Gradient estimates of very weak solutions to nonlinear equations with nonstandard growth

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Standard growth

- Consider variational integrals

$$w \in W^{1,1} \to \int_\Omega f(x, Dw(x))dx,$$

$f(x,\xi)$ denoting a given Lagrangian and Ω a bounded domain in $\mathbb{R}^n$ with $n \geq 2$.

- Standard growth condition is

$$|\xi|^p \lesssim f(x,\xi) \lesssim |\xi|^p + 1$$

for $1 < p < \infty$.

Nonstandard growth

- Nonstandard growth condition is

$$|\xi|^p \lesssim f(x,\xi) \lesssim |\xi|^q + 1$$

  for $1 < p < q < \infty$.
- Regularity is to be expected if p and q are not too far away, as observed by Paolo Marcellini in the late '80s.
- A natural and the best/simplist example is the G-Laplacian considered by Gary M. Lieberman in the early '90s.

Nonstandard growth

- Assume $G \in C^2(0,\infty)$, $g = G'$, is an N-function such that $0 < \delta_0 \leq \frac{g'(t)t}{g(t)} \leq g_0$ for some constants δ_0 and g_0.
- The problem under consideration is

$$\begin{cases} \operatorname{div} a(x, Dx) = \operatorname{div}\left(\frac{g(|F|)}{|F|}F\right) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

- The given Carathéodory function $a = a(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to satisfy
  1. $|a(x,\xi)| + |\xi||D_\xi a(x,\xi)| \leq Lg(|\xi|)$ and
  2. $D_\xi a(x,\xi)z \cdot z \geq \nu \frac{g(|\xi|)}{|\xi|}|z|^2$

  for all x, $\xi \neq 0$, $z \in \mathbb{R}^n$ and some constants $0 < \nu \leq 1 \leq L$.
- $F \in L^g(\Omega, \mathbb{R}^n)$ is given.

Nonlinear elliptic problems of G-Laplacian type

- $G \in \Delta_2 \cap \nabla_2$.
- There exists a small constant $\delta_1 = \delta_1(\delta_0, g_0) \in (0, \min\{1, g_0\})$ such that $G^{1-\delta_1} \in \Delta_2 \cap \nabla_2$.
- For $g(t) = t^{p-1}$, it becomes the p-Laplacian.

Very weak solution

Definition

$u \in W_0^{1,g}(\Omega)$ is a very weak solution of (1) if

$$\int_\Omega a(x, Du) D\varphi \, dx = \int_\Omega \frac{g(|F|)}{|F|} F D\varphi \, dx$$

for all $\varphi \in C_0^\infty(\Omega)$.

Main result (B. - Lim)

Theorem

There exists a small positive constant $\tilde{\delta} = \tilde{\delta}(n, \delta_0, g_0, \nu, L)$ *such that if* $F \in W^{1,G^{1-\delta}}(\Omega, \mathbb{R}^n)$ *for all* $\delta \in \left(0, \frac{1}{2}\tilde{\delta}\right)$, *then any very weak solution* $u \in W^{1,G^{1-\tilde{\delta}}}_{loc}(\Omega)$ *to the problem* (1) *satisfies*

$$u \in W^{1,G^{1-\delta}}_{loc}(\Omega)$$

and for each open $\Omega' \subset\subset \Omega$ *we have the estimate*

$$\|u\|_{W^{1,G^{1-\delta}}(\Omega')} \le c\left(\|u\|_{W^{1,G^{1-\tilde{\delta}}}(\Omega)} + \|F\|_{W^{1,G^{1-\delta}}(\Omega)}\right),$$

the constant c *independent of* F *and* u.

History

- We look at
$$\mathrm{div} a(x, Dx) = \mu \text{ in } \Omega$$
whose distributional formulation is
$$\int_\Omega a(x, Du)D\varphi \, dx = \langle \mu, \varphi \rangle \qquad (\varphi \in C_0^\infty(\Omega)) \, .$$
- Due to monotone operators theory, one can assert existence, uniqueness and regularity of weak solutions in $W_0^{1,G}(\Omega)$ when $\mu \in \left(W_0^{1,G}(\Omega)\right)^*$. This is the dual case and it lies in the realm of weak solutions.

The dual case

- We first look at the case when $a(x,\xi) \approx |\xi|^{p-2}\xi$, and $g(t) = t^{p-1}$.
- We consider when the right hand side is in divergence form

$$\mu = \mathrm{div}(|F|^{p-2}F) \qquad (F \in L^q(\Omega) \text{ with } p \leq q < \infty).$$

- An issue is to show that

$$F \in L^q \Longrightarrow Du \in L^q.$$

- Iwaniec, Studia Math. '83 for the elliptic case.
- Acerbi and Giuseppe, Duke Math. J. '07 for the parabolic case.

The dual case

- We next look at the case when $a(x,\xi) \approx \frac{g(|F|)}{|F|}F$.
- The right hand side is in divergence form

$$\mu = \mathrm{div}\left(\frac{g(|\xi|)}{|\xi|}\xi\right) \qquad \left(F \in L^H(\Omega) \text{ with } G \prec H\right).$$

- An issue is to show that

$$F \in L^H \Longrightarrow Du \in L^H.$$

- Anna Verde, J. Convex Anal. '05 for the elliptic case.
- Yumi Cho, J. Evol. Equ. '18 & Fengping Yao, J. Differential Equations '19 for the parabolic case.

Below the duality exponent

- We return again to the case when $a(x,\xi) \approx |\xi|^{p-2}\xi$, and $g(t) = t^{p-1}$.
- The distributional formulation,

$$\int_\Omega a(x, Du)D\varphi \; dx = \langle \mu, \varphi \rangle \qquad (\varphi \in C_0^\infty(\Omega)),$$

  is well-defined even when $Du \in L^{p-1}$ and we may not have finite L^p-energy.
- This is the case below the duality exponent.

Below the duality exponent

- We consider again when the right hand side is in divergence form

$$\mu = \mathrm{div}(|F|^{p-2}F) \qquad (F \in L^q(\Omega) \text{ with } p-1 < q < p)\,.$$

- An issue is to show that

$$F \in L^q \Longrightarrow Du \in L^q.$$

$p - \delta \leq q$ with δ very small

- Iwaniec and Sbordone, J. Reine Angew. Math. '94: Hodge decomposition.
- John L. Lewis, Comm. Partial Differential Equations '93: Whitney extension theorem.
- Adimurthi and Phuc, Calc. Var. Partial Differential Equations '15 for a global estimate: Uniform p-thick complements.
- Bulicek, Diening and Schwarzacher, Anal. PDE '16 & Bulicek, and Schwarzacher, Calc. Var. Partial Differential Equations '16.
- Adimurthi and Byun, J. Math. Pures Appl. '19 for the parabolic case.

$\mathrm{div} a(x, Du) = \mu$

- When $\mu \in L^\gamma$ for $1 < \gamma < (p^*)'$, Mingione, Math. Ann. '10 for the p-Laplacian type.
- Chlebicka, Nonlinear Analysis 20 for the G-Laplacian type.
- Of course, there have many noteworthy works when μ is a bounded Radon measure or L^1-data.

$\operatorname{div} a(x, du) = \operatorname{div}\left(\frac{g(|F|)}{|F|}F\right)$

- There are few results when $F \in L^{G^{1-\delta}}$.
- References:
    - Adimurthi abd Phuc, Global Lorentz and Lorentz-Morrey estimates below the natural exponent for quasilinear equations. Calc. Var. Partial Differential Equations 54 (2015).
    - Baroni, Riesz potential estimates for a general class of quasilinear equations. Calc. Var. Partial Differential Equations 53 (2015).
    - Cianchi and Ferone, Hardy inequalities with non-standard remainder terms. Ann. Inst. H. Poincare Anal. Non Lineaire 25 (2008).
    - Diening, Malek and Steinhauer, On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications. ESAIM Control Optim. Calc. Var. 14 (2008).
    - Harjulehto and Hasto, Orlicz spaces and generalized Orlicz spaces. Lecture Notes in Mathematics, 2236. Springer, Cham, 2019. x+167 pp.

Maximal function

- Let $0 < \delta \leq \frac{1}{2}\delta_1$, being determined later.
- Let $u \in W_0^{1,G^{1-\delta}}(\Omega)$ be a very weak solution.
- Let $B_{2r} = B_{2r}(x_0) \subset\subset \Omega$ be a ball.
- Choose a standard cut-off function $\eta \in C_0^1(B_{2r})$ such that $\eta = 1$ on B_r, $0 \leq \eta \leq 1$ and $|D\eta| \leq \frac{2}{r}$.
- Set $\tilde{u} = (u - \overline{u}_{B_{2r}})\,\eta \in W_0^{1,G^{1-2\delta}}(B_{2r})$.
- For any $q \in (1-\delta_1, 1-2\delta]$, define $h(x) = M^{\frac{1}{q}}\left(G^q(|D\tilde{u}|)\right)(x)$ for $x \in \Omega$, where M is the Hardy-Littlewood maximal operator.
- Note that $\frac{1}{q} > 1$ and $h^{-\delta}$ is in the Muckenhoupt class $A_{\frac{1}{q}}$.

Lipschitz truncation

Lemma

Let $\lambda > 0$ *and set*

$$E_\lambda = E_\lambda(h, B_{2r}) = \{x \in B_{2r} : h(x) \leq \lambda\}.$$

Then there exists a Lipschitz truncation $\tilde{u}_\lambda$ *of* $\tilde{u}$ *such that*

- $\tilde{u}_\lambda = \tilde{u}$ *and* $D\tilde{u}_\lambda = D\tilde{u}$ *a.e. in* E_λ,
- $\tilde{u}_\lambda$ *has support within* E_λ, *and*
- $G(|D\tilde{u}_\lambda|) \leq c\lambda$ *a.e. in* $\mathbb{R}^n$ *and for some constant* $c(\delta_0, g_0, n) > 1$.

Estimates

Take $\tilde{u}_\lambda$ as a test function, multiply the resulting distributional formulation by $\lambda^{-(1+\delta)}$, and integrate with respect to λ, to discover

$$\underbrace{\int_0^\infty \lambda^{-(1+\delta)} \int_{E_\lambda} a(Du,x) D\tilde{u}_\lambda dx d\lambda}_{I_1}$$

$$= \underbrace{-\int_0^\infty \lambda^{-(1+\delta)} \int_{B_{2r}\setminus E_\lambda} a(Du,x) D\tilde{u}_\lambda dx d\lambda}_{I_2}$$

$$+ \underbrace{\int_0^\infty \lambda^{-(1+\delta)} \int_{E_\lambda} \frac{g(|F|)}{|F|} F D\tilde{u}_\lambda dx d\lambda}_{I_3}$$

$$+ \underbrace{\int_0^\infty \lambda^{-(1+\delta)} \int_{B_{2r}\setminus E_\lambda} \frac{g(|F|)}{|F|} F D\tilde{u}_\lambda dx d\lambda}_{I_4}.$$

Estimate of I_3

$$
\begin{aligned}
I_3 &\le \int_0^\infty \lambda^{-(1+\delta)} \int_{E_\lambda} g(|F|)|D\tilde{u}|\,dx\,d\lambda \\
&= \int_{B_{2r}} g(|F|)|D\tilde{u}| \int_{h(x)}^\infty \lambda^{-(1+\delta)}\,d\lambda\,dx \\
&= \frac{1}{\delta}\int_{B_{2r}} g(|F|)|D\tilde{u}|h^{-(1+\delta)}\,dx \\
&\le \frac{1}{\delta}\int_{B_{2r}} g(|F|)|D\tilde{u}|G^{-(1+\delta)}(|D\tilde{u}|)\,dx \\
&\lesssim \epsilon\frac{1}{\delta}\int_{B_{2r}} G^{1-\delta}(|Du|)\,dx \\
&+c(\epsilon)\frac{1}{\delta}\int_{B_{2r}} G^{1-\delta}(|F|)\,dx.
\end{aligned}
$$

Estimate of I_4

$$\begin{aligned}
I_4 &\leq \int_0^\infty \lambda^{-(1+\delta)} \int_{B_{2r}\setminus E_\lambda} g(|F|)|D\tilde{u}_\lambda| dx d\lambda \\
&= \int_{B_{2r}} g(|F|) \int_0^{h(x)} \lambda^{-(1+\delta)} |D\tilde{u}_\lambda| d\lambda dx \\
&\lesssim \int_{B_{2r}} g(|F|) \int_0^{h(x)} \lambda^{-(1+\delta)} G^{-1}(\lambda) d\lambda dx \\
&\lesssim \int_{B_{2r}} g(|F|) h^{-\delta} G^{-1}(h) dx \\
&\lesssim \int_{B_{2r}} G^{1-\delta}(|Du|) dx + \int_{B_{2r}} G^{1-\delta}(|F|) dx.
\end{aligned}$$

Estimate of I_2

$$
\begin{aligned}
I_2 &\le \int_0^\infty \lambda^{-(1+\delta)} \int_{B_{2r}\setminus E_\lambda} g(|Du|)|D\tilde{u}_\lambda| dx d\lambda \\
&= \int_{B_{2r}} g(|Du|) \int_0^{h(x)} \lambda^{-(1+\delta)} |D\tilde{u}_\lambda| d\lambda dx \\
&\lesssim \int_{B_{2r}} g(|Du|) \int_0^{h(x)} \lambda^{-(1+\delta)} G^{-1}(\lambda) d\lambda dx \\
&\lesssim \int_{B_{2r}} g(|Du|) h^{-\delta} G^{-1}(h) dx \\
&\lesssim \int_{B_{2r}} G^{1-\delta}(|Du|) dx.
\end{aligned}
$$

Estimate of I_1

Partition B_{2r} into B_r,
$D_1 = \{x \in B_{2r} \setminus B_r : M^{\frac{1}{q}}(G^q(|D\tilde{u}|)(x) \leq \delta M^{\frac{1}{q}}(G^q(|Du|))(x)\}$ and
$D_2 = \{x \in B_{2r} \setminus B_r : M^{\frac{1}{q}}(G^q(|D\tilde{u}|)(x) > \delta M^{\frac{1}{q}}(G^q(|Du|))(x)\}$ to see

$$\begin{aligned}\delta I_1 &= \int_{B_{2r}} a(x, Du) D\tilde{u} h^{-\delta} dx \\ &= \underbrace{\int_{B_r} a(x, Du) D\tilde{u} h^{-\delta} dx}_{I_{11}} \\ &\quad + \underbrace{\int_{D_1} a(x, Du) D\tilde{u} h^{-\delta} dx}_{I_{12}} + \underbrace{\int_{D_2} a(x, Du) D\tilde{u} h^{-\delta} dx}_{I_{13}} .\end{aligned}$$

Estimate of I_{11}

Recall $h(x) = M^{\frac{1}{q}}(G^q(|D\tilde{u}|)(x)$ and that $h^{-\delta}$ is in the Muckenhoupt class $A_{\frac{1}{q}}$. Consequently,

$$
\begin{aligned}
I_{11} &= \int_{B_r} a(x, Du) Du h^{-\delta} dx \\
&\geq \nu \int_{B_r} G(|Du|) h^{-\delta} dx \\
&\geq \nu \int_{B_r} M^{\frac{1}{q}}(G^q(|D\tilde{u}|) h^{-\delta} dx \\
&\gtrsim \int_{B_{\frac{r}{2}}} G^{1-\delta}(|Du|) dx \\
&\quad - r^n \left[\frac{1}{r^n} \int_{B_{2r}} G^q(|Du|) dx \right]^{\frac{1-\delta}{q}} .
\end{aligned}
$$

Estimates of I_{12} & I_{12}

- $$|I_{12}| \lesssim \delta^q \int_{B_{2r}} G^{1-\delta}(|Du|)dx.$$

- $$\begin{aligned} |I_{13}| &\lesssim \epsilon \int_{B_{2r}} G^{1-\delta}(|Du|)dx \\ &\quad + c(\epsilon) r^n \left[\frac{1}{r^n} \int_{B_{2r}} G^q(|Du|)dx \right]^{\frac{1-\delta}{q}}. \end{aligned}$$

Interior $G^{1-\delta}(|Du|)$ estimates

We combine all estimates to conclude that there exists a positive constant $c_* = c_*(n, \nu, L, \delta_0, g_0)$ such that

$$\int_{B_{\frac{r}{2}}} G^{1-\delta}(|Du|)dx \leq c_*(\epsilon + \delta^q) \int_{B_{2r}} G^{1-\delta}(|Du|)dx$$

$$+ [1 + c(\epsilon)]\, r^n \left[\frac{1}{r^n} \int_{B_{2r}} G^q(|Du|)dx \right]^{\frac{1-\delta}{q}}$$

$$+ c(\epsilon) \int_{B_{2r}} G^{1-\delta}(|F|)dx$$

First choose $\epsilon = \delta^q$ and then select a small constant δ so that

$$0 < 2c_* \delta^p < 1$$

to derive the desired estimate.

Related works

- Boundary $G^{1-\delta}(|Du|)$ estimates.
- Parabolic problems.