Higher regularity in congested traffic dynamics

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Monday's Nonstandard Seminar

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Model case of a very degenerate PDE

$$\operatorname{div}\left((|\nabla u| - 1)_+^{p-1} \frac{\nabla u}{|\nabla u|} \right) = f$$

$\Leftrightarrow$ Minimizing the variational integral

$$F(u) = \int_\Omega \left[\tfrac{1}{p} (|\nabla u| - 1)_+^p + fu \right] \mathrm{d}x$$

$h(t) = (t-1)_+^p$

$h(t) = \frac{(t-1)_+^{p-1}}{t}$

$p = \frac{3}{2}$, $p = 2$, $p = 4$

Very degenerate PDEs

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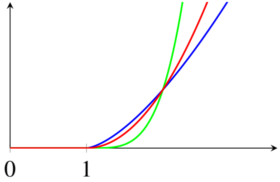
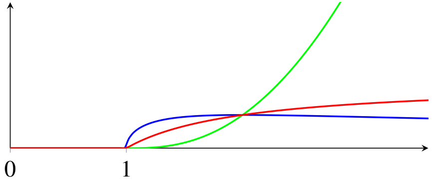


$h(t)=(t-1)_+^p$



$h(t)=\frac{(t-1)_+^{p-1}}{t}$

$p=\frac{3}{2}$, $p=2$, $p=4$

Motivation by traffic congestion problems

Wardrop equilibrium (Wardrop 1952)

Relies on two principles:

- **User equilibrium:** each user chooses the route that is the best $\Rightarrow$ journey times in all routes actually used are equal and less than those that would be experienced by a single vehicle on any unused route
- **System optimality:** average journey time is at a minimum (in particular, users behave cooperatively in choosing their routes to ensure the most efficient use of the whole system)

Model by Monge-Kantorovich problem

- $\Omega \subset \mathbb{R}^n$ ($\overline{\Omega}$ models the city for $n = 2$)
- μ_0, μ_1 probability measures on $\overline{\Omega}$ (distribution of residents and services in the city $\overline{\Omega}$)
- $\Pi(\mu_0, \mu_1)$: set of transportation plans (probability measures on $\overline{\Omega} \times \overline{\Omega}$ having μ_0 and μ_1 as marginals)
- $c \in C(\overline{\Omega} \times \overline{\Omega}, \mathbb{R})$ cost function

Monge-Kantorovich optimal transportation problem

$$\inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} c(x,y)\, \mathrm{d}\gamma(x,y).$$

What is not realistic in this model:

- model is path independent (individual's travelling strategies are irrelevant)
- congestion effects are not considered (the cost $c(x,y)$ is independent of "how crowded" the used path is)

Traffic congestion model I

Carlier, Jimenez, Santambrogio (2008) introduced the notion of a transportation strategy taking into account

- different possible paths
- congestion effects

Traffic congestion model II

This model results in the following minimization problem:

$$\min\left\{\int_\Omega \mathscr{H}(\sigma)\,\mathrm{d}x\colon \sigma\in L^q(\Omega,\mathbb{R}^n),\ \operatorname{div}\sigma=\mu_0-\mu_1,\ \sigma\cdot\nu_{\partial\Omega}=0\right\},$$

where σ represents the traffic flow and

$$\mathscr{H}(\sigma)=H(|\sigma|),\qquad \text{with } H(t)=t+\tfrac{1}{q}t^q \quad \text{and} \quad \tfrac{1}{p}+\tfrac{1}{q}=1.$$

The function $g(t)=H'(t)=1+t^{q-1}$ models the congestion effect.

Traffic congestion model III

By duality one can show that $\sigma = \nabla \mathscr{H}^*(\nabla u)$

- $\mathscr{H}^*$ is the Legendre transform of $\mathscr{H}$
- u solves the Neumann problem

$$\begin{cases} \operatorname{div} \nabla \mathscr{H}^*(\nabla u) = \mu_0 - \mu_1 & \text{in } \Omega, \\ \nabla \mathscr{H}^*(\nabla u) \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

Literature

Literature on traffic congestion problem models:

- Warprop (1952)
- Carlier, Jimenez, Santambrogio (2008)
  Derivation of the model and existence of minimizers
- Brasco, Carlier, Santambrogio (2010)
  Characterization by the very degenerate elliptic PDE
- ...

Very degenerate PDEs

Since

$$\mathscr{H}(\zeta) = \frac{1}{q}|\zeta|^q + |\zeta|$$

we compute

$$\mathscr{H}^*(z) = \frac{1}{p}(|z| - 1)_+^p, \quad \text{where } p = \tfrac{q}{q-1}.$$

This results in the very degenerate PDE

$$\operatorname{div}\left((|\nabla u| - 1)_+^{p-1}\frac{\nabla u}{|\nabla u|}\right) = f$$

Lipschitz continuity

- Weak solutions are **Lipschitz continuous**
  - Scalar setting: Brasco & Carlier & Santambrogio, Brasco
  - Vectorial setting: Clop & Giova & Hathami & Passarelli di Napoli
  - As special case of an asymptotically regular problem: Chipot & Evans, Raymond, Foss, Foss & Passarelli di Papoli & Verde, . . .
- Even if $f \equiv 0$: better than Lipschitz is not possible:

$$(|\nabla u| - 1)_+ = 0 \quad \text{if } |\nabla u| \le 1$$

Higher regularity

- **Sobolev regularity:** $\mathscr{G} \in W^{1,2}$ for $\mathscr{G} = (|\nabla u| - 1)_+^{\frac{p}{2}} \frac{\nabla u}{|\nabla u|}$
  - Brasco & Carlier & Santambrogio (2010)
  - Clop & Giova & Hatami & Passarelli di Napoli (2019): Vector valued case
- **Continuity:** $g(\nabla u)$ is continuous for any continuous function $g\colon \mathbb{R}^n \to \mathbb{R}$ with $g = 0$ on B_1
  - Santambrogio & Vespri (2010): $n = 2$
  - Colombo & Figalli (2014): $n \geq 2$

Vectorial setting

Consider weak solutions $u\colon \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ with $N \geq 1$ of

$$\operatorname{div}\left((|Du|-1)_+^{p-1}\frac{Du}{|Du|}\right) = f,$$

where $p > 1$ and $f\colon \Omega \to \mathbb{R}^N$.

What is the optimal regularity?

- Contra: Solutions are less regular in the vectorial case (even unbounded; counterexample by De Giorgi)
- Pro: Solutions of the p-Laplace system

$$\Delta_p u = 0$$

  are of class $C^{1,\alpha}$ for some $\alpha > 0$ (first proof by Uhlenbeck)
- $\frac{(|Du|-1)_+^{p-1}}{|Du|}$ depends only on the modulus of Du

$\Rightarrow$ there is some hope for regularity

Regularity in the vectorial setting

Consider weak solutions $u\colon \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ with $N \geq 1$ of

$$\operatorname{div}\left((|Du|-1)_+^{p-1}\frac{Du}{|Du|}\right) = f.$$

Theorem (B., Duzaar, Giova, Passarelli di Napoli)

Let $p > 1$ and

$$f \in L^{n+\sigma}(\Omega,\mathbb{R}^N) \quad \textit{for some } \sigma > 0.$$

Then

$$g(Du) \textit{ is continuous}$$

for any continuous function $g\colon \mathbb{R}^{Nn} \to \mathbb{R}$ vanishing on $\{|\xi| \leq 1\}$.

Optimality of the result

- We treat any $p > 1$
- On the set where $|Du| \leq 1$, Du could be discontinuous
- Hölder continuity of $g(Du)$ is not true: counterexample
- $f \in L^n$ is not enough: Du possibly unbounded

Scetch of the Proof I

- **Lipschitz regularity.** Du is bounded on any compact subset of Ω
- **Regularization.** Consider solution u_ε of

$$\begin{cases} \operatorname{div}\left((|Du_\varepsilon| - 1)_+^{p-1} \frac{Du_\varepsilon}{|Du_\varepsilon|}\right) + \varepsilon \Delta u_\varepsilon = f, & \text{in } B_R \subset \Omega, \\ u_\varepsilon = u, & \text{on } \partial B_R. \end{cases}$$

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Scetch of the Proof II

- **Hölder-continuity.** For any $\delta \in (0, 1]$

$$G_\delta(Du_\varepsilon) \text{ is Hölder continuous with exponent } \alpha_\delta$$

where $G_\delta(\xi) := \frac{(|\xi|-1-\delta)_+}{|\xi|}\,\xi.$

Constants are independent of ε!

- **Passage to the limit.**
  - $\varepsilon \to 0$: $G_\delta(Du)$ is Hölder continuous with exponent α_δ
  - $\delta \to 0$: Continuity of

$$\frac{(|Du|-1)_+}{|Du|}\,Du$$

- **Continuity of $g(Du)$.**
  - $\xi \mapsto \frac{(|\xi|-1)_+}{|\xi|}\,\xi$ is invertible on the set $\{|\xi| > 1\}$

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Hölder-continuity of $G_\delta(Du_\varepsilon)$

Our goal (abbreviate $Du = Du_\varepsilon$):

$$\fint_{B_r(x_o)} |G_\delta(Du) - \Gamma_{x_o}|^2 \,\mathrm{d}x \le c \left(\frac{r}{\varrho}\right)^{2\alpha} \qquad \forall B_r(x_o) \subset B_\varrho(x_o) \subset B_R$$

- Suppose

$$\sup_{B_\varrho(x_o)} |G_\delta(Du)| \le \mu$$

- Distinguish between two regimes ($0 < \nu \ll 1$):

(D) $$|E^\nu_\varrho(x_o)| \le (1-\nu)|B_\varrho(x_o)|,$$

(ND) $|E^\nu_\varrho(x_o)| > (1-\nu)|B_\varrho(x_o)|$ and $\mu \ge \delta,$

where

$$E^\nu_\varrho(x_o) := B_\varrho(x_o) \cap \big\{|G_\delta(Du)| > (1-\nu)\mu\big\}.$$

Universal energy inequality

In the weak form use the test-function

$$\varphi = \zeta\phi(|Du|)D_\beta u,$$

where $\beta \in \{1, \dots, n\}$, ζ cut-off function, ϕ non-negative and non-decreasing. We obtain

$$\int_{B_R} \Big[\mathscr{A}\big(D^2u, D^2u\big)\phi(|Du|) + \mathscr{B}\big(\nabla|Du|, \nabla|Du|\big)\phi'(|Du|)|Du|\Big]\zeta \mathrm{d}x \\ + \int_{B_R} \mathscr{B}\big(\nabla|Du|, \nabla\zeta\big)\phi(|Du|)|Du| \,\mathrm{d}x \le 0,$$

where $\mathscr{A} = \mathscr{A}(Du)$ and $\mathscr{B} = \mathscr{B}(Du)$ are bilinear forms.

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$$+ \int_{B_R} \mathscr{B}\big(\nabla|Du|, \nabla\zeta\big) \phi(|Du|)|Du| \, \mathrm{d}x \leq 0,$$

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$$\int_{B_R}\Big[\underbrace{\mathscr{A}\big(D^2u, D^2u\big)\phi(|Du|)}_{\geq 0} + \underbrace{\mathscr{B}\big(\nabla|Du|, \nabla|Du|\big)\phi'(|Du|)|Du|}_{\geq 0}\Big]\zeta\,\mathrm{d}x$$

$$+ \underbrace{\int_{B_R} \mathscr{B}\big(\nabla|Du|, \nabla\zeta\big)\phi(|Du|)|Du|\,\mathrm{d}x}_{\leq 0 \,\Rightarrow\, \nabla|Du| \text{ is subsolution}} \leq 0,$$

where $\mathscr{A} = \mathscr{A}(Du)$ and $\mathscr{B} = \mathscr{B}(Du)$ are bilinear forms.

Degenerate regime

- $\nabla|Du|$ is a subsolution to an elliptic equation
- Reduction of the supremum by a De Giorgi type argument:

$$\sup_{B_{\varrho/2}(x_o)} |G_\delta(Du)| \le \kappa\mu, \qquad \kappa < 1$$

Non-degenerate regime

- Define the excess

$$\Phi(x_o,\varrho) := \fint_{B_\varrho(x_o)} \big|Du - (Du)_{x_o,\varrho}\big|^2 \,\mathrm{d}x$$

- The measure theoretic information yields

$$\Phi(x_o,\varrho) \ll 1 \quad \text{and} \quad |(Du)_{x_o,\varrho}| \ge 1 + \delta + \tfrac{1}{2}\mu$$

- Compare u with the solution v of a linear elliptic system

$$\fint_{B_{\varrho/2}(x_o)} |Du - Dv|^2 \,\mathrm{d}x \le c\,\Phi(x_o,\varrho)^{1+\vartheta} \qquad \text{for some } \vartheta > 0$$

- Excess decay

$$\Phi(x_o,\tau\varrho) \le c\,\tau^2\Phi(x_o,\varrho) \qquad \text{for } \tau \in (0,1)$$

- Iteration
  - The limit $\Gamma_{x_o} := \lim_{i\to\infty} \big(G_\delta(Du)\big)_{\tau^i\varrho}$ exists
  - Campanato-type estimate: $\fint_{B_r} |G_\delta(Du) - \Gamma_{x_o}|^2 \,\mathrm{d}x \le c\left(\frac{r}{\varrho}\right)^{2\beta}$

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Combining both regimes

- Define $\varrho^i = 2^{-i}\varrho$
- Suppose that (D) is satisfied on $B_{\varrho_i}(x_o)$ for $i = 0, \dots, i_o - 1$:

$$\sup_{B_{\varrho_i}(x_o)} |G_\delta(Du)| \le \kappa^i \mu =: \mu_i$$

- Suppose that (D) is not satisfied on $B_{\varrho_{i_o}}(x_o)$

$$⨍_{B_r} |G_\delta(Du) - \Gamma_{x_o}|^2 \, dx \le c \left(\frac{r}{\varrho_{i_o}}\right)^{2\beta} \quad \text{for } r \le \varrho_{i_o}$$

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