Sharp inequalities for the Ornstein-Uhlenbeck operator

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The **Ornstein-Uhlenbeck operator** $\mathcal{L} = \Delta - x\cdot\nabla$ is the natural counterpart of the Laplace operator when the ambient Euclidean space is replaced by the probability space $(\mathbb{R}^n, \gamma_n)$, where γ_n denotes the **Gauss measure** with the density

$$\mathrm{d}\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}}\,\mathrm{d}x.$$

For any $f \in L^1(\mathbb{R}^n, \gamma_n)$ satisfying $\int_{\mathbb{R}^n} f\,\mathrm{d}\gamma_n = 0$ a unique solution to

$$\mathcal{L}u = -f \quad \text{in } \mathbb{R}^n \quad \text{and} \quad \operatorname{med}(u) = 0$$

exists (in a suitable weak sense) and the **optimal transfer of integrability** from f to u is available. More precisely, for a given r.i. space X, we characterise the optimal (smallest) r.i. space Y such that

$$\|u\|_{Y(\mathbb{R}^n,\gamma_n)} \le C\|f\|_{X(\mathbb{R}^n,\gamma_n)}$$

for some $C > 0$ and every $f \in X(\mathbb{R}^n, \gamma_n)$. Unlike in the Euclidean case, the **gain** of integrability is **not always guaranteed**. For instance, if f belongs to the exponential space $\exp L^\beta$, $\beta > 0$, the Orlicz space built upon a Young function equivalent to e^{t^β} near infinity, the increase of integrability of u deteriorates so that only $u \in \exp L^\beta$, i.e.

$$\|u\|_{\exp L^\beta(\mathbb{R}^n,\gamma_n)} \le C\|f\|_{\exp L^\beta(\mathbb{R}^n,\gamma_n)}.$$

Our specific concern is the **sharp form** of the last inequality. We identify the **largest constant** θ in the integral inequality

$$\sup_u \int_{\mathbb{R}^n} \exp^\beta(\theta|u|)\,\mathrm{d}\gamma_n < \infty,$$

where the supremum is extended over all u subject to a constraint

$$\int_{\mathbb{R}^n} \exp^\beta(|\mathcal{L}u|)\,\mathrm{d}\gamma_n \le M \quad \text{and} \quad \operatorname{med}(u) = 0$$

for some $M > 1$. We also show that the **maximizers exist** in relevant cases, i.e. that the supremum is in fact attained.

This problem can be regarded as a Gaussian analogue of that solved by Adams for the classical Laplacian in the Euclidean setting, which is in turn a second order version of the famous **Moser's inequality**.