Minty transform
Problem
Musielak–Orlicz space
We come back to the problem
Results

On renormalized solutions to elliptic inclusions with nonstandard growth

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Talk based on the paper (under the same title: "On renormalized solutions to elliptic inclusions with nonstandard growth") with A. Denkowska and P. Gwiazda.

To appear in Calculus of Variations and Partial Differential Equations, doi: 10.1007/s00526-020-01893-4.

Also availabie on arxiv. https://arxiv.org/abs/1912.12729

Monotone and maximally monotone multifunctions

Definition (Monotone multifunction)

The multifunction $a : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ is monotone if for every $\mu_1, \mu_1 \in \mathbb{R}^d$ and every $\lambda_1 \in a(\mu_1)$, $\lambda_2 \in a(\mu_2)$ there holds

$$(\lambda_1 - \lambda_2) \cdot (\mu_1 - \mu_2) \geqslant 0.$$

Definition (Maximal monotone multifunction)

The monotone multifunction $a : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ is maximal monotone if and only if whenever $(\mu, \lambda) \in \mathbb{R}^d \times \mathbb{R}^d$ is such that

$$(\lambda - \lambda_1) \cdot (\mu - \mu_1) \geqslant 0 \text{ for every } \mu_1 \in \mathbb{R}^d, \lambda_1 \in a(\mu_1),$$

then $\lambda \in a(\mu)$.

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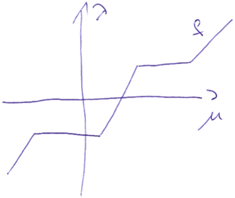
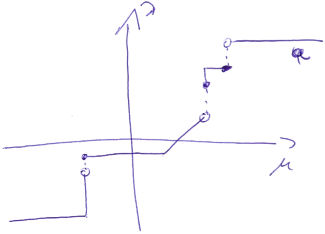


Monotone function

$\lambda_1 = f(\mu_1)$

$\lambda_2 = f(\mu_2)$

$(\lambda_1 - \lambda_2)\cdot(\mu_1 - \mu_2) \geq 0$



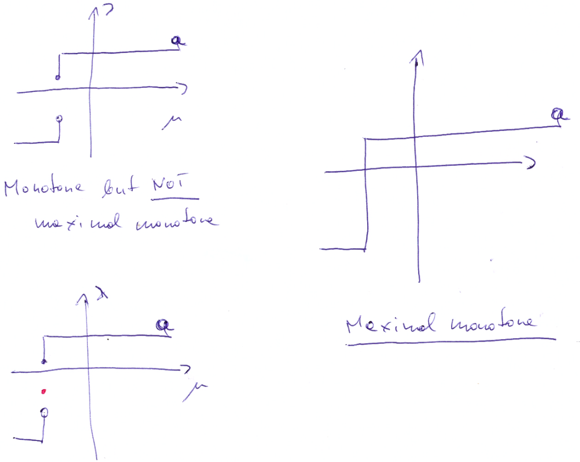
Monotone multifunction

$\lambda_1 \in \varphi(\mu_1)$

$\lambda_2 \in \varphi(\mu_2)$

$(\lambda_1 - \lambda_2)(\mu_1 - \mu_2) \geq 0$

Monotone and maximally monotone multifunctions



Minty transformation

Let $a : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ be a multifunction. We define another multifunction $\varphi : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ by the formula.

$$e \in \varphi(d) \Leftrightarrow \text{ there exists } \mu \in \mathbb{R}^d, \lambda \in a(\mu) : \mu+\lambda = d, \mu-\lambda = e.$$

Minty transform of monotone multifunction

If a is monotone then φ is a 1-Lipschitz function.

Proof. $e_1 \in \varphi(d_1), e_2 \in \varphi(d_2)$. Denote corresponding μ and λ by $\mu_1, \lambda_1, \mu_2, \lambda_2$.

$$|(\mu_1-\mu_2)+(\lambda_1-\lambda_2)|^2-|(\mu_1-\mu_2)-(\lambda_1-\lambda_2)|^2 = 4(\mu_1-\mu_2)\cdot(\lambda_1-\lambda_2).$$

$$|d_1 - d_2|^2 \geqslant |e_1 - e_2|^2$$

$$|e_1 - e_2| \leqslant |d_1 - d_2|$$

Minty transform of maximal monotone multifunction

If a is maximal monotone then φ is a 1-Lipschitz function with $\operatorname{dom}\varphi = \mathbb{R}^d$.

Proof. Suppose $d \notin \operatorname{dom}\varphi$. By Kirszbraun theorem extend φ to 1-Lipshitz function $\widetilde{\varphi}$ defined on whole $\mathbb{R}^d$. Denote $e = \widetilde{\varphi}(d)$. Calculate

$$\mu + \lambda = d, \mu - \lambda = e.$$

If $\lambda_1 \in a(\mu_1)$, then

$$|(\lambda_1 - \mu_1) - (\lambda - \mu)| = |\varphi(\lambda_1 + \mu_1) - \widetilde{\varphi}(\lambda + \mu)|$$

$$= |\widetilde{\varphi}(\lambda_1 + \mu_1) - \widetilde{\varphi}(\lambda + \mu)| \leqslant |(\lambda_1 + \mu_1) - (\lambda + \mu)|.$$

$$|(\lambda_1 - \lambda) - (\mu_1 - \mu)| \leqslant |(\lambda_1 - \lambda) + (\mu_1 - \mu)|.$$

$$4(\lambda_1 - \lambda) \cdot (\mu_1 - \mu) \geqslant 0.$$

Which contradicts maximality.

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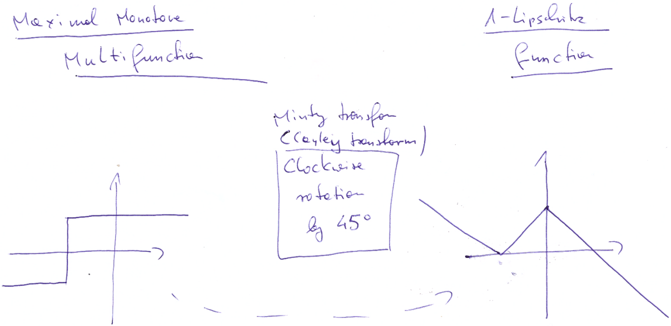
Maximal Monotone Multifunction
1-Lipschitz function
Minty transform
(Cayley transform)
Clockwise rotation by 45°

Some literature on Minty transform

[1] George J. Minty, Monotone (nonlinear) operators in Hilbert space. Duke Math. J. 29 1962, 341–346.
[2] R. Tyrrell Rockafellar, Roger J.-B. Wets, Variational analysis, Springer 2009 (3rd ed) Minty parameterization.
[3] Giovanni Alberti, Luigi Ambrosio, A geometrical approach to monotone functions in R^n. Math. Z. 230 (2) 1999, 259–316.
Cayley transformation.
[4] Gilles Francfort, Francois Murat, Luc Tartar, Monotone operators in divergence form with x–dependent multivalued graphs, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8), 7 (2004), 23–59.
[5] Piotr Gwiazda, Anna Zatorska-Goldstein, On elliptic and parabolic systems with x-dependent multivalued graphs, Mathematical Methods in the Applied Sciences 30 (2), 213-236.
(concept: one needs continuous nonlinearities to work with Young measures)

Elliptic inclusion

Problem under consideration

Let

- $\Omega \subset \mathbb{R}^d$ be a bounded domain with sufficiently smooth boundary,
- $f \in L^1(\Omega)$ be a function,
- $A : \Omega \times \mathbb{R}^d \to 2^{\mathbb{R}^d}$ be a multifunction such that $A(x, \cdot)$ is maximally monotone for a.e. x (+nonstandard growth! and measurability w.r. to x)

We want to find the function $u : \Omega \to \mathbb{R}$ with $u = 0$ on $\partial\Omega$ such that

$$-\mathrm{div}\, A(x, \nabla u(x)) \ni f(x).$$

Find function u and the selection $\eta(x) \in A(x, \nabla u(x))$ a.e. x such that in appropriate weak sense

$$-\mathrm{div}\, \eta(x) = f(x).$$

Results

1. Existence of solution understood in renormalized sense (as $f \in L^1(\Omega)$).
2. Uniqueness of renormalized solution if A is in addition strictly monotone.
3. If f satisfies appropriate Orlicz type regularity assumption then renormalized solution $u \in L^\infty(\Omega)$ (and hence it is also a weak solution because we can drop $h \in C_c^1(\mathbb{R})$ from the definition of renormalized solution).

[3.] and in part [2.] are obtained for those renormalized solutions which are limits of the approximation procedure used in the proof of existence.

Problem setup

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A. We assume nonstandard, Musielak–Orlicz type growth and coercivity conditions on A which leads to ∇u (or more, precisely gradients of truncations) in some Musielak–Orlicz space.

B. To get optimal "L^p type" Musielak–Orlicz space to which f should belong (in order to work with the weak solution) one needs to know the dual of the "best L^p type" space in which the "$W^{1,p}$ -type" Musielak–Orlicz–Sobolev space embeds.

C. To avoid answering this (difficult) question we choose $f \in L^1$ and we work with well established notion of renormalized solutions.

D. It appears to us that there were no previous works on renormalized solutions for problems with multivalued leading term. So we define the correct notion of such solution.

N-function

The function $M : \Omega \times \mathbb{R}^d \to [0,\infty)$ is an N-function if

(N1) M is Carathéodory, that is, $M(\cdot,\xi)$ is measurable for every $\xi \in \mathbb{R}^d$ and $M(x,\cdot)$ is continuous for almost every $x \in \Omega$,

(N2) $M(x,\xi) = M(x,-\xi)$ for every $\xi \in \mathbb{R}^d$ a.e. in Ω and $M(x,\xi) = 0$ is and only if $\xi = 0$ a.e. in Ω,

(N3) $M(x,\cdot)$ is convex for almost every $x \in \Omega$,

(N4) Growth of M in ξ at zero and infinity, that is,

$$\lim_{|\xi|\to 0} \operatorname*{ess\,sup}_{x\in\Omega} \frac{M(x,\xi)}{|\xi|} = 0 \quad \text{and} \quad \lim_{|\xi|\to\infty} \operatorname*{ess\,inf}_{x\in\Omega} \frac{M(x,\xi)}{|\xi|} = \infty.$$

(N5) $\operatorname{ess\,inf}_{x\in\Omega} \inf_{|\xi|=s} M(x,\xi) > 0$ for every $s \in (0,\infty)$ and $\operatorname{ess\,sup}_{x\in\Omega} M(x,\xi) < \infty$ for every $\xi \neq 0$.

Assuming (N1)-(N4), (N5) is equivalent to existence of one dimensional N-functions m_1, m_2 such that $m_1(|\xi|) \leqslant M(x,\xi) \leqslant m_2(|\xi|)$.

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One extra assumption

One of the following two assumptions holds

Either
(C1) The Fenchel conjugate $\widetilde{M}$ of M satisfies the Δ_2 condition. (which implies that $(L_{\widetilde{M}}(\Omega))^* = (E_{\widetilde{M}}(\Omega))^* = L_M(\Omega)$ and this allows us to "work with" weak-* convergence in L_M)

or

(C2) "Balance condition" (on dependance of M on x) of [1,2] which guarantees modular density of $C_0^\infty(\Omega)$ functions in $L_M(\Omega)$ hold (and hence it is enough to have $(E_{\widetilde{M}}(\Omega))^* = L_M(\Omega)$ to deal with weak-* convergence in L_M), cf.

[1] I. Chlebicka, P. Gwiazda, and A. Zatorska-Goldstein, Parabolic equation in time and space dependent anisotropic Musielak– Orlicz spaces in absence of Lavrentievs phenomenon, Annales de l'Institut Henri Poincaré, C, Analyse non linéaire 36 (2019), 1431–1465.

[2] I. Chlebicka, P. Gwiazda, and A. Zatorska-Goldstein, Renormalized solutions to parabolic equations in time and space dependent anisotropic Musielak–Orlicz spaces in absence of Lavrentievs phenomenon, J. Differ. Equations 267 (2019), 1129–1166.

Extra assumptions on M - comments

- We nowhere assume that both $M, \widetilde{M}$ satisfy Δ_2. So we work in nonreflexive and nonseparable spaces.
- If $\widetilde{M}$ does not satisfy Δ_2 then our approach works if (C2) holds.
  Condition (C2) covers:
  - Any pure Orlicz setting.
  - Variable exponent spaces with log-Hölder condition.
  - Double phase spaces with optimal closeness condition.

  Condition (C1) covers:
  - Variable exponent spaces without log-Hölder condition.
  - Double phase spaces without optimal closeness condition.

Examples of covered cases

- Pure isotropic Orlicz

$$M(x,\xi) = |\xi| \ln(1+|\xi|),$$
$$M(x,\xi) = |\xi|(\exp|\xi| - 1).$$

- Anisotropic Orlicz

$$M(x,\xi) = |\xi| \ln(1+|\xi|),$$
$$M(x,\xi) = |\xi|(\exp|\xi| - 1),$$
$$M(x,\xi) = \sum_{i=1}^{d} B_i(\xi_i),$$

- Variable exponent without log-Hölder

$$M(x,\xi) = |\xi|^{p(x)}, 1 \ll p \ll \infty$$

- Doubling (with a touching zero) without optimal closeness

$$M(x,\xi) = |\xi|^p + a(x)|\xi|^q, \quad 1 < p < q < \infty,$$
$$M(x,\xi) = |\xi|^p + a(x)|\xi|^p \ln(e+|\xi|), \quad 1 < p < q < \infty,$$

- Combination of the above.

Multivalued map

$A : \Omega \times \mathbb{R}^d \to 2^{\mathbb{R}^d}$ satisfies.

(A1) A is measurable with respect to the σ-algebra $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^d)$ on its domain $\Omega \times \mathbb{R}^d$ and the σ-algebra $\mathcal{B}(\mathbb{R}^d)$ on its range. Here $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra and $\mathcal{L}(\Omega)$ is the Lebesgue σ-algebra.

(A2) the multivalued map $A(x, \cdot)$ is maximally monotone for a.e. $x \in \Omega$.

[1] V. Chiado'Piat, G. Dal Maso, and A. Defrancheschi, G-convergence of monotone operators, Annales de l'H.P., section C 7 (1990), 123–160.

Coercivity and growth

(A3) there exists an N-function M and a nonnegative function $m \in L^1(\Omega)$ such that

$$\eta \cdot \xi \geqslant M(x,\xi) + \widetilde{M}(x,\eta) - m(x).$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^d, \eta \in A(x,\xi)$.

Encompasses growth and coercivity in one condition. Weaker then (in doubling case equivalent) to

$$c_A M(x,\xi) - m_A(x) \leqslant \eta \cdot \xi,$$
$$\widetilde{M}(x,\eta) \leqslant c_G M(x,\xi) + m_G(x),$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^d, \eta \in A(x,\xi)$.

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Definition of renormalized solution

$$V_0^M = \{v \in W_0^{1,1}(\Omega) \ : \ \nabla v \in L_M(\Omega)\},$$

$$T_k f(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| \leqslant k, \\ k\frac{f(x)}{|f(x)|} & \text{otherwise.} \end{cases}$$

Definition

1. For every $k > 0$ there holds $T_k(u) \in V_0^M \cap L^\infty(\Omega)$.
2. There exists a measurable selection $\alpha : \Omega \to \mathbb{R}^d$ of $A(\cdot, \nabla u(\cdot))$ such that for any $h \in C_c^1(\mathbb{R})$ and for any test function $w \in W_0^{1,\infty}(\Omega)$ there holds

$$\int_\Omega \alpha \cdot \nabla(h(u)w)\, dx = \int_\Omega fh(u)w\, dx.$$

3. There holds

$$\lim_{k\to\infty} \int_{\{k<|u(x)|<k+1\}} \alpha \cdot \nabla u\, dx = 0.$$

Renormalized solutions

We follow and generalize (to our knowledge for the first time to case of inclusion with multivalued leading term), now standard, framework of

[1] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vazquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze 22 (1995), 241–273.

- The generalized gradient of u such that $T_k u \in V_0^M$ is a measurable function $v : \Omega \to \mathbb{R}^d$ such that $v\chi_{\{|v|<k\}} = v\chi_{\{|v|\leqslant k\}} = \nabla T_k(u)$ for almost every $x \in \Omega$ for each $k > 0$.
- The selection $\alpha : \Omega \to \mathbb{R}^d$ is a measurable function such that for every $k > 0$ there exists the selection $\alpha_k \in L_{\widetilde{M}}(\Omega)$ of the multifunction $A(\cdot, \nabla T_k u)$ such that $\alpha_k \chi_{\{|u|<k\}} = \alpha\chi_{\{|u|<k\}}$.

Theorem 1 - existence

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Existence result

Suppose that an N-function M satisfies either (C1) or (C2). If $f \in L^1(\Omega)$ and A satisfies (A1)–(A3) then a renormalized solution exists.

Proof. 1. Construct auxiliary problem governed by

$$-\operatorname{div} a^\epsilon(x, \nabla u_\epsilon) = T_{1/\epsilon} f \qquad \text{in} \qquad \Omega,$$
$$u_\epsilon(x) = 0 \qquad \text{on} \qquad \partial\Omega.$$

2. Compactness method and Minty trick similar as in [1].
3. In the course of the proof we must show equiintegrability of $a^\epsilon(x, \nabla T_k u_\epsilon) \cdot \nabla T_k u_\epsilon$ w.r. to ϵ. To get this we need compactness theorem on Young measures, which uses the fact that $L^\infty_w(\Omega; \mathcal{M}(\mathbb{R}^d))$ (mappings which are weak-* measurable and essentialy bounded in $\mathcal{M}(\mathbb{R}^d)$) is $(L^1(\Omega; C_0(\mathbb{R}^d)))^*$. Nonlinearity must be continuous. We achieve this by the Minty transform.

[1] P. Gwiazda, I. Chlebicka, A. Zatorska-Goldstein, Existence of renormalized solutions to elliptic equation in Musielak-Orlicz space, J. Differ. Equations 264 (1) (2018), 341-377.

Theorem 2 - uniqueness

Theorem

Assume, in addition, that A is strictly monotone, i.e. if $\xi \neq \eta$, then for every $g \in A(x, \xi), h \in A(x, \eta)$ and a.e. $x \in \Omega$ there holds $(g - h) \cdot (\xi - \eta) > 0$.

- *If M satisfies (C1) ($\widetilde{M}$ satisfies Δ_2) then the renormalized solution to the problem is unique in the class of solutions obtained as the limit as $\epsilon \to 0$ of solutions of the approximative problems.*
- *If an N-function M satisfies (C2) (modular density of smooth functions) then the renormalized solution is unique.*

Proof. Test by $T_k(T_l u_1 - T_l u_2)$.

Theorem 3 - boundedness

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Theorem

Assume, (in addition to assumptions of Theorem 1)

(W1) *There exists a constant $\lambda > 1$ such that*

$$\int_0^{|\Omega|} s^{\frac{1}{d}-1} \Psi_{\blacklozenge}^{-1}\left(\frac{\lambda}{d\omega_d^{\frac{1}{d}}} s^{\frac{1}{d}} f^{**}(s)\right) ds < \infty,$$

where ω_d is the Lebesgue measure on one dimensional unit ball in $\mathbb{R}^d$, i.e., $\omega_d = \pi^{d/2}/\Gamma(1+\frac{d}{2})$.

(W2) *The function m in corecivity/growth condition $\eta \cdot \xi \geqslant M(x,\xi) + \widetilde{M}(x,\eta) - m(x)$ belongs to $L^\infty(\Omega)$.*

Then every renormalized solution u obtained as the limit of solutions to approximative problems belongs to $L^\infty(\Omega)$.

Comment. In such a case we can drop $h \in C_0^1(\mathbb{R})$ from the definition of renormalized solution and renormalized solutions are also weak.

Assumption (W1)

Assumption

$$\int_0^{|\Omega|} s^{\frac{1}{d}-1}\Psi_{\blacklozenge}^{-1}\left(\frac{\lambda}{d\omega_d^{\frac{1}{d}}}s^{\frac{1}{d}}f^{**}(s)\right)\,ds<\infty,$$

is the Orlicz type regularity requirement on f in spirit of

[1] A. Cianchi, Symmetrization in anisotropic elliptic problems, Comm. Partial Differential Equations 32 (2007), 693–717.
[2] A. Alberico, I. Chlebicka, A. Cianchi, and A. Zatorska-Goldstein, Fully anisotropic elliptic problems with minimally integrable data, Calc. Var. PDEs 58 (2019), 186.

It is sharp in (anisotropic) Orlicz setting. Proof follow by the the concept from [1] i.e. the symmetrization method. f^{**} is the maximal rearrangement of f, and

$$|\{\xi\in\mathbb{R}^d\,:\,L_\circ(|\xi|)\leqslant t\}|=|\{\xi\in\mathbb{R}^d\,:\,L(\xi)\leqslant t\}|,\;\;L_\bigstar(\xi)=L_\circ(|\xi|).$$

$$m_1(|\xi|)\leqslant M_1(\xi)\leqslant M(x,\xi),\quad (M_1)_\blacklozenge(|\xi|)=\left(\widetilde{(\widetilde{M_1})_\bigstar}\right)(\xi)$$

$$\Psi_\blacklozenge(s)=\frac{(M_1)_\blacklozenge(s)}{s}.$$

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Thank you!!!