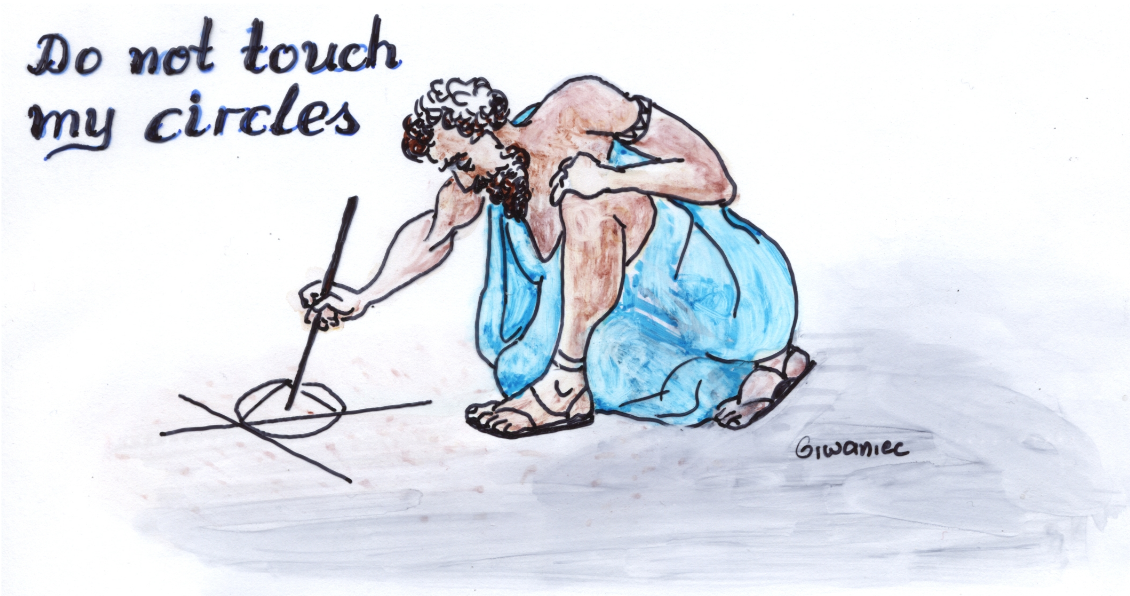
Monotone Hopf-Harmonics

Tadeusz Iwaniec (Syracuse University, USA)
joint work with Jani Onninen

(Monday's Nonstandard Seminar (online), Warsaw, May 24, 2021)

Archimedes of Siracusa (287BC - 212BC)

Father of the application of scientific knowledge



Quasiconformal Geometry and Nonlinear Elasticity share compelling beauty through variational integrals.

- *Tadeusz of SyracUSA*

My special thanks go to Iwona Chlebicka

Good ”morning”; it is 9:00 am here in Syracuse. I am happy to welcome many distinguished participants and friends of mine.

No doubt, your seminars are fabulous and stimulating.

It is a pleasure and honor that I can speak on this occasion about my joint studies with Jani Onninen of the mathematical foundations of Nonlinear Elasticity (NE)

Indeed, our passion for Nonlinear Elasticity, and our rather modest engagement in this area, grew out of the very early pioneering "ABC" papers by S.S. Antman, J. Ball and P.G. Ciarlet

(folklore nowadays)

Nonlinear Hyperelasticity
(in a few words)

One enquires into deformations $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of **smallest stored energy**

$$\mathscr{E}[h] = \int_{\mathbb{X}} \mathbf{E}(x, h, Dh)\, \mathrm{d}x\ , \qquad \mathbf{E} : \mathbb{X} \times \mathbb{Y} \times \mathbb{R}^{m\times n} \to \mathbb{R}$$

The accustomed hypothesis on $\mathbf{E}$ is **convexity, polyconvexity or quasiconvexity** with respect to the deformation gradient $Dh \in \mathbb{R}^{m\times n}$. This secures lower semicontinuity of the energy functional; that is,

$$\int_{\mathbb{X}} \mathbf{E}(x, h, Dh)\, \mathrm{d}x \;\leqslant \liminf \int_{\mathbb{X}} \mathbf{E}(x, h_k, Dh_k)\, \mathrm{d}x$$

whenever $h_k \rightharpoonup h$, weakly in a relevant Sobolev space, usually in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$.

p-Harmonic Energy

(widely used model of convex energy functionals)

Think of two particular cases of convex energy-functionals in the planar domains (plates) or surfaces (thin films); the Dirichlet and the p-harmonic integrals.

$$\mathscr{E}_2[h] = \int_{\mathbb{X}} |Dh(x)|^2 \, dx \qquad \mathscr{E}_p[h] = \int_{\mathbb{X}} |Dh(x)|^p \, dx$$

These are ideal examples, good enough to demonstrate the essence of the problems and the ideas of solving them. Thus the natural domain of definition is the space of mappings:

$$h : \mathbb{X} \overset{\text{onto}}{\longrightarrow} \mathbb{Y}\,, \text{ of class } \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})\,, \ \ 1 < p < \infty.$$

Non-interpenetration of Matter

It is a common struggle in mathematical theory of NE to establish existence of the energy-minimal deformations which are invertible.. Direct method in the Calculus of Variations reveals that injectivity is lost when passing to the limit of an energy-minimizing sequence of homeomorphisms. Some parts of the material body are squeezed to lower dimensional pieces of the deformed configuration. From the topological point of view this property is characteristic of monotone mappings. It is for this reason that we must adopt:

Sobolev Monotone Mappings

as legitimate deformations in the Theory of Hyperelasticity.

Such deformations turn out to be weak $\mathscr{W}^{1,p}$-limits of homeomorphisms.

Monotone Mappings

A topological concept due to C.B. Morrey (1935)

A continuous map $h : \mathbf{A} \overset{\text{onto}}{\longrightarrow} \mathbf{B}$ between compact metric spaces is said to be monotone if every fibre $h^{-1}\{b\}$ of a point $b \in \mathbf{B}$ is connected.

In fact, the preimage $h^{-1}(\mathcal{B})$ of any connected set $\mathcal{B} \subset \mathbf{B}$ turns out to be connected in $\mathbf{A}$ as well (G.T. Whyburn) .

Theorem (Kuratowski-Lacher, 1968) If the target space $\mathbf{B}$ is locally connected, then the space of all monotone mappings from $\mathbf{A}$ onto $\mathbf{B}$ is closed under uniform convergence.

Theorem of J. W. T. Youngs (1948)

Homeomorphic approximation to monotone mappings

Let $\mathbf{A}$ and $\mathbf{B}$ be topologically equivalent compact 2-manifolds (with or without boundary). Then a continuous map $h : \mathbf{A} \overset{\text{onto}}{\longrightarrow} \mathbf{B}$ is monotone if and only if there is a sequence of homeomorphisms $h_j : \mathbf{A} \overset{\text{onto}}{\longrightarrow} \mathbf{B}$ converging uniformly to h.

It is in this way that we went into exciting adventures. Myriad challenging problems of Nonlinear Elasticity (NE) and numerous elegant conjectures are very appealing to us. It is from the point of view of GFT (generalization of Riemann's mapping theorem, in particular) that we were especially interested in **traction free** energy-minimal deformations (deformations that are sliding freely along the boundary). We were able to answer some of the basic questions and solve long standing conjectures.

In what follows the abbreviation **[IO]** refers to a joint work of T. Iwaniec and J. Onninen.

Let us begin with some prerequisites for the discussion.

Diffeomorphic Approximation of Sobolev Homeomorphisms

[IO] and L. Kovalev

Arch. Ration. Mech. Anal. (2011)

Every homeomorphism $h : \mathbb{X} \to \mathbb{Y}$ between planar open sets that belongs to the Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 < p < \infty$, can be approximated uniformly and in $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, with $\mathscr{C}^\infty$-smooth diffeomorphisms $h_j \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$, $j = 1, 2, \ldots$
Actually $h_j - h \in \mathscr{W}_\circ^{1,p}(\mathbb{X}, \mathbb{R}^2)$, for $j = 1, 2, \ldots$ [1]

[1] Somewhat later the case $p = 1$ has been handled by **S. Hencl and A. Pratelli**.

Diffeomorphic Approximation of Monotone Sobolev Mappings
([IO], ARMA, 2016)

Let $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^2$ be Jordan domains (multi connected) of the same topological type, $\mathbb{Y}$ being Lipschitz. Then for every monotone (continuous) map $h : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ of Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $1 < p < \infty$, there exists a sequence of homeomorphisms $h_j : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ converging to h uniformly and in the norm topology of $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^2)$. Actually, the mappings $h_j : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ can be $\mathscr{C}^\infty$-diffeomorphisms.

Existence of Traction Free Energy-Minimal Deformations

[IO], ARMA (2016)

Let $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^2$ **be bounded Lipschitz domains of the same topological type. Among all monotone mappings** $h : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ **of Sobolev class** $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $p \geqslant 2$, **there exists** $h_\circ : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ **of smallest energy.**

This also holds when a monotone boundary data is prescribed on some of the components of $\partial\mathbb{X}$.

No Lavrentiev Discrepancy [IO]

$$\int_{\mathbb{X}} |Dh_\circ(x)|^p \, dx \quad =$$

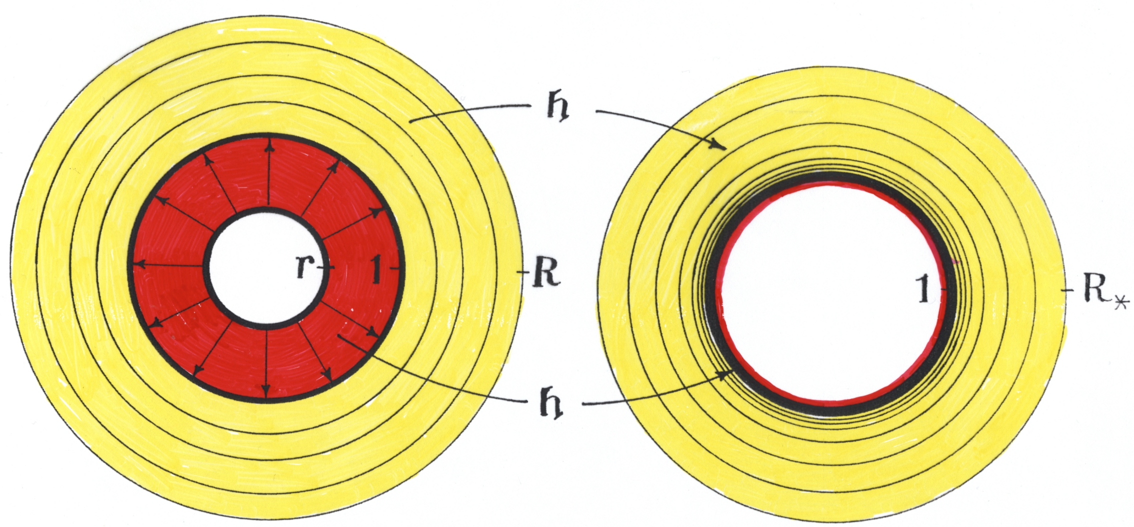
$$\inf \left\{ \int_{\mathbb{X}} |Dh|^p \, ; \text{ homeomorphisms } h : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}} \right\}$$

Behind this result is the following fact:

The weak sequential closure and strong closure of homeomorphisms $h : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ of Sobolev class $\mathscr{W}^{1,p}(\mathbb{X}, \mathbb{Y})$, $p \geqslant 2$, are the same

Squeezing Phenomenon (and Nitsche Conjecture,

IO and L. Kovalev, JAMS (2011)) $R_* > \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right)$



$$\mathfrak{h}(z) = \begin{cases} \frac{z}{|z|}, & r < |z| < 1 \quad \left(\text{squeezing into concave boundary}\right) \\ \frac{1}{2}\left(z + \frac{1}{\bar{z}}\right), & 1 < |z| < R \quad \left(\text{critical harmonic Nitsche map}\right) \end{cases}$$

Among all traction free monotone mappings the Nitsche map $\mathfrak{h}(z)$ is a unique (up to rotation) of smallest energy. Surprisingly, $\mathfrak{h}$ is $\mathscr{C}^{1,1}$-smooth in the entire annulus and its Hopf quadratic differential is analytic.

$$\mathcal{Q}(z)\, \mathrm{d}z \otimes \mathrm{d}z \; = \; h_z \, \overline{h_{\bar z}}\, \mathrm{d}z \otimes \mathrm{d}z$$

$$= \; \frac{-1}{4z^2}\, \mathrm{d}z \otimes \mathrm{d}z \, , \quad r \leqslant |z| \leqslant R$$

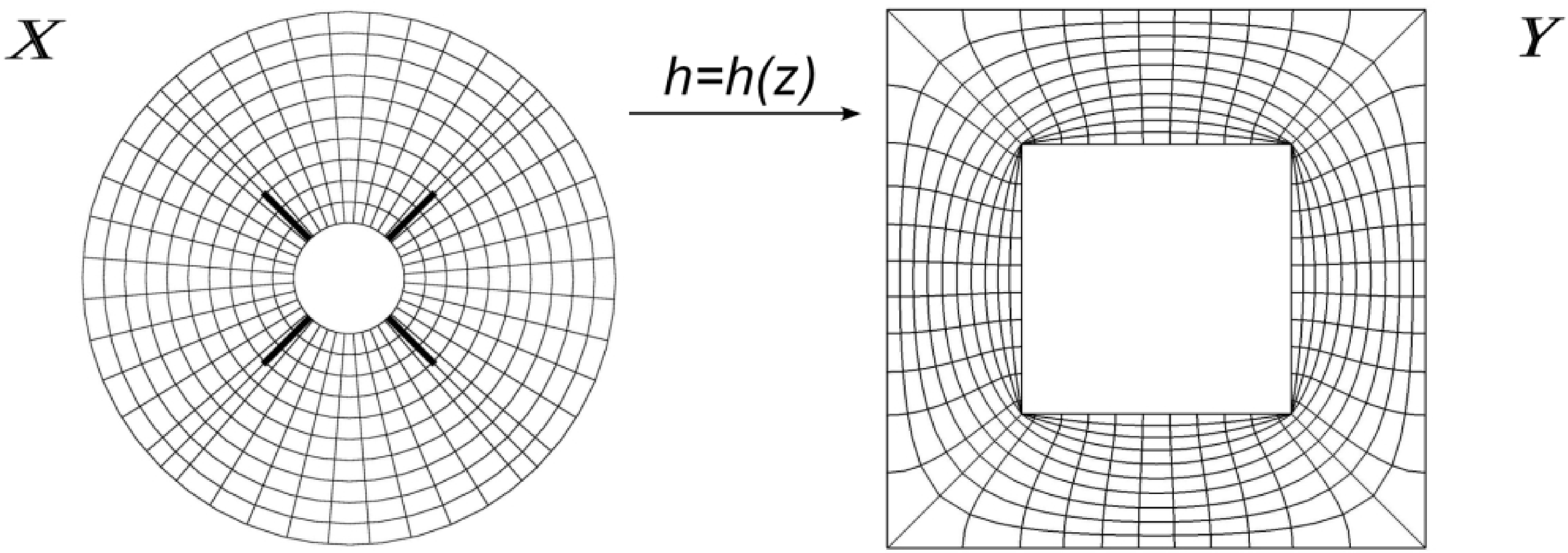
Inner Variation of the Dirichlet Energy

Hopf-Laplace Equation $\dfrac{\partial}{\partial \bar{z}}\left(h_z \overline{h_{\bar{z}}}\right) = 0$

Hopf Quadratic Differential $h_z \overline{h_{\bar{z}}}\ \mathrm{d}z \otimes \mathrm{d}z$

Injectivity of an energy-minimal map is lost exactly in a neighborhood where it fails to be harmonic. (**IO**, Cal.Var. PDE (2015))

An Example [IO], ARMA (2013)



The round annulus is too fat. Consequently, cracks are inevitable along vertical trajectories of the Hopf quadratic differential.

No Formation of Cracks

THEOREM (**IO, Koh, Kovalev, Invent. Math. 2011**) **Among all homeomorphisms** $h \colon \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ **between bounded doubly connected domains such that**

$$\operatorname{Mod} \mathbb{X} \leqslant \operatorname{Mod} \mathbb{Y}$$

there exists one of smallest Dirichlet energy. This is a harmonic diffeomorphism, unique up to conformal automorphisms of $\mathbb{X}$**.**

Nevertheless, an energy-minimal monotone map that fails to be invertible tells us when to stop the minimizing sequence of homeomorphisms prior to the conditions favorable to the formation of cracks.

Theoretical prediction of failure of bodies caused by cracks is a good motivation that should appeal to MATHEMATICAL ANALYSTS and researchers in the ENGINEERING FIELDS.

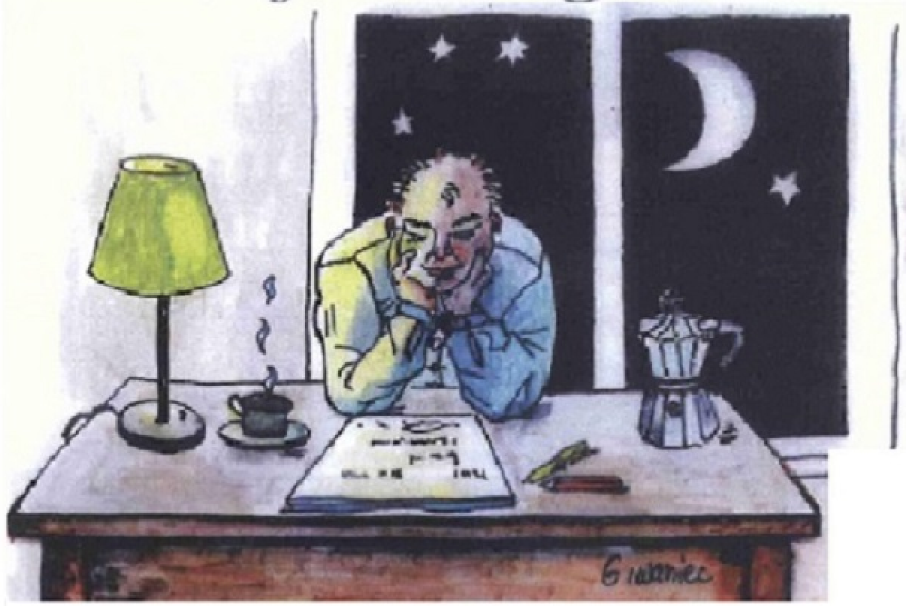
Three blacksmiths are hammering a piece of hot iron to create a new shape.

Consecutive strikes define an energy-minimizing sequence of Sobolev homeomorphisms. The limit satisfies the inner variational

Hopf-Laplace equation

Hopf quadratic differential and its trajectories come into play.

Afterthought



Should I tell the blacksmiths when to stop hammering prior to the permanent damage caused by cracks?

Dirichlet Problem for Hopf-Laplace Equation

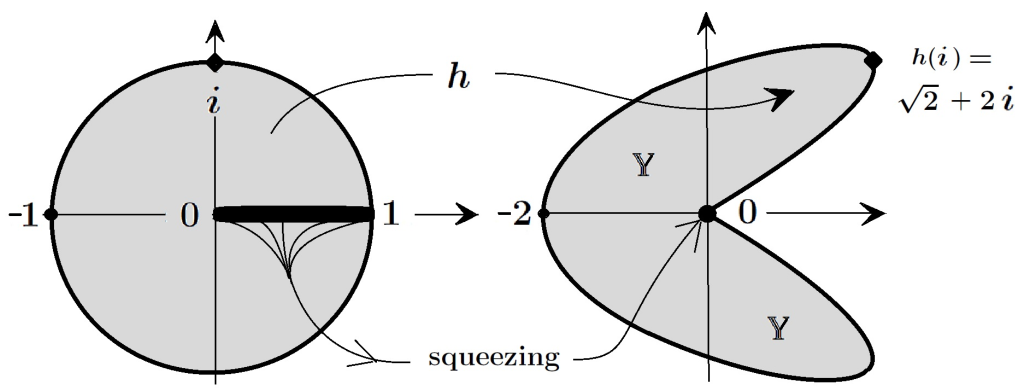
It is clear that every harmonic homeomorphism $h : \mathbb{X} \xrightarrow{\text{onto}} \mathbb{Y}$ of Sobolev class $\mathscr{W}^{1,2}_{\text{loc}}(\mathbb{X}, \mathbb{Y})$ solves the Hopf-Laplace equation.

$$\frac{\partial}{\partial \bar{z}} \left(h_z \, \overline{h_{\bar{z}}} \right) = 0$$

Conversely (which is a non-trivial fact) $\mathscr{W}^{1,2}$- homeomorphic solutions are harmonic. There are, however, Lipschitz continuous bizarre solutions.

A well posed boundary value problem for the Hopf-Laplace equation is to find $\mathscr{W}^{1,2}$- monotone solutions subject to a given monotone boundary data. Without restriction to monotone mappings the problem would be ill posed.

THEOREM. **Suppose that $\mathbb{X}, \mathbb{Y} \subset \mathbb{R}^2$ are simply connected Jordan domains, $\mathbb{Y}$ being Lipschitz regular. To every monotone map $g : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ of Sobolev class $\mathscr{W}^{1,2}(\mathbb{X}, \mathbb{Y})$ there corresponds a monotone Hopf-harmonic $h : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ of Sobolev class $\mathscr{W}^{1,2}(\mathbb{X}, \mathbb{Y})$ which agrees with g on $\partial\mathbb{X}$. Actually h is locally Lipschitz on $\mathbb{X}$ (non-trivial).**



$$h(z) = z - \overline{z} - i[z^{3/2} - \overline{z}^{3/2}]$$

This monotone Hopf-harmonic map $h : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ squeezes the segment $[0, 1]$ into a point in $\partial\mathbb{Y}$, exactly where $\partial\mathbb{Y}$ fails to be convex.

Global Invertibility versus Non-interpenetration of Matter

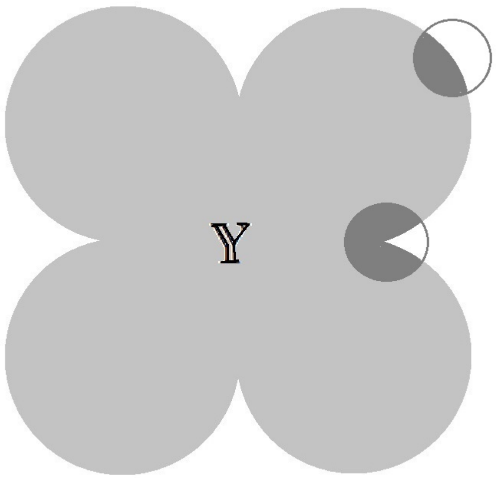
In connection with the results by J. Ball (1981) , P. G. Ciarlet & J. Nečas (1987) on *global invertibility*, regarded as non-interpenetration of matter, let me bring on stage the following:

THEOREM (Injectivity of $h : h^{-1}(\mathbb{Y}) \overset{\text{onto}}{\longrightarrow} \mathbb{Y}$) **[IO]**
Any monotone Hopf-harmonic solution $h : \overline{\mathbb{X}} \overset{\text{onto}}{\longrightarrow} \overline{\mathbb{Y}}$ of Lipschitz domains is a diffeomorphism of $h^{-1}(\mathbb{Y})$ <u>into</u> $\mathbb{X}$.

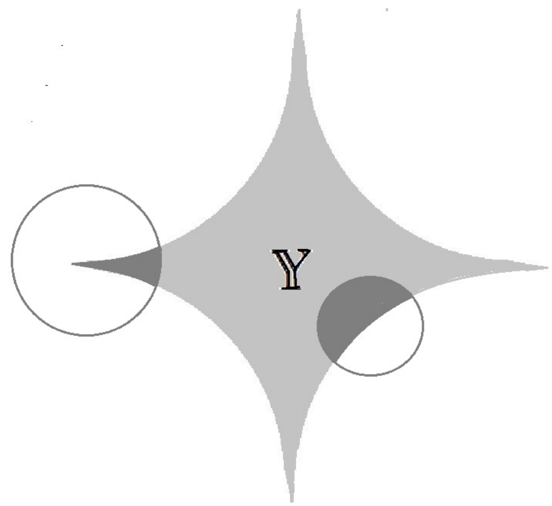
For a comparison, the term global invertibility (by J. Ball, P.G.Ciarlet and J. Nečas) refers to mappings $h : \overline{\mathbb{X}} \overset{\text{onto}}{\longrightarrow} \overline{\mathbb{Y}}$ such that the preimage $h^{-1}(y_\circ)$ of almost every point $y_\circ \in \mathbb{Y}$ is a singleton.

Uniqueness

THEOREM (Uniqueness) [IO]. **The monotone Hopf-harmonic boundary value problem admits unique solution whenever $\mathbb{Y}$ is somewhere convex; for example, when $\mathbb{Y}$ is $\mathscr{C}^2$- smooth.**



somewhere convex at $\partial\mathbb{Y}$



nowhere convex at $\partial\mathbb{Y}$

Monotone Hopf-harmonics are the energy minimizers.

Let $\mathbb{Y} \subset \mathbb{R}^2$ be a simply connected Lipschitz domain and $g : \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ a homeomorphism of Sobolev class $\mathscr{W}^{1,2}(\mathbb{X}, \mathbb{R}^2)$ (Dirichlet data). Then a monotone Sobolev mapping $h \in g + \mathscr{W}_{\circ}^{1,2}(\mathbb{X}, \mathbb{R}^2)$ satisfies the Hopf-Laplace equation if and only if

$$\int_{\mathbb{X}} |Dh(x)|^2 \, \mathrm{d}x \; =$$

$$\inf_{H \in \mathrm{Diff}_g(\overline{\mathbb{X}}, \overline{\mathbb{Y}})} \int_{\mathbb{X}} |DH(x)|^2 \, \mathrm{d}x$$

Afterthought

Planar Monotone Sobolev Mappings (Cellular Sobolev Mappings in higher dimensions) are profoundly insightful and as such should take legitimate place in the theory of Nonlinear Elasticity.

Thank You
for Listenning

Abstract: We are primary concerned with Sobolev homeomorphisms and their weak and strong limits. Such limits turn out to be monotone mappings. This includes the weak limits of energy-minimizing sequences of homeomorphisms (hyperelastic deformations). Usually, the injectivity is lost when passing to the limit of homeomorphisms. Call such circumstance *weak collapse of matter*. In case of Dirichlet energy, we shall provide plausible mathematical explanation of the collapsing phenomena and, consequently, formation of cracks.

- The interpenetration of matter occurs exactly in the region of the body where the energy-minimal monotone Sobolev map fails to satisfy the Lagrange-Euler equation.
- Cracks propagate along trajectories of the Hopf quadratic differential associated with the inner variation of the energy integral.
- We believe that planar Monotone Sobolev Mappings (Cellular Sobolev Mappings in higher dimensions) are profoundly insightful and as such should take legitimate place in the theory of Nonlinear Elasticity.