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# Regularity results for bounded minimizers

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# The Problem

Regularity results for local bounded minimizers of integral functionals of the type

$$\mathcal{F}(v,\Omega) = \int_{\Omega} f(x, Dv)\, dx \qquad\qquad \Omega \subset \mathbb{R}^n$$

in case

- *unconstrained* problem
- *constrained* problem

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In both cases the integrand $f$

- $\xi \to f(x,\xi)$ $\quad p$-growth
- can be discontinuous with respect to the $x$-variable.

- **R. G. & A. Passarelli di Napoli** *Regularity results for a priori bounded minimizers of non autonomous functionals with discontinuous coefficients* Adv. Calc. Var. **12** (2019)

- **M. Caselli, A. Gentile, R. G.** *Regularity results for solutions to obstacle problems with Sobolev coefficients.* J. Differential Equations **269** (2020)

# Assumptions

Let us consider

$$\mathcal{F}(v,\Omega) = \int_\Omega f(x, Dv)\,dx \qquad \text{(F)}$$

$\Omega$ open bounded set in $\mathbb{R}^n$, $n > 2$

- $v : \Omega \to \mathbb{R}^N$ $\qquad N \geq 2$
- $f : \Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}$ is a Carathéodory mapping satisfying

# ASSUMPTIONS W.R.T $\xi$-VARIABLE

there exist $p \geq 2$ and positive constants $L, \ell, \nu > 0$ s.t.

$$\frac{1}{L}|\xi|^p \leq f(x,\xi) \leq L(1+|\xi|^p). \tag{F1}$$

$$\langle D_\xi f(x,\xi) - D_\xi f(x,\eta), \xi - \eta \rangle \geq \nu(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}|\xi-\eta|^2 \tag{F2}$$

$$|D_\xi f(x,\xi) - D_\xi f(x,\eta)| \leq \ell(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}|\xi-\eta| \tag{F3}$$

for all $\xi, \eta \in \mathbb{R}^{n\times N}$ and for almost every $x \in \Omega$.

# ASSUMPTIONS W.R.T $x$-VARIABLE

There exists $g(x) \in L^{\sigma}(\Omega)$, $\sigma > 1$ s.t.

$$|D_\xi f(x,\xi) - D_\xi f(y,\xi)| \leq (|g(x)| + |g(y)|)|x-y|(1+|\xi|^2)^{\frac{p-1}{2}} \quad \text{(F4)}$$

for a.e. $x, y \in \Omega$ and for all $\xi \in \mathbb{R}^{n \times N}$.

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for a.e. $x, y \in \Omega$ and for all $\xi \in \mathbb{R}^{n\times N}$.

Assumption (F4) with $g \in \mathrm{L}^{\sigma}_{\mathrm{loc}}(\Omega)$ implies that

$$x \to D_\xi f(x,\xi) \in \mathrm{W}^{1,\sigma}_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times N})$$

(see Hajlasz, Potential Anal. **5** (1996))

(see Kristensen–Mingione, Arch. Ration. Mech. Anal. (2006)-Arch. Ration. Mech. Anal.(2010))

# Model case

$$\int_\Omega a(x)(1+|Du|^2)^{\frac{p}{2}}\,dx \qquad \text{with} \qquad a(x) \in L^\infty \cap W^{1,\sigma}(\Omega)$$

$p \geq 2$ and $\sigma > 1$

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$p \geq 2$ and $\sigma > 1$

Question:

How does the regularity of $a(x)$ transfer to $Du$?

# Unconstrained case

# ABOUT THE ASSUMPTION ON $x$-VARIABLE

**Classical Theory**

- $x \mapsto D_\xi f(x,\xi) \in Lip(\Omega)$

i.e. there exists a constant $K > 0$

$$|D_\xi f(x,\xi) - D_\xi f(y,\xi)| \leq K|x-y|(1+|\xi|^2)^{\frac{p-1}{2}}$$

$$\Downarrow$$

$$(1+|Du|^2)^{\frac{p-2}{4}} Du \in W^{1,2}$$

# Sobolev assumption

**More recent Developments**

- $x \mapsto D_\xi f(x,\xi) \in W^{1,n}$

i.e. there exists a non negative function $g \in L^n$ such that

$$|D_\xi f(x,\xi) - D_\xi f(y,\xi)| \le (|g(x)| + |g(y)|)|x-y|(1+|\xi|^2)^{\frac{p-1}{2}}$$

$$\Downarrow$$

**Higher differentiability results with integer order**

# $W^{1,n}$ assumption: Higher differentiability results with integer order

# $W^{1,n}$ ASSUMPTION: HIGHER DIFFERENTIABILITY RESULTS WITH INTEGER ORDER

**Beltrami Equations**

- Clop, Faraco, Mateu, Orobitg & Zhong - Publ. Mat. (2009)
  ( $n = 2$ and $A(x,\xi) = A(x) \cdot \xi$ with $\det A = 1$ )
  in connections with **planar mappings with finite distortion**

## $W^{1,n}$ ASSUMPTION: HIGHER DIFFERENTIABILITY RESULTS WITH INTEGER ORDER

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- Clop, Faraco, Mateu, Orobitg & Zhong - Publ. Mat. (2009) ( $n = 2$ and $A(x,\xi) = A(x) \cdot \xi$ with $\det A = 1$ ) in connections with **planar mappings with finite distortion**

**Systems and integral functionals**

- Passarelli di Napoli - Pot. Anal.(2014), Adv. Cal. Var.(2014) $\mathbf{p = n = 2 \quad 2 \leq p < n}$
- Giannetti & Passarelli di Napoli - Math. Z.(2015) **variable exponents**
- G. - J. Differential Equation (2015) $\mathbf{p = n > 2}$
- G. - NoDEA (2016) $\mathbf{Orlicz - Sobolev\ coefficients}$
- Cruz Uribe, Moen & Rodney - Ann. Math. Pura Appl.(2016) **Dirichlet problem**

## $W^{1,n}$ ASSUMPTION: HIGHER DIFFERENTIABILITY RESULTS WITH INTEGER ORDER

- Giannetti, Passarelli di Napoli & Scheven - J. Lond. Math. Soc. (2016) **parabolic case**- Proc. Roy. Soc. Edinburgh Sect. A (2019) **p-q growth**
- Cupini, Giannetti, G. & Passarelli di Napoli - J. Differential Equation (2018) **convexity only at** $\infty$
- Gentile - Adv. Calc. Var. (2020) **sub-quadratic growth**
- Capone & Radice - Journal of Elliptic and Parabolic Equations (2020) - preprint(2021)**lower order terms.**
- Cupini, Marcellini, Mascolo & Passarelli di Napoli , Preprint (2021) **degenerate ellipticity**

## Further Results in case of Sobolev coefficients

- Kristensen & Mingione - Arch. Ration. Mech. Anal. (2010)
- Kuusi & Mingione - J. Funct. Anal. (2012)
- Eleuteri, Marcellini & Mascolo
  - Ann. Mat. Pura Appl. (2016),
  - Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. (2016)
  - Discrete Contin. Dyn. Syst. (2019)
  - Adv. Calc. Var. (2020)
- Giannetti & Passarelli di Napoli J. Differential Equation (2015)
- Cupini, Giannetti, G. & Passarelli di Napoli Nonlinear Anal.(2017)
- De Filippis & Mingione, Preprint (2020)
- Clop, G., Hatami & Passarelli di Napoli Forum Math. (2020)
- Cupini, Marcellini, Mascolo & Passarelli di Napoli , Preprint (2021)

$W^{1,n}$

$$W^{1,n} \hookrightarrow VMO$$

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- Iwaniec & Sbordone J. Anal. Math. (1998)
- Kinnunen & Zhou Comm. Partial Differential Equations (1999)
- Bögelein, Duzaar, Habermann & Scheven, Proc. Lond. Math. Soc. (2011)
- Bögelein, J. Differential Equation (2012)
- Di Fazio, Fanciullo & Zamboni, Algebra i Analiz (2013)
- Goodrich & Ragusa , Nonlinear Anal (2019)
- Goodrich, Scilla & Stroffolini , Preprint (2021)

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- Di Fazio, Fanciullo & Zamboni, Algebra i Analiz (2013)
- Goodrich & Ragusa , Nonlinear Anal (2019)
- Goodrich, Scilla & Stroffolini , Preprint (2021)
- Balci, Diening, G. & Passarelli di Napoli preprint (2020)

Question:

What happens if we weaken the assumption on $g$?

# A PRIORI BOUNDED MINIMIZERS

Theorem. [ G.- Passarelli di Napoli (2019)]

Let $f : \Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}$ be an integrand satisfying the assumptions (F1)–(F4) for a function $g \in L^{p+2}_{\mathrm{loc}}(\Omega)$. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap L^{\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local minimizer of the functional $\mathcal{F}$, then

$$(1 + |Du|^2)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times N})$$

Moreover, for every balls $B_R \subset B_{2R} \subset \Omega$, we have that

$$\int_{B_R} \Big| D\Big( (1 + |Du|^2)^{\frac{p-2}{4}} Du \Big) \Big|^2 dx \leq c \int_{B_{2R}} (1 + |Du|^2)^{\frac{p}{2}} dx + c \int_{B_{2R}} |g(x)|^{p+2} dx,$$

where $c = c(\|u\|_\infty, R, p, n, N, L, \nu)$.

# REMARKS

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1. assumption on the summability of the function $g(x)$ that is independent of the dimension $n$.

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1. assumption on the summability of the function $g(x)$ that is independent of the dimension $n$.

2. this is a weaker assumption with respect to previous papers when $2 \leq p < n-2$

# Proof of the Theorem

**Step 1: The approximation.** We constract the approximating problems:

Fix a compact set $\Omega' \Subset \Omega$, and for a smooth kernel $\phi \in \mathbb{C}_c^\infty(B_1(0))$ with $\phi \geq 0$ and $\int_{B_1(0)} \phi = 1$, let us consider the corresponding family of mollifiers $(\phi_\varepsilon)_{\varepsilon>0}$. Put

$$g_\varepsilon = g * \phi_\varepsilon$$

and

$$f_\varepsilon(x,\xi) = \int_{B_1} \phi(\omega) f(x + \varepsilon\omega, \xi)\, \mathrm{d}\omega$$

on $\Omega'$, for each positive $\varepsilon < \operatorname{dist}(\Omega', \Omega)$.

Fix a real number $a \geq ||u||_{L^\infty(\Omega')}$ and, for $m > \frac{p}{2}$, let $u_{\varepsilon,m}$ be a minimizer to the functional

$$\mathfrak{F}_{\varepsilon,m}(v,\Omega') = \int_{\Omega'} \Big( f_\varepsilon(x, Dv) + \big(|v| - a\big)_+^{2m} \Big)$$

(Carozza – Kristensen – Passarelli di Napoli, Annales Inst. H. Poincaré (C) Non Linear Analysis , (2011))

# Proof of the Theorem

**Step 2: Uniform higher differentiability estimates (by using interpolation inequality)**

$$\tau_{s,h} u_{\varepsilon,m}(x) = u_{\varepsilon,m}(x + he_s) - u_{\varepsilon,m}(x)$$

Choosing $\varphi = \tau_{s,-h}(\rho^{p+2}\tau_{s,h} u_{\varepsilon,m})$ as test function in the Euler–Lagrange system associated to the functional $\mathfrak{F}_{\varepsilon,m}(v, \Omega')$ and using the assumptions and some properties of the difference quotients we obtain

$$\begin{aligned} \int_{B_{2R}} & |\tau_{s,h}(\rho^{\frac{p+2}{2}} V(Du_{\varepsilon,m}))|^2 \\ & \leq c|h|^2 \int_{B_{2R}} \rho^{p+2} \big(g_\varepsilon(x) + g_\varepsilon(x+h)\big)^2 (1 + |Du_{\varepsilon,m}|^2)^{\frac{p}{2}} \\ & \quad + c\frac{|h|^2}{R^2} \int_{B_{3R}} (1 + |Du_{\varepsilon,m}|^2)^{\frac{p}{2}}. \end{aligned}$$

# PROOF OF THE THEOREM

By a suitable interpolation inequality we have

$$Du_{\varepsilon,m} \in \mathrm{L}^{\frac{m}{m+1}(p+2)}$$

## Proof of the Theorem

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we can use Hölder's inequality with exponents $\frac{m}{m+1}\frac{p+2}{p}$ and $\frac{m(p+2)}{2m-p}$ to get

$$\begin{aligned}
&\int_{B_{2R}} |\tau_{s,h}(\rho^{\frac{p+2}{2}} V(Du_{\varepsilon,m}))|^2 \\
&\qquad \leq c|h|^2 \left(\int_{B_{2R}} \rho^{p+2}(g_\varepsilon(x) + g_\varepsilon(x+h))^{\frac{2m(p+2)}{2m-p}}\right)^{\frac{2m-p}{m(p+2)}} \\
&\qquad\quad \cdot \left(\int_{B_{2R}} \rho^{p+2}(1+|Du_{\varepsilon,m}|^2)^{\frac{m}{m+1}\frac{(p+2)}{2}}\right)^{\frac{m+1}{m}\frac{p}{p+2}}, \\
&\qquad + c\frac{|h|^2}{R^2} \int_{B_{3R}} (1+|Du_{\varepsilon,m}|^2)^{\frac{p}{2}}
\end{aligned}$$

# Proof of the Theorem

**Step 3:** we show that such estimates are preserved in passing to the limit.

# Systems under suitable structure assumptions

We consider elliptic systems of the form

$$\mathrm{div} A(x, Du) = \sum_{i=1}^{n} D_{x_i}\Big(\sum_{j=1}^{n} a_{ij}(x, Du) u^{\alpha}_{x_j}\Big) = 0,\ \ 1 \le \alpha \le N, \text{ in } \Omega \subset \mathbb{R}^n \ (*)$$

satisfying

$$A(x, 0) = 0 \qquad \text{(A0)}$$

$$\langle A(x,\xi) - A(x,\eta), \xi - \eta \rangle \ge \alpha |\xi - \eta|^2 (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \qquad \text{(A1)}$$

$$|A(x,\xi) - A(x,\eta)| \le \beta |\xi - \eta| (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \qquad \text{(A2)}$$

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There exists a nonnegative function $g \in L^{p+2}_{\mathrm{loc}}(\Omega)$, such that

$$|A(x,\xi) - A(y,\xi)| \le (g(x) + g(y))|x - y| \ (1 + |\xi|^2)^{\frac{p-1}{2}}; \qquad \text{(A3)}$$

for every $\xi \in \mathbb{R}^{n \times N}$ and for almost every $x, y \in \Omega$.

Theorem. [ G.- Passarelli di Napoli (2019)]

Let $A\colon \Omega\times\mathbb{R}^{N\times n}\to\mathbb{R}^{N\times n}$ be a Carathéodory function satisfying the assumptions (A0)–(A3). If $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a local solution of the system (*) , then

$$(1+|Du|^2)^{\frac{p-2}{4}}Du\in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^{n\times N})$$

Moreover, for every ball $B_r\Subset\Omega$

$$\int_{B_{r/4}}(1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2\,\mathrm{d}x\le\frac{c}{r^2}\int_{B_r}(1+|Du|^2)^{\frac{p}{2}}\,\mathrm{d}x$$
$$\frac{c}{r^n}||u||^p_{L^{p^*}(B_{2r})}\left(\int_{B_r}(1+g(x))^{p+2}\,\mathrm{d}x\right),$$

for a constant $c=c(\alpha,\beta,p,n)$.

# Proof of the Theorem

**Step 1** A priori estimate

- difference quotient method
- local boundedness of the solutions $u \in W^{1,p}_{\text{loc}}(\Omega)$ of the system and following estimate

$$\sup_{B_{\frac{R}{2}}(x_0)} |u| \leq c \left\{ \fint_{B_R(x_0)} (|u|+1)^{p^*}\, dx \right\}^{\frac{1}{p^*}}$$

(see Cupini, Marcellini & Mascolo,
Manuscripta Math. (2012) J. Optim. Theory Appl.(2015)-
Nonlinear Anal.(2017))
(see also Leonetti Boll. Un. Mat. Ital. (1991))

- interpolation inequality

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**Step 2** Approximation procedure

# REMARK

If is assumed a priori

$$u \in \mathrm{L}^q, \quad \text{with} \quad q > \frac{np}{n-p-2} \qquad (\text{instead of} \quad u \in \mathrm{L}^\infty)$$

the interpolation inequality gives

$$Du \in \mathrm{L}^{\frac{q}{q+2}(p+2)} \qquad (\text{instead of} \quad Du \in \mathrm{L}^{p+2})$$

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the interpolation inequality gives

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Such higher integrability allow us to obtain the same higher differentiability result assuming $g \in \mathrm{L}^{\frac{q}{q-p}(p+2)}$.

We'd like to point out that for $p < n-2$ it results $\frac{q}{q-p}(p+2) < n$.

# Constrained case

# Obstacle Problem

We consider the following *obstacle problem*

$$\min\left\{\int_\Omega f(x,Dv(x))\;:\;v\in\mathcal{K}_\psi(\Omega)\right\}, \tag{1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open set,

- $\psi:\Omega\mapsto[-\infty,+\infty)$ belonging to $W^{1,p}_{\mathrm{loc}}$ is the *obstacle*,

- $\mathcal{K}_\psi(\Omega)=\{v\in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}):v\geq\psi \text{ a.e. in } \Omega\}$ is the class of the admissible functions.

# Obstacle problems and variational formulation

We observe that

$$u \in W^{1,p}_{\mathrm{loc}}(\Omega) \text{ is a solution to the obstacle problem in } \mathcal{K}_\psi$$

$$\Updownarrow$$

$$u \in \mathcal{K}_\psi(\Omega) \text{ is a solution to the variational inequality}$$

$$\int_\Omega \langle A(x, Du), D(\varphi - u)\rangle \, dx \geq 0 \qquad \forall \varphi \in \mathcal{K}_\psi(\Omega),$$

where $A(x,\xi) = D_\xi f(x,\xi)$.

# REGULARITY

It is well known that:

*the regularity of solutions to the obstacle problems depends on the regularity of the obstacle itself*

Analysis of the extra differentiability of the solutions of the obstacle problems

$$\int_\Omega \langle A(x, Du(x)), D(\varphi(x) - u(x))\rangle \, dx \geq 0 \qquad \forall \varphi \in \mathcal{K}_\psi(\Omega),$$

assuming that the gradient of the obstacle $D\psi$ has some differentiability property

# ASSUMPTIONS

Let us fix $\psi \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and consider

$$\int_\Omega \langle A(x, Du), D(\varphi - u)\rangle \, dx \geq 0, \qquad (**)$$

for every $\varphi \in \mathcal{K}_\psi(\Omega) = \{v \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}) : v \geq \psi \text{ a.e. in } \Omega\}$

There exist constants $\nu, L > 0$ and an exponent $p \geq 2$ such that

$$\langle A(x,\xi) - A(x,\eta), \xi - \eta\rangle \geq \nu|\xi - \eta|^2(1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \qquad \text{(A1)}$$

$$|A(x,\xi) - A(x,\eta)| \leq L|\xi - \eta|(1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \qquad \text{(A2)}$$

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There exists a nonnegative function $g \in L^{p+2}_{\mathrm{loc}}(\Omega)$, such that

$$|A(x,\xi) - A(y,\xi)| \leq (g(x) + g(y))|x - y| \, (1 + |\xi|^2)^{\frac{p-1}{2}}; \qquad \text{(A3)}$$

for all $\xi, \eta \in \mathbb{R}^n$ and for almost every $x, y \in \Omega$.

# Remark

The regularity of the solutions to the obstacle problem (**) is strictly connected to the regularity of the solutions to PDE's of the form

$$\mathrm{div}A(x, Du) = \mathrm{div}A(x, D\psi).$$

It is well known that no extra differentiability properties for the solutions of equations of the type

$$\mathrm{div}A(x, Du) = \mathrm{div}G$$

can be expected even if $G$ is smooth, unless some assumption is given on the $x$-dependence of the operator $A$.

# Some Results

$$x \mapsto A(x, \xi) \in W^{1,r} \quad \text{with} \quad r \geq n$$

- Eleuteri & Passarelli di Napoli - Calc. Var. Partial Differential Equations.(2018) - Nonlinear Anal. (2020)
- Gavioli - Forum Math. (2019)
- Ma & Zhang - J. Math. Anal. Appl. (2019)
- De Filippis - J. Math. Anal. Appl. (2019)
- Chlebicka& De Filippis - Ann. Mat. Pura Appl. (2019)
- De Filippis & Mingione - (2020)
- Gentile - Forum Math. (2021)

Theorem. [Caselli – Gentile – G.(2020)]

Let $A(x,\xi)$ satisfy the conditions (A1)–(A4) for an exponent $p \geq 2$ and let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to the obstacle problem. Then, if $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$ the following implication holds

$$D\psi \in W^{1,\frac{p+2}{2}}_{\mathrm{loc}}(\Omega) \ \Rightarrow\ \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega),$$

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Remark: the assumption $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$ is needed to get the boundedness of the solution. Therefore if we deal with a priori bounded minimizers, then the result holds without the hypothesis $\psi \in L^\infty$.

(see Caselli – Eleuteri – Passarelli di Napoli, ESAIM - Control. Optim. Calc. Var. (2021))

# Proof of the Theorem

- A priori estimate
- Approximation procedure

# Test functions

The main point is the choice of suitable test functions $\varphi$:

1. involving the difference quotient of the solution
2. belonging to the class of the admissible functions $\mathcal{K}_\psi(\Omega)$,

Let us consider $\varphi := u + \tau v$ for a suitable $v \in W_0^{1,p}(\Omega)$ such that

$$u - \psi + \tau v \geq 0 \qquad \forall \tau \in [0,1], \qquad (***)$$

Then $\varphi \in \mathcal{K}_\psi(\Omega)$ for all $\tau \in [0,1]$, since $\varphi = u + \tau v \geq \psi$.

# Test functions

Let $\eta$ be a cut off function, we consider

$$v_1(x) = \eta^2(x)\left[(u-\psi)(x+h) - (u-\psi)(x)\right],$$

$v_1$ satisfies (***). Indeed, for a.e. $x \in \Omega$ and for any $\tau \in [0,1]$

$$\begin{aligned} & u(x) - \psi(x) + \tau v_1(x) = \\ = \quad & u(x) - \psi(x) + \tau\eta^2(x)\left[(u-\psi)(x+h) - (u-\psi)(x)\right] \\ = \quad & \tau\eta^2(x)(u-\psi)(x+h) + (1-\tau\eta^2(x))(u-\psi)(x) \geq 0, \end{aligned}$$

since $u \in \mathcal{K}_\psi(\Omega)$ and $0 \leq \eta \leq 1$.
So we can use $\varphi = u + \tau v_1$ as a test function in variational inequality.

# TEST FUNCTIONS

In a similar way, we consider

$$v_2(x) = \eta^2(x)\left[(u-\psi)(x-h) - (u-\psi)(x)\right],$$

and we have (***) still is satisfied for any $\tau \in [0,1]$, since

$$\begin{aligned} & u(x) - \psi(x) + \tau v_2(x) = \\ = \quad & u(x) - \psi(x) + \tau\eta^2(x)\left[(u-\psi)(x-h) - (u-\psi)(x)\right] \\ = \quad & \tau\eta^2(x)(u-\psi)(x-h) + (1-\tau\eta^2(x))(u-\psi)(x) \geq 0. \end{aligned}$$

So we can use $\varphi = u + \tau v_2$ as a test function in variational inequality .

Thanks for your attention!