth functions are in general **not** dense in $W^{1,p(x)}(\Omega)$. If $p$ is logarithmic Hölder continuous (**notation:** $p \in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$), that is,

$$|p(x) - p(y)| \leqslant \frac{C}{|\log|x-y||} \quad \forall x, y \in \overline{\Omega},\ |x-y| \leqslant 1/2,$$

then the smooth functions are dense in $W^{1,p(x)}(\Omega)$ and so the Sobolev space $W_0^{1,p(x)}(\Omega)$ with zero boundary values is the closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|$. Edmunds and Rakosnik (1992) derived the same conclusion under a local monotonicity condition on $p$.

As shown by Zhikov (1987), the smooth functions are in general **not** dense in $W^{1,p(x)}(\Omega)$. If $p$ is logarithmic Hölder continuous (**notation:** $p \in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$), that is,

$$|p(x) - p(y)| \leqslant \frac{C}{|\log |x-y||} \quad \forall\, x, y \in \overline{\Omega},\ |x-y| \leqslant 1/2,$$

then the smooth functions are dense in $W^{1,p(x)}(\Omega)$ and so the Sobolev space $W_0^{1,p(x)}(\Omega)$ with zero boundary values is the closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|$. Edmunds and Rakosnik (1992) derived the same conclusion under a local monotonicity condition on $p$.
Since $\Omega$ is bounded and $p \in C_+(\overline{\Omega})$ is logarithmic Hölder continuous, then

$$|u|_{p(x)} \leqslant C\,|\nabla u|_{p(x)} \quad \forall\, u \in W_0^{1,p(x)}(\Omega) \ \ \text{[Poincaré inequality]},$$

where $C = C(p, |\Omega|, \mathrm{diam}\,(\Omega), N)$. Poincaré's inequality holds under a much weaker assumption on $p$ than the Sobolev inequality and embedding, namely if the exponent $p$ is **not too discontinuous**.

Remarks. 1. If $\Omega$ is **bounded** then

$$C^{0,1}(\overline{\Omega}) \;\subset\; W^{1,q}(\Omega)\;(q>N)\;\subset C^{0,\frac{1}{|\log t|}}(\overline{\Omega}).$$

Remarks. 1. If $\Omega$ is **bounded** then

$$C^{0,1}(\overline{\Omega}) \ \subset \ W^{1,q}(\Omega)\ (q > N) \ \subset C^{0,\frac{1}{|\log t|}}(\overline{\Omega}).$$

2. If $\Omega$ is unbounded, $p$ is said logarithmic Hölder continuous if

$$|p(x) - p(y)| \leqslant \frac{C}{|\log|x-y||} \quad \forall\, x, y \in \overline{\Omega},\ |x-y| \leqslant 1/2$$

and

$$|p(x) - p(y)| \leqslant \frac{C}{\log(e + |x|)\,|} \quad \forall\, x, y \in \Omega,\ |y| \geqslant |x|.$$

Remarks. 1. If $\Omega$ is **bounded** then

$$C^{0,1}(\overline{\Omega}) \;\subset\; W^{1,q}(\Omega)\;(q > N)\;\subset C^{0,\frac{1}{|\log t|}}(\overline{\Omega}).$$

2. If $\Omega$ is unbounded, $p$ is said logarithmic Hölder continuous if

$$|p(x)-p(y)| \leqslant \frac{C}{|\log|x-y||} \quad \forall\, x,y \in \overline{\Omega},\; |x-y| \leqslant 1/2$$

and

$$|p(x)-p(y)| \leqslant \frac{C}{\log(e+|x|)\,|} \quad \forall\, x,y \in \Omega,\; |y| \geqslant |x|.$$

In such a case we cannot require $p \in W^{1,q}(\Omega)$ (since $\int_\Omega |p(x)|^q dx = \infty$).

Let

$$W^{1,(\infty,q(\cdot))}(\Omega) := \{u \in L^\infty(\Omega);\ |\nabla u| \in L^{q(\cdot)}(\Omega)\},$$

where $N < q_- \leqslant q_+ < \infty$.

Let

$$W^{1,(\infty,q(\cdot))}(\Omega) := \{u \in L^{\infty}(\Omega);\ |\nabla u| \in L^{q(\cdot)}(\Omega)\},$$

where $N < q_- \leqslant q_+ < \infty$.

Conclusion. If $\Omega$ is unbounded then the hypotheses
(i) $p \in C^{0,1}(\overline{\Omega})$;
(ii) $p \in W^{1,(\infty,q(\cdot))}(\Omega)$ with $N < q_- \leqslant q_+ < \infty$;
(iii) $p \in C^{0,\frac{1}{|\log t|}}(\overline{\Omega})$
are **independent** each other.

# Features of spaces with variable exponent

The function spaces with variable exponent $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ have some curious properties, for instance:

# Features of spaces with variable exponent

The function spaces with variable exponent $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ have some curious properties, for instance:
(i) If $1 < p^- \leqslant p^+ < \infty$ and $p : \overline{\Omega} \to [1, \infty)$ is smooth, then the formula $\int_\Omega |u(x)|^p dx = p \int_0^\infty t^{p-1} \left|\{x \in \Omega;\ |u(x)| > t\}\right| dt$ has no variable exponent analogue.

# Features of spaces with variable exponent

The function spaces with variable exponent $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ have some curious properties, for instance:
(i) If $1 < p^- \leqslant p^+ < \infty$ and $p : \overline{\Omega} \to [1, \infty)$ is smooth, then the formula $\int_\Omega |u(x)|^p dx = p \int_0^\infty t^{p-1} \left|\{x \in \Omega;\ |u(x)| > t\}\right| dt$ has no variable exponent analogue.
(ii) Variable exponent Lebesgue spaces do not have the “mean continuity property". More precisely, if $p$ is continuous and nonconstant in an open ball $B$, then there exists a function $u \in L^{p(x)}(B)$ such that $u(x+h) \notin L^{p(x)}$ for all $h \in \mathbb{R}^N$ with arbitrary small norm.

# Features of spaces with variable exponent

The function spaces with variable exponent $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ have some curious properties, for instance:
(i) If $1 < p^- \leqslant p^+ < \infty$ and $p : \overline{\Omega} \to [1, \infty)$ is smooth, then the formula $\int_\Omega |u(x)|^p dx = p \int_0^\infty t^{p-1} \left|\{x \in \Omega;\ |u(x)| > t\}\right| dt$ has no variable exponent analogue.
(ii) Variable exponent Lebesgue spaces do not have the "mean continuity property". More precisely, if $p$ is continuous and nonconstant in an open ball $B$, then there exists a function $u \in L^{p(x)}(B)$ such that $u(x+h) \notin L^{p(x)}$ for all $h \in \mathbb{R}^N$ with arbitrary small norm.
(iii) The function spaces with variable exponent are never translation invariant. The use of convolution is also limited, for instance the Young inequality $|f * g|_{p(x)} \leqslant C\, |f|_{p(x)}\, \|g\|_{L^1}$ holds if and only if $p$ is constant.

# Features of spaces with variable exponent

The function spaces with variable exponent $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ have some curious properties, for instance:
(i) If $1 < p^- \leqslant p^+ < \infty$ and $p : \overline{\Omega} \to [1, \infty)$ is smooth, then the formula $\int_\Omega |u(x)|^p dx = p \int_0^\infty t^{p-1} \left|\{x \in \Omega;\ |u(x)| > t\}\right| dt$ has no variable exponent analogue.
(ii) Variable exponent Lebesgue spaces do not have the “mean continuity property". More precisely, if $p$ is continuous and nonconstant in an open ball $B$, then there exists a function $u \in L^{p(x)}(B)$ such that $u(x+h) \notin L^{p(x)}$ for all $h \in \mathbb{R}^N$ with arbitrary small norm.
(iii) The function spaces with variable exponent are never translation invariant. The use of convolution is also limited, for instance the Young inequality $|f * g|_{p(x)} \leqslant C\,|f|_{p(x)}\,\|g\|_{L^1}$ holds if and only if $p$ is constant.
(iv) Generally, the space of smooth functions with compact support is no longer dense in $W^{1,p(x)}(\Omega)$.

## Pioneers in the field of double phase problems

[1] P. Marcellini, *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, 1986.

[2] P. Marcellini, *J. Differential Equations*, 1991.

[3] V. Bögelein, F. Duzaar, P. Marcellini, *Arch. Ration. Mech. Anal.*, 2013.

[4] V. Bögelein, F. Duzaar, P. Marcellini, *J. Math. Pures Appl.*, 2013.

[5] G. Cupini, P. Marcellini, E. Mascolo, *Adv. Differential Equations*, 2014.

[6] G. Cupini, P. Marcellini, E. Mascolo, *Nonlinear Anal.*, 2018.

[7] P. Marcellini, *Nonlinear Anal.*, 2020.

[8] P. Marcellini, *DCDS-S*, 2020.

[1] P. Baroni, M. Colombo, G. Mingione, *Algebra i Analiz*, 2015.

[2] M. Colombo, G. Mingione, *Arch. Ration. Mech. Anal.*, 2015.

[3] M. Colombo, G. Mingione, *J. Funct. Anal.*, 2016.

[4] P. Baroni, M. Colombo, G. Mingione, *Calc. Var. PDE*, 2018.

[5] L. Beck, G. Mingione, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 2019.

[6] L. Beck, G. Mingione, *Comm. Pure Appl. Math.*, 2020.

[7] C. De Filippis, G. Mingione, *St. Petersburg Math. J.*, 2020.

[8] C. De Filippis, G. Mingione, *J. Geom. Anal.*, 2020.

# 2. Double phase versus a discontinuity property of the spectrum

Consider the following nonlinear eigenvalue problem:

$$\left.\begin{cases} -\alpha\Delta_p u(z) - \beta\Delta_q u(z) = \lambda|u(z)|^{q-2}u(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \alpha > 0, \beta > 0, \lambda > 0, 1 < p, q < \infty, p \neq q. \end{cases}\right\} \quad (P_\lambda)$$

## 2. Double phase versus a discontinuity property of the spectrum

Consider the following nonlinear eigenvalue problem:

$$\left.\begin{cases} -\alpha\Delta_p u(z) - \beta\Delta_q u(z) = \lambda|u(z)|^{q-2}u(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \alpha > 0, \beta > 0, \lambda > 0, 1 < p, q < \infty, p \neq q. \end{cases}\right\} \quad (P_\lambda)$$

Particular case: $\alpha = 1 - \beta,\ \beta \in (0,1)$. Let $L_\beta = -(1-\beta)\Delta_p - \beta\Delta_q$ and let $\hat{\sigma}(\beta)$ be the spectrum of $L_\beta$. We obtain that

$$\hat{\sigma}(\beta) = (\beta\hat{\lambda}_1(q), +\infty).$$

## 2. Double phase versus a discontinuity property of the spectrum

Consider the following nonlinear eigenvalue problem:

$$\left\{ \begin{array}{l} -\alpha\Delta_p u(z) - \beta\Delta_q u(z) = \lambda |u(z)|^{q-2} u(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \alpha > 0, \beta > 0, \lambda > 0, 1 < p, q < \infty, p \neq q. \end{array} \right\} \quad (P_\lambda)$$

Particular case: $\alpha = 1 - \beta,\ \beta \in (0,1)$. Let $L_\beta = -(1-\beta)\Delta_p - \beta\Delta_q$ and let $\hat{\sigma}(\beta)$ be the spectrum of $L_\beta$. We obtain that

$$\hat{\sigma}(\beta) = (\beta\hat{\lambda}_1(q), +\infty).$$

The multivalued map $\beta \mapsto \hat{\sigma}(\beta)$ is Hausdorff and Vietoris continuous on $(0,1)$, but at $\beta = 1$, it exhibits a discontinuity since

$$\hat{\sigma}(1) = \text{the spectrum of } (-\Delta_q, W_0^{1,q}(\Omega))$$

and $\hat{\lambda}_1(q) > 0$ is isolated and so $\hat{\sigma}(1) \neq (\hat{\lambda}_1(q), +\infty)$.

## 2. Double phase versus a discontinuity property of the spectrum

Consider the following nonlinear eigenvalue problem:

$$\left\{ \begin{array}{l} -\alpha\Delta_p u(z) - \beta\Delta_q u(z) = \lambda|u(z)|^{q-2}u(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \alpha > 0, \beta > 0, \lambda > 0, 1 < p, q < \infty, p \neq q. \end{array} \right\} \quad (P_\lambda)$$

Particular case: $\alpha = 1 - \beta,\ \beta \in (0, 1)$. Let $L_\beta = -(1 - \beta)\Delta_p - \beta\Delta_q$ and let $\hat{\sigma}(\beta)$ be the spectrum of $L_\beta$. We obtain that

$$\hat{\sigma}(\beta) = (\beta\hat{\lambda}_1(q), +\infty).$$

The multivalued map $\beta \mapsto \hat{\sigma}(\beta)$ is Hausdorff and Vietoris continuous on $(0, 1)$, but at $\beta = 1$, it exhibits a discontinuity since

$$\hat{\sigma}(1) = \text{the spectrum of } (-\Delta_q, W_0^{1,q}(\Omega))$$

and $\hat{\lambda}_1(q) > 0$ is isolated and so $\hat{\sigma}(1) \neq (\hat{\lambda}_1(q), +\infty)$.
This is more emphatically illustrated when $q = 2$. Then

$$\hat{\sigma}(\beta) = (\beta\hat{\lambda}_1(2), +\infty) \text{ for all } \beta \in (0, 1)$$

but at $\beta = 1$, we have $\hat{\sigma}(1) = \{\hat{\lambda}_k(2)\}_{k\geqslant 1}$ (discrete spectrum).

Consider the nonlinear eigenvalue problem

$$-\Delta_r u(z) = \hat{\lambda}|u(z)|^{r-2}u(z) \text{ in } \Omega, \ u|_{\partial\Omega} = 0. \tag{2}$$

Consider the nonlinear eigenvalue problem

$$-\Delta_r u(z) = \hat{\lambda}|u(z)|^{r-2}u(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0. \tag{2}$$

We say that $\hat{\lambda}$ is an eigenvalue of $(-\Delta_r, W_0^{1,r}(\Omega))$ if problem (2) admits a nontrivial solution $\hat{u} \in W_0^{1,r}(\Omega)$, known as an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. Then $\hat{u} \in C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\right\}$ and there is a smallest eigenvalue $\hat{\lambda}_1(r)$ such that:

- $\hat{\lambda}_1(r)$ is isolated (that is, there exists $\epsilon > 0$ such that the interval $(\hat{\lambda}_1(r), \hat{\lambda}_1(r) + \epsilon)$ contains no eigenvalue of $(-\Delta_r, W_0^{1,r}(\Omega))$).
- $\hat{\lambda}_1(r)$ is simple (that is, if $\hat{u}, \hat{v}$ are eigenfunction corresponding to $\hat{\lambda}_1(r)$, then $\hat{u} = \mu\hat{v}$ with $\mu \in \mathbb{R}\backslash\{0\}$).
- $\hat{\lambda}_1(r) > 0$ and admits the following variational characterization

$$\hat{\lambda}_1(r) = \inf\left\{\frac{||Du||_r^r}{||u||_r^r} : u \in W_0^{1,r}(\Omega), u \neq 0\right\} \tag{3}$$

Let $r = \max\{p, q\}$ and $\lambda > 0$. The energy (Euler) functional for problem $(P_\lambda)$ is defined by

$$\varphi_\lambda(u) = \frac{\alpha}{p}||Du||_p^p + \frac{\beta}{q}||Du||_q^q - \frac{\lambda}{q}||u||_q^q \text{ for all } u \in W_0^{1,r}(\Omega).$$

Let $r = \max\{p, q\}$ and $\lambda > 0$. The energy (Euler) functional for problem $(P_\lambda)$ is defined by

$$\varphi_\lambda(u) = \frac{\alpha}{p}||Du||_p^p + \frac{\beta}{q}||Du||_q^q - \frac{\lambda}{q}||u||_q^q \text{ for all } u \in W_0^{1,r}(\Omega).$$

The Nehari manifold for the functional $\varphi_\lambda$ is the set

$$N_\lambda = \{u \in W_0^{1,r}(\Omega) : \langle \varphi'_\lambda(u), u \rangle = 0, u \neq 0\}.$$

Let $r = \max\{p, q\}$ and $\lambda > 0$. The energy (Euler) functional for problem $(P_\lambda)$ is defined by

$$\varphi_\lambda(u) = \frac{\alpha}{p}||Du||_p^p + \frac{\beta}{q}||Du||_q^q - \frac{\lambda}{q}||u||_q^q \text{ for all } u \in W_0^{1,r}(\Omega).$$

The Nehari manifold for the functional $\varphi_\lambda$ is the set

$$N_\lambda = \{u \in W_0^{1,r}(\Omega) : \left\langle \varphi_\lambda'(u), u \right\rangle = 0, u \neq 0\}.$$

We denote by $\hat{\sigma}(\alpha, \beta)$ the spectrum of

$$u \to -\alpha\Delta_p u - \beta\Delta_q u \text{ for all } u \in W_0^{1,r}(\Omega).$$

So, $\lambda \in \hat{\sigma}(\alpha, \beta)$ if and only if problem $(P_\lambda)$ has a nontrivial solution $\hat{u} \in C_0^1(\overline{\Omega})$. This solution is an eigenvector for the eigenvalue $\lambda$.

**Theorem (Papageorgiou, R., Repovš).** *If* $\lambda > \beta\hat{\lambda}_1(q)$ *then* $\lambda$ *is an eigenvalue of problem* $(P_\lambda)$ *with eigenfunction* $\hat{\lambda} \in C_0^1(\overline{\Omega})$.

**Theorem (Papageorgiou, R., Repovš).** *If* $\lambda > \beta\hat{\lambda}_1(q)$ *then* $\lambda$ *is an eigenvalue of problem* $(P_\lambda)$ *with eigenfunction* $\hat{\lambda} \in C_0^1(\overline{\Omega})$.

*Case 1 (easy)*: $1 < q < p$. Then $\varphi_\lambda(\cdot)$ is coercive and we use the direct method of the calculus of variations.

**Theorem (Papageorgiou, R., Repovš).** *If* $\lambda > \beta\hat{\lambda}_1(q)$ *then* $\lambda$ *is an eigenvalue of problem* $(P_\lambda)$ *with eigenfunction* $\hat{\lambda} \in C_0^1(\overline{\Omega})$.

*Case 1 (easy)*: $1 < q < p$. Then $\varphi_\lambda(\cdot)$ is coercive and we use the direct method of the calculus of variations.

*Case 2*: $1 < p < q$. Then the energy functional is no longer coercive. We minimize $\varphi_\lambda$ on the Nehari manifold $N_\lambda$.

**Theorem (Papageorgiou, R., Repovš).** *If* $\lambda > \beta\hat{\lambda}_1(q)$ *then* $\lambda$ *is an eigenvalue of problem* $(P_\lambda)$ *with eigenfunction* $\hat{\lambda} \in C_0^1(\overline{\Omega})$.

*Case 1 (easy)*: $1 < q < p$. Then $\varphi_\lambda(\cdot)$ is coercive and we use the direct method of the calculus of variations.

*Case 2*: $1 < p < q$. Then the energy functional is no longer coercive. We minimize $\varphi_\lambda$ on the Nehari manifold $N_\lambda$.

**Lemma 1.** $\lambda > \beta\hat{\lambda}_1(q)$ *if and only if* $N_\lambda \neq \emptyset$.

**Theorem (Papageorgiou, R., Repovš).** *If $\lambda > \beta\hat{\lambda}_1(q)$ then $\lambda$ is an eigenvalue of problem $(P_\lambda)$ with eigenfunction $\hat{\lambda} \in C_0^1(\overline{\Omega})$.*

*Case 1 (easy)*: $1 < q < p$. Then $\varphi_\lambda(\cdot)$ is coercive and we use the direct method of the calculus of variations.

*Case 2*: $1 < p < q$. Then the energy functional is no longer coercive. We minimize $\varphi_\lambda$ on the Nehari manifold $N_\lambda$.

**Lemma 1.** *$\lambda > \beta\hat{\lambda}_1(q)$ if and only if $N_\lambda \neq \emptyset$.*

We define

$$m_\lambda = \inf\{\varphi_\lambda(u) : u \in N_\lambda\}. \tag{4}$$

For $u \in N_\lambda$, we have

$$\alpha||Du||_p^p + \beta||Du||_q^q = \lambda||u||_q^q. \tag{5}$$

Therefore

$$\begin{aligned} \varphi_\lambda(u) &= \frac{\alpha}{p}||Du||_p^p + \frac{\beta}{q}||Du||_q^q - \frac{1}{q}[\alpha||Du||_p^p + \beta||Du||_q^q] \\ &= \alpha\left[\frac{1}{p} - \frac{1}{q}\right]||Du||_p^p \Rightarrow\ m_\lambda \geqslant 0. \end{aligned} \tag{6}$$

From (6) we infer that $\varphi_\lambda|_{N_\lambda}$ is coercive on $W_0^{1,p}(\Omega)$.

From (6) we infer that $\varphi_\lambda|_{N_\lambda}$ is coercive on $W_0^{1,p}(\Omega)$.

**Lemma 2.** *If $\lambda > \beta\hat{\lambda}_1(q)$, then every minimizing sequence of (4) is bounded in $W_0^{1,q}(\Omega)$.*

From (6) we infer that $\varphi_\lambda|_{N_\lambda}$ is coercive on $W_0^{1,p}(\Omega)$.

**Lemma 2.** *If* $\lambda > \beta\hat{\lambda}_1(q)$*, then every minimizing sequence of (4) is bounded in* $W_0^{1,q}(\Omega)$.

**Lemma 3.** *If* $\lambda > \beta\hat{\lambda}_1(q)$*, then* $m_\lambda > 0$.

From (6) we infer that $\varphi_\lambda|_{N_\lambda}$ is coercive on $W_0^{1,p}(\Omega)$.

**Lemma 2.** *If $\lambda > \beta\hat{\lambda}_1(q)$, then every minimizing sequence of (4) is bounded in $W_0^{1,q}(\Omega)$.*

**Lemma 3.** *If $\lambda > \beta\hat{\lambda}_1(q)$, then $m_\lambda > 0$.*

**Lemma 4.** *If $\lambda > \beta\hat{\lambda}_1(q)$, then there exists $\hat{u}_\lambda \in N_\lambda$ such that $m_\lambda = \varphi_\lambda(\hat{u}_\lambda)$.*

## 3. Double phase problems with mixed regime

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded regular connected open set and assume that $p, q \in (1, \infty)$. Consider the Lane-Emden problem

$$\begin{cases} -\Delta_p u = |u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \not\equiv 0 & \text{in } \Omega. \end{cases} \tag{7}$$

## 3. Double phase problems with mixed regime

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded regular connected open set and assume that $p, q \in (1, \infty)$. Consider the Lane-Emden problem

$$\begin{cases} -\Delta_p u = |u|^{q-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \not\equiv 0 & \text{in } \Omega. \end{cases} \tag{7}$$

Usually, this analysis is developed in relationship with the values of $q$ with respect to the Sobolev critical exponent $p^*$ of $p$, which is defined by

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } 1 < p < N \\ +\infty & \text{if } p \geqslant N. \end{cases}$$

The following three basic situations can occur:

(i) $q < p^*$ (*subcritical case*). Then the associated energy functional is either coercive (if $q < p$) or has a mountain pass geometry and satisfies the Palais-Smale condition (if $q > p$), hence problem (7) has at least one solution. The case $p = q$ corresponds to an eigenvalue problem, so we cannot exclude a nonexistence property.

The following three basic situations can occur:

(i) $q < p^*$ (*subcritical case*). Then the associated energy functional is either coercive (if $q < p$) or has a mountain pass geometry and satisfies the Palais-Smale condition (if $q > p$), hence problem (7) has at least one solution. The case $p = q$ corresponds to an eigenvalue problem, so we cannot exclude a nonexistence property.

(ii) $q = p^*$, provided that $1 < p < N$ (*critical case*). In this case, the topology of $\Omega$ plays a crucial role. In particular, if $p = 2$, $N = 3$, $q = 6$ and $\Omega$ is not contractible, then problem (7) has at least one positive solution.

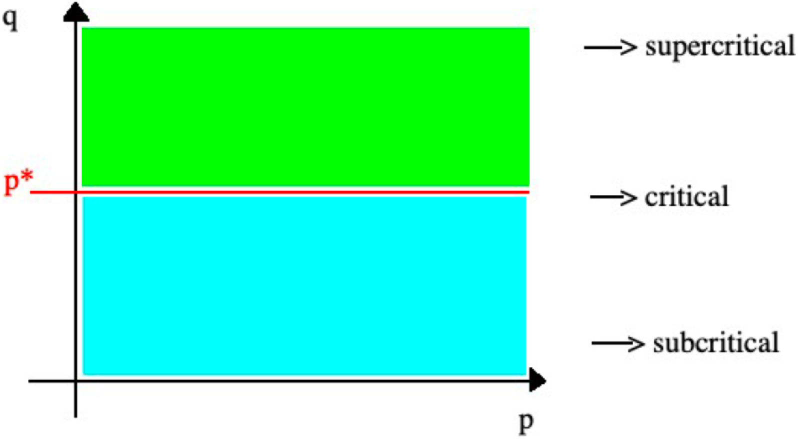
The following three basic situations can occur:

(i) $q < p^*$ (*subcritical case*). Then the associated energy functional is either coercive (if $q < p$) or has a mountain pass geometry and satisfies the Palais-Smale condition (if $q > p$), hence problem (7) has at least one solution. The case $p = q$ corresponds to an eigenvalue problem, so we cannot exclude a nonexistence property.

(ii) $q = p^*$, provided that $1 < p < N$ (*critical case*). In this case, the topology of $\Omega$ plays a crucial role. In particular, if $p = 2$, $N = 3$, $q = 6$ and $\Omega$ is not contractible, then problem (7) has at least one positive solution.

(iii) $q > p^*$, provided that $1 < p < N$ (*supercritical case*). This situation is delicate and a major role is played by the geometry of $\Omega$. For instance, if $\Omega$ is starshaped then problem (7) does not have any solution (by Pohozaev's identity). Also, if $\Omega$ is an annulus, problem (7) always has at least one solution.

# Isotropic case: $p$ and $q$ are constant



In the case of variable exponents, the Lane-Emden problem (7) becomes

$$\begin{cases} -\Delta_{p(x)} u = |u|^{q(x)-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \not\equiv 0 & \text{in } \Omega, \end{cases} \tag{8}$$

where $\Delta_{p(x)} u := \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right)$.

In the case of variable exponents, the Lane-Emden problem (7) becomes

$$\begin{cases} -\Delta_{p(x)} u = |u|^{q(x)-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \not\equiv 0 & \text{in } \Omega, \end{cases} \tag{8}$$

where $\Delta_{p(x)} u := \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right)$.
In this case, the critical exponent of $p(x)$ depends on the point and it is defined by

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{if } 1 < p(x) < N \\ +\infty & \text{if } p(x) \geqslant N. \end{cases}$$

An example in the subcritical setting. Consider the problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda \left|u\right|^{q(x)-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \not\equiv 0 & \text{in } \Omega \end{cases} \tag{9}$$

under the following hypotheses:
(h1) $1 < \min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x) < \max_{x\in\overline{\Omega}} q(x)$;
(h2) $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

An example in the subcritical setting. Consider the problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda\, |u|^{q(x)-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \not\equiv 0 & \text{in } \Omega \end{cases} \tag{9}$$

under the following hypotheses:
(h1) $1 < \min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x) < \max_{x\in\overline{\Omega}} q(x)$;
(h2) $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

*Case of small perturbations*: there exists $\lambda^* > 0$ such that problem (9) has at least one solution for all $\lambda \in (0, \lambda^*)$.

An example in the subcritical setting. Consider the problem

$$\begin{cases} -\Delta_{p(x)} u = \lambda \, |u|^{q(x)-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \not\equiv 0 & \text{in } \Omega \end{cases} \tag{9}$$

under the following hypotheses:

(h1) $1 < \min_{x \in \overline{\Omega}} q(x) < \min_{x \in \overline{\Omega}} p(x) < \max_{x \in \overline{\Omega}} q(x)$;

(h2) $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

*Case of small perturbations*: there exists $\lambda^* > 0$ such that problem (9) has at least one solution for all $\lambda \in (0, \lambda^*)$.

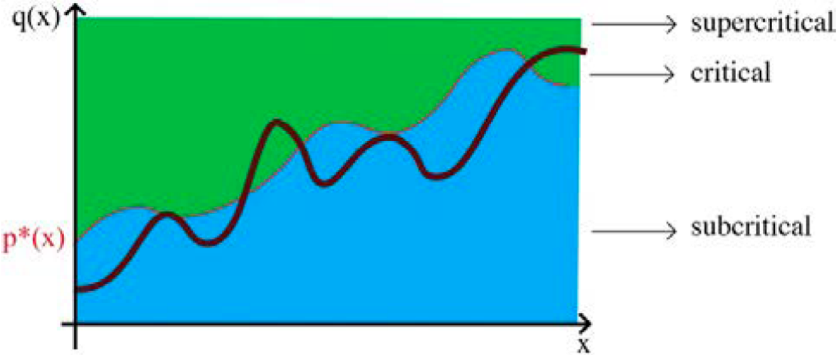
Problem (8) can fulfill even a “subcritical-critical-supercritical" triple regime, in the sense that $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ and

$$q(x) < p^*(x) \quad \text{if } x \in \Omega_1;$$

$$q(x) = p^*(x) \quad \text{if } x \in \Omega_2;$$

$$q(x) > p^*(x) \quad \text{if } x \in \Omega_3.$$

# Anisotropic case: $p$ and $q$ are variable



Problem 1: the radial case.

Let $p, q, m, a : \overline{B}_R(0) \to \mathbb{R}$ be continuous functions satisfying :

$$\begin{cases} 1 < p_- = \min_{x\in\overline{B}_R(0)} p(x) \leqslant \max_{x\in\overline{B}_R(0)} p(x) = p_+ < N. \\ 1 < m_- = \min_{x\in\overline{B}_R(0)} m(x) \leqslant \max_{x\in\overline{B}_R(0)} m(x) = m_+ < N. \end{cases} \tag{H1}$$

$$0 \leqslant a(x) \leqslant L, \quad \forall x \in \overline{B}_R(0). \tag{H2}$$

$$p(x) = p(|x|), a(x) = a(|x|), \quad q(x) = q(|x|) \quad \forall x \in \overline{B}_R(0). \tag{H3}$$

Problem 1: the radial case.

Let $p, q, m, a : \overline{B}_R(0) \to \mathbb{R}$ be continuous functions satisfying :

$$\begin{cases} 1 < p_- = \min_{x\in\overline{B}_R(0)} p(x) \leqslant \max_{x\in\overline{B}_R(0)} p(x) = p_+ < N. \\ 1 < m_- = \min_{x\in\overline{B}_R(0)} m(x) \leqslant \max_{x\in\overline{B}_R(0)} m(x) = m_+ < N. \end{cases} \tag{H1}$$

$$0 \leqslant a(x) \leqslant L, \quad \forall x \in \overline{B}_R(0). \tag{H2}$$

$$p(x) = p(|x|), a(x) = a(|x|), \quad q(x) = q(|x|) \quad \forall x \in \overline{B}_R(0). \tag{H3}$$

Assume that there exists $0 < r < R$ such that

$$q \geqslant 0 \text{ in } \Omega \text{ and } p_+ < q^r_- = \min_{x\in\overline{B}_r(0)} q(x) \leqslant \max_{x\in\overline{B}_r(0)} q(x) = q^r_+ < \min_{x\in\overline{\Omega}} p^*(x). \tag{H4}$$

Note that $q$ is subcritical in $\overline{B}_r(0)$, but there is no hypotheses on the function $q$ in the annulus $A_{R,r} = \overline{B}_R(0) \setminus B_r(0)$, hence $q$ can have a supercritical growth close to the boundary. However, note that for any $t \in (0, R)$ we have the continuous embedding

$$W^{1,p(x)}(B_R(0)) \hookrightarrow W^{1,p_-}(A_{R,t})$$

and the compact embedding (Strauss)

$$W^{1,p_-}_{rad}(A_{R,t}) \hookrightarrow C(\overline{A}_{R,t}).$$

Therefore the embedding

$$W^{1,p(x)}_{rad}(B_R(0)) \hookrightarrow C(\overline{A}_{R,t}), \tag{10}$$

is compact, where

$$W^{1,p(x)}_{rad}(B_R(0)) = \{u \in W^{1,p(x)}(B_R(0)) :\ u(x) = u(|x|) \quad \text{a.e. in}\, B_R(0)\}.$$

It follows that the embedding

$$W^{1,p(x)}_{rad}(B_R(0)) \hookrightarrow L^{q(x)}(B_R(0)), \tag{11}$$

is also compact.

Denote

$$\Delta_{m(x),a(x)}u = \operatorname{div}\left(a(x)|\nabla u|^{m(x)-2}\nabla u\right).$$

If $a \neq 0$, we set

$$E = W_0^{1,p(x)}(B_R(0)) \cap W_{a(x),0}^{1,m(x)}(B_R(0)),$$

where $W_{a(x),0}^{1,m(x)}(B_R(0))$ is the space $W_0^{1,m(x)}(B_R(0))$ endowed with the norm

$$\|\nabla u\|_{m(x),a(x)} = \inf\left\{\lambda > 0 \,\middle|\, \int_{\mathbb{R}^N} a(x)\left|\frac{|\nabla u|}{\lambda}\right|^{m(x)} dx \leqslant 1\right\}.$$

Hereafter, we endow $E$ with the norm

$$\|u\| = \|\nabla u\|_{p(x)} + \|\nabla u\|_{m(x),a(x)}.$$

We observe that if $a = 0$, then $E = W_0^{1,p(x)}(B_R(0))$ and $\| \quad \|$ is exactly the usual norm in $W_0^{1,p(x)}(B_R(0))$.

From the definition of $E$, we have the continuous embedding

$$E \hookrightarrow W_0^{1,p(x)}(B_R(0)).$$

This fact combined with (11) implies that the embedding

$$E_{rad}(B_R(0)) \hookrightarrow L^{q(x)}(B_R(0)), \tag{12}$$

is also compact, where

$$E_{rad} = W_{rad,0}^{1,p(x)}(B_R(0)) \cap W_{rad,0}^{1,m(x)}(B_R(0)).$$

### Theorem

*Assume that conditions $(H1)-(H4)$ are fulfilled. Then the following nonhomogeneous boundary value problem*

$$\begin{cases} -\Delta_{p(x)}u - \Delta_{m(x),a(x)}u = |u|^{q(x)-2}u \quad in \quad B_R, \\ u = 0 \quad on \quad \partial B_R \end{cases} \tag{$P_1$}$$

*has a nontrivial solution in E.*

Theorem

*Assume that conditions $(H1)-(H4)$ are fulfilled. Then the following nonhomogeneous boundary value problem*

$$\begin{cases} -\Delta_{p(x)}u - \Delta_{m(x),a(x)}u = |u|^{q(x)-2}u \quad in \quad B_R, \\ u = 0 \quad on \quad \partial B_R \end{cases} \tag{$P_1$}$$

*has a nontrivial solution in E.*

Palais' principle of symmetric criticality, 1979:

Theorem

*Assume that conditions* $(H1)-(H4)$ *are fulfilled. Then the following nonhomogeneous boundary value problem*

$$\begin{cases} -\Delta_{p(x)}u-\Delta_{m(x),a(x)}u=|u|^{q(x)-2}u \quad in \quad B_R, \\ u=0 \quad on \quad \partial B_R \end{cases} \qquad (P_1)$$

*has a nontrivial solution in E.*

Palais' principle of symmetric criticality, 1979:

Critical symmetric points are symmetric critical points.

Theorem

*Assume that conditions* $(H1)-(H4)$ *are fulfilled. Then the following nonhomogeneous boundary value problem*

$$\begin{cases} -\Delta_{p(x)}u-\Delta_{m(x),a(x)}u=|u|^{q(x)-2}u \quad in \quad B_R, \\ u=0 \quad on \quad \partial B_R \end{cases} \qquad (P_1)$$

*has a nontrivial solution in* $E$.

Palais' principle of symmetric criticality, 1979:

Critical symmetric points are symmetric critical points.

Let $X$ be a Banach space on which a symmetry group $G$ linearly acts and let $J$ be a $G$-invariant functional defined on $X$. Then every critical point of $J$ restricted on the subspace of symmetric points becomes also a critical point of $J$ on the whole space $X$.

## *Proof.*

Let

$$I(u) = \int_{B_R} \left( \frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{m(x)} |\nabla u|^{m(x)} \right) dx - \int_{B_R} \frac{1}{q(x)} |u|^{q(x)} \, dx.$$

## *Proof.*

Let

$$I(u) = \int_{B_R} \left( \frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{m(x)} |\nabla u|^{m(x)} \right) dx - \int_{B_R} \frac{1}{q(x)} |u|^{q(x)}\, dx.$$

This functional is not well defined on the whole space $E$ because we do not assume any growth condition on $q$ in the annulus $A_{R,r}$. In the sequel we will restrict $I$ to $E_{rad}$, because $I \in C^1(E_{rad}, \mathbb{R})$ and

$$I'(u)v = \int_{B_R} (|\nabla u|^{p(x)-2} \nabla u \nabla v + a(x) |\nabla u|^{m(x)-2} \nabla u \nabla v)\, dx - \int_{B_R} |u|^{q(x)-2} uv\, dx.$$

Then $I$ satisfies the mountain pass geometry and also the $(PS)$ condition, because we have the compact embedding (11). Thus, we find a nontrivial critical point $u \in E_{rad}$.

Then $I$ satisfies the mountain pass geometry and also the $(PS)$ condition, because we have the compact embedding (11). Thus, we find a nontrivial critical point $u \in E_{rad}$.
Our goal is to prove that $u$ is in fact a critical point of $I$ in the whole space $E$. However, we cannot applied directly the Palais principle of symmetric criticality, because $I$ is not well defined in whole $E$. In order to overcome this difficulty, we will use the following trick: consider the function

$$g(x,t) = \xi(|x|)|t|^{q(x)} + (1-\xi(|x|))|u(x)|^{q(x)}, \quad \forall x \in B_R,$$

where $\xi \in C^\infty([0,R],\mathbb{R})$ satisfies

$$\xi(x) = \begin{cases} 1, & x \in \overline{B}_{\frac{r}{2}}(0) \\ 0, & x \in \overline{B}_R(0) \setminus \overline{B}_{\frac{3r}{5}}(0). \end{cases}$$

Then $I$ satisfies the mountain pass geometry and also the $(PS)$ condition, because we have the compact embedding (11). Thus, we find a nontrivial critical point $u \in E_{rad}$.
Our goal is to prove that $u$ is in fact a critical point of $I$ in the whole space $E$. However, we cannot applied directly the Palais principle of symmetric criticality, because $I$ is not well defined in whole $E$. In order to overcome this difficulty, we will use the following trick: consider the function

$$g(x,t) = \xi(|x|)|t|^{q(x)} + (1-\xi(|x|))|u(x)|^{q(x)}, \quad \forall x \in B_R,$$

where $\xi \in C^\infty([0,R],\mathbb{R})$ satisfies

$$\xi(x) = \begin{cases} 1, & x \in \overline{B}_{\frac{r}{2}}(0) \\ 0, & x \in \overline{B}_R(0) \setminus \overline{B}_{\frac{3r}{5}}(0). \end{cases}$$

Since $u \in C(\overline{A}_{R,\frac{r}{2}})$, it follows from $(H4)$ that

$$|g(x,t)| \leqslant C(|t|^{q^r_+} + 1), \quad \forall (x,t) \in B_R \times \mathbb{R}.$$

This fact implies that $g$ has a subcritical growth.

Consider the nonlinear problem

$$\begin{cases} -\Delta_{p(x)} w - \Delta_{m(x),a(x)} w = g(x,w) \quad \text{in} \quad B_R, \\ w = 0 \quad \text{on} \quad \partial B_R, \end{cases} \tag{$P_g$}$$

whose associated energy is given by

$$J(w) = \int_{B_R} \left( \frac{1}{p(x)} |\nabla w|^{p(x)} + \frac{a(x)}{m(x)} |\nabla w|^{m(x)} \right) dx - \int_{B_R} G(x,w)\, dx,$$

where $G(x,t) = \int_0^t g(x,s)\, ds$.

Since $g$ is subcritical, it follows that $J$ is well defined in the whole space $E$, $J \in C^1(E, \mathbb{R})$ and

$$\begin{aligned} J'(u)v = & \int_{B_R} (|\nabla w|^{p(x)-2} \nabla w \nabla v + a(x)|\nabla w|^{m(x)-2} \nabla w \nabla v)\, dx \\ & - \int_{B_R} g(x, w) v\, dx, \quad \forall u, v \in E. \end{aligned}$$

Since

$$g(x, u(x)) = |u|^{q(x)-2} u(x), \quad \forall x \in B_R,$$

we see that $u$ is a critical point of $J$ restricted to $E_{rad}$. Now we can apply the Palais principle of symmetric criticality to conclude that $u$ is a nontrivial critical point of $J$ in the whole $E$.

Problem 2: the non-radial case.

Consider the problem:

$$\begin{cases} -\Delta_{p(x)}u - \Delta_{m(x),a(x)}u = |u|^{q(x)-2}u \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{$P_2$}$$

Problem 2: the non-radial case.

Consider the problem:

$$\begin{cases} -\Delta_{p(x)}u - \Delta_{m(x),a(x)}u = |u|^{q(x)-2}u \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{$P_2$}$$

We assume that there exist positive numbers $r < R$ such that $B_R \subset \Omega$ and $a(x) = a_0$ for all $x \in A_{R,r}$.

## Problem 2: the non-radial case.

Consider the problem:

$$\begin{cases} -\Delta_{p(x)}u - \Delta_{m(x),a(x)}u = |u|^{q(x)-2}u \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{$P_2$}$$

We assume that there exist positive numbers $r < R$ such that $B_R \subset \Omega$ and $a(x) = a_0$ for all $x \in A_{R,r}$.

Assume that $p, q, m, a : \overline{\Omega} \to \mathbb{R}$ are continuous and

$$\begin{cases} 1 < p_- = \min_{x\in\overline{\Omega}} p(x) \leqslant \max_{x\in\overline{\Omega}} p(x) = p_+ < N, \\ 1 < m_- = \min_{x\in\overline{\Omega}} m(x) \leqslant \max_{x\in\overline{\Omega}} m(x) = m_+ < N. \end{cases} \tag{$H5$}$$

$$0 \leqslant a(x) \leqslant L, \quad \forall x \in \overline{\Omega} \tag{$H6$}$$

$$p(x) = p(|x|) \quad \text{and} \quad q(x) = q(|x|), \quad \forall x \in \overline{A}_{R,r}. \tag{$H7$}$$

Assume that the variable exponent $q$ satisfies

$$q \geqslant 0 \text{ in } \overline{\Omega} \text{ and } p_+ < q_-^A = \min_{x\in\overline{\Omega}\setminus A_{R,r}} q(x) = q_+^A \leqslant \max_{x\in\overline{\Omega}\setminus A_{R,r}} q(x) < \min_{x\in\overline{\Omega}} p^*(x). \tag{H8}$$

Important: we do not assume any growth condition on $q$ in the annulus $A_{R,r}$, hence $q$ can have a supercritical growth in that region.

Assume that the variable exponent $q$ satisfies

$$q \geqslant 0 \text{ in } \overline{\Omega} \text{ and } p_+ < q_-^A = \min_{x\in\overline{\Omega}\setminus A_{R,r}} q(x) = q_+^A \leqslant \max_{x\in\overline{\Omega}\setminus A_{R,r}} q(x) < \min_{x\in\overline{\Omega}} p^*(x). \tag{H8}$$

Important: we do not assume any growth condition on $q$ in the annulus $A_{R,r}$, hence $q$ can have a supercritical growth in that region.

**Theorem**

*Assume that hypotheses $(H5) - (H8)$ are fulfilled. Then problem $(P_2)$ has a nontrivial solution in $E$.*

## *Sketch of the proof.*

1. The energy associated to problem $(P_2)$ is

$$I(u) = \int_\Omega \left( \frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{m(x)} |\nabla u|^{m(x)} \right) dx - \int_\Omega \frac{1}{q(x)} |u|^{q(x)} \, dx.$$

## *Sketch of the proof.*

1. The energy associated to problem $(P_2)$ is

$$I(u) = \int_\Omega \left( \frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{a(x)}{m(x)} |\nabla u|^{m(x)} \right) dx - \int_\Omega \frac{1}{q(x)} |u|^{q(x)} \, dx.$$

Since we do not assume any growth condition on $q$ in the annulus $A_{R,r}$ $I$ is not well defined on the whole $E$.

2. We restrict $I$ to the closed subspace $X \subset E$ given by

$$X = \{u \in E \,:\, u(x) = u(|x|) \quad \text{a.e.} \quad x \in \overline{A}_{R,r}\}.$$

3. By the mountain pass theorem, there is a nontrivial critical point $u_0 \in X$ of $I$.

4. Next, we show that $u_0$ is, in fact, a critical point of $I$.

4. Next, we show that $u_0$ is, in fact, a critical point of $I$. For this purpose we cannot use the Palais principle, because $\Omega$ is not a ball. Here, the trick is the following: for all $\varphi \in X_0(A_{R,r}) = \{u \in X \,:\, u = 0 \text{ on } \partial(A_{R,r})\}$ we have

$$\int_{A_{R,r}} (|\nabla u_0|^{p(x)-2}\nabla u_0 \nabla\varphi + a(x)|\nabla u_0|^{m(x)-2}\nabla u_0 \nabla\varphi)\, dx - \\ \int_{A_{R,r}} |u_0|^{q(x)-2} u_0 \varphi \, dx = 0.$$

4. Next, we show that $u_0$ is, in fact, a critical point of $I$. For this purpose we cannot use the Palais principle, because $\Omega$ is not a ball. Here, the trick is the following: for all $\varphi \in X_0(A_{R,r}) = \{u \in X \,:\, u = 0 \text{ on } \partial(A_{R,r})\}$ we have

$$\int_{A_{R,r}} (|\nabla u_0|^{p(x)-2}\nabla u_0 \nabla\varphi + a(x)|\nabla u_0|^{m(x)-2}\nabla u_0\nabla\varphi)\,dx - \int_{A_{R,r}} |u_0|^{q(x)-2}u_0\varphi\,dx = 0.$$

5. Finally, by using cut-off functions and density arguments, we conclude that $u_0$ is a nontrivial solution.

Problem 3: The case where $q$ vanishes close to the boundary.

Consider the problems

$$\begin{cases} -\Delta_{p(x)}u - \Delta_{m(x),a(x)}u = \lambda|u|^{q(x)-2}u \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{$P_3$}$$

Problem 3: The case where $q$ vanishes close to the boundary.

Consider the problems

$$\begin{cases} -\Delta_{p(x)}u - \Delta_{m(x),a(x)}u = \lambda|u|^{q(x)-2}u \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{$P_3$}$$

Assume that there exist positive numbers $r < R$ such that $B_R(0) \subset \Omega$,

$$A_{R,r} \subset \Omega_\delta \quad \text{and} \quad a(x) = a_0 \quad \forall x \in A_{R,r},$$

where

$$\Omega_\delta = \{x \in \Omega \; : \; \text{dist}\,(x, \partial\Omega) > \delta\}.$$

Assume that $p, q, m, a : \overline{\Omega} \to \mathbb{R}$ are continuous and satisfy

$$\begin{cases} 1 < p_- = \min_{x\in\overline{\Omega}} p(x) \leqslant \max_{x\in\overline{\Omega}} p(x) = p_+ < N, \\ 1 < m_- = \min_{x\in\overline{\Omega}} m(x) \leqslant \max_{x\in\overline{\Omega}} m(x) = m_+ < N. \end{cases} \tag{$H9$}$$

$$\max\{p_+, m_+\} < q_-^A = \min_{x\in\overline{\Omega}_\delta \setminus A_{R,r}} q(x) \leqslant q_+^A = \max_{x\in\overline{\Omega}_\delta \setminus A_{R,r}} q(x) < \min_{x\in\overline{\Omega}} p^*(x). \tag{$H10$}$$

$$0 \leqslant a(x) \leqslant L, \quad \forall x \in \overline{\Omega} \tag{$H11$}$$

$$q(x) \geqslant 0 \quad \forall x \in \overline{\Omega} \quad \text{and} \quad \lim_{\text{dist}\,(x,\partial\Omega)\to 0} q(x) = 0. \tag{$H12$}$$

### Theorem

*Assume that hypotheses* $(H9)-(H12)$ *are fulfilled. Then there exists* $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*)$ *problem* $(P_3)$ *has at least two nontrivial solutions in* $E$.

### Theorem

*Assume that hypotheses* $(H9)-(H12)$ *are fulfilled. Then there exists* $\lambda^*>0$ *such that for all* $\lambda\in(0,\lambda^*)$ *problem* $(P_3)$ *has at least two nontrivial solutions in* $E$.

*Sketch of the proof.* The associated energy functional is

$$I(u)=\int_\Omega\left(\frac{1}{p(x)}|\nabla u|^{p(x)}+\frac{a(x)}{m(x)}|\nabla u|^{m(x)}\right)\,dx-\int_\Omega\frac{\lambda}{q(x)}|u|^{q(x)}\,dx.$$

### Theorem

*Assume that hypotheses* $(H9)-(H12)$ *are fulfilled. Then there exists* $\lambda^*>0$ *such that for all* $\lambda\in(0,\lambda^*)$ *problem* $(P_3)$ *has at least two nontrivial solutions in* $E$*.*

*Sketch of the proof.* The associated energy functional is

$$I(u)=\int_\Omega\left(\frac{1}{p(x)}|\nabla u|^{p(x)}+\frac{a(x)}{m(x)}|\nabla u|^{m(x)}\right)dx-\int_\Omega\frac{\lambda}{q(x)}|u|^{q(x)}\,dx.$$

Again, $I$ is not well defined in the whole space $E$. That is why we restrict the functional $I$ to the closed subspace $X\subset E$ given by

$$X=\{u\in E\,:\,u(x)=u(|x|)\text{ a.e. }x\in\overline{A}_{R,r}\}.$$

Then $I\in C^1(X,\mathbb{R})$ and $I$ satisfies the $(PS)$ condition in $X$.

## Lemma

*Given* $\tau > 0$*, there are* $\rho = \rho(\tau) > 0$ *and* $\lambda^* = \lambda^*(\tau)$ *such that*

$$I_\lambda(u) \geqslant \rho \quad \text{for} \quad \|u\| = \tau \quad \text{and} \quad \lambda \in (0, \lambda^*).$$

**Lemma**

*Given* $\tau > 0$*, there are* $\rho = \rho(\tau) > 0$ *and* $\lambda^* = \lambda^*(\tau)$ *such that*

$$I_\lambda(u) \geqslant \rho \quad \textit{for} \quad \|u\| = \tau \quad \textit{and} \quad \lambda \in (0, \lambda^*).$$

**Lemma**

*Setting* $A_\lambda = \inf\{I_\lambda(u) \, : \, \|u\| \leqslant \tau\}$*, we have that* $A_\lambda < 0$ *for all* $\lambda \in (0, \lambda^*)$.

**Lemma**

*Given* $\tau > 0$*, there are* $\rho = \rho(\tau) > 0$ *and* $\lambda^* = \lambda^*(\tau)$ *such that*

$$I_\lambda(u) \geqslant \rho \quad \textit{for} \quad \|u\| = \tau \quad \textit{and} \quad \lambda \in (0, \lambda^*).$$

**Lemma**

*Setting* $A_\lambda = \inf\{I_\lambda(u) \ : \ \|u\| \leqslant \tau\}$*, we have that* $A_\lambda < 0$ *for all* $\lambda \in (0, \lambda^*)$.

The last two lemmas permit to apply the Ekeland variational principle to conclude that there exists $u_\lambda \in X$ such that

$$I'_\lambda(u_\lambda)v = 0, \quad \forall v \in X \quad \text{and} \quad I_\lambda(u_\lambda) = A_\lambda < 0.$$

It follows that $u_\lambda$ is a critical point of $I_\lambda$ in $E$ for all $\lambda \in (0, \lambda^*)$.

## Lemma

*For any fixed* $\phi \in C_0^\infty(\Omega_\delta \setminus \overline{A}_{R,r})$*, we have*

$$I_\lambda(t\phi) \to -\infty \quad \textit{as} \quad t \to +\infty.$$

## Lemma

*For any fixed* $\phi \in C_0^\infty(\Omega_\delta \setminus \overline{A}_{R,r})$*, we have*

$$I_\lambda(t\phi) \to -\infty \quad as \quad t \to +\infty.$$

*Proof of Theorem 3.* By the previous results, $I_\lambda$ satisfies the mountain pass geometry. Then for almost every $\lambda \in (0, \lambda^*)$ there is a bounded $(PS)_{c_\lambda}$ sequence for $I_\lambda$, where $c_\lambda$ is the mountain level of $I_\lambda$. Since $I_\lambda$ verifies the $(PS)$ condition, it follows that for almost every $\lambda \in (0, \lambda^*)$ the level $c_\lambda$ is a critical level, that is, there is $u^\lambda \in X$ such that

$$I'_\lambda(u^\lambda) = 0 \quad \text{and} \quad I_\lambda(u^\lambda) = c_\lambda > 0.$$

We conclude that problem $(P_3)$ has at least two solutions $u_\lambda$ and $u^\lambda$ for almost every $\lambda \in (0, \lambda^*)$ with

$$I_\lambda(u_\lambda) = A_\lambda < 0 \quad \text{and} \quad I_\lambda(u^\lambda) = c_\lambda > 0.$$

Finally, we conclude that $u_\lambda$ and $u^\lambda$ are, in fact, critical points of $I_\lambda$ in $E$, hence two nontrivial solutions of problem $(P_3)$.

# 4. Open problems

## 4. Open problems

1. Baouendi-Grushin operators:

$$\mathrm{div}_x \left(G(x,y)|\nabla_x|^{G(x,y)-2}\nabla_x\right) + \mathrm{div}_y \left(G(x,y)|x|^{\gamma}\,|\nabla_y|^{G(x,y)-2}\nabla_y\right).$$

## 4. Open problems

1. Baouendi-Grushin operators:

$$\operatorname{div}_x\left(G(x,y)|\nabla_x|^{G(x,y)-2}\nabla_x\right)+\operatorname{div}_y\left(G(x,y)|x|^{\gamma}\,|\nabla_y|^{G(x,y)-2}\nabla_y\right).$$

2. Biharmonic problems with mixed regime

## 4. Open problems

1. Baouendi-Grushin operators:

$$\operatorname{div}_x \left( G(x,y) |\nabla_x|^{G(x,y)-2} \nabla_x \right) + \operatorname{div}_y \left( G(x,y) |x|^{\gamma} |\nabla_y|^{G(x,y)-2} \nabla_y \right).$$

2. Biharmonic problems with mixed regime

3. Double-phase fractional anisotropic Kirchhoff problems

## 4. Open problems

1. Baouendi-Grushin operators:

$$\operatorname{div}_x \left( G(x,y) |\nabla_x|^{G(x,y)-2} \nabla_x \right) + \operatorname{div}_y \left( G(x,y) |x|^{\gamma} |\nabla_y|^{G(x,y)-2} \nabla_y \right).$$

2. Biharmonic problems with mixed regime
3. Double-phase fractional anisotropic Kirchhoff problems
4. Choquard problems with mixed regime

## 4. Open problems

1. Baouendi-Grushin operators:

$$\operatorname{div}_x \left( G(x,y) |\nabla_x|^{G(x,y)-2} \nabla_x \right) + \operatorname{div}_y \left( G(x,y) |x|^{\gamma} |\nabla_y|^{G(x,y)-2} \nabla_y \right).$$

2. Biharmonic problems with mixed regime
3. Double-phase fractional anisotropic Kirchhoff problems
4. Choquard problems with mixed regime
4. Heat equations with mixed regime

# References

[1] N. Papageorgiou, V.D. Rădulescu, D. Repovs, Double-phase problems and a discontinuity property of the spectrum, *Proc. Amer. Math. Soc.* **147** (2019), 2899-2910.

[2] C. Alves, V.D. Rădulescu, The Lane-Emden equation with variable double-phase and multiple regime, *Proc. Amer. Math. Soc.* **148** (2020), 2937-2952.

# References

[1] N. Papageorgiou, V.D. Rădulescu, D. Repovs, Double-phase problems and a discontinuity property of the spectrum, *Proc. Amer. Math. Soc.* **147** (2019), 2899-2910.

[2] C. Alves, V.D. Rădulescu, The Lane-Emden equation with variable double-phase and multiple regime, *Proc. Amer. Math. Soc.* **148** (2020), 2937-2952.

[3] V. Ambrosio, V.D. Rădulescu, Fractional double-phase patterns: concentration and multiplicity of solutions, *J. Math. Pures Appl.* **142** (2020), 101-145.

[4] N. Papageorgiou, V.D. Rădulescu, D. Repovs, Existence and multiplicity of solutions for double-phase Robin problems, *Bull. London Math. Soc.* **52** (2020), 546-560.

[5] D. Kumar, V.D. Rădulescu, K. Sreenadh, Singular elliptic problems with unbalanced growth and critical exponent, *Nonlinearity* **33** (2020), 3336-3369.

Thank you !