Orlicz spaces and generalized Orlicz spaces

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Abstract

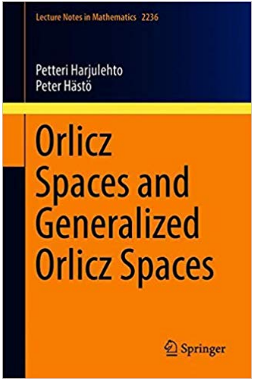
Generalized Orlicz spaces include as special cases a wide range of function spaces, such as Lebesgue space, Orlicz spaces, variable exponent spaces, double phase spaces and logarithmic perturbations of the aforementioned. Working in generalized Orlicz spaces involves some operations such as splicing the Orlicz functions that are not commonplace in the traditional Orlicz setting. In this talk, I explain some extensions to the Orlicz space theory which enable these operations and show that they may be useful even when there in the non-generalized Orlicz case, sometimes even yielding new results for classical Lebesgue spaces.

https://sites.google.com/site/varexpspa/
http://cc.oulu.fi/~phasto/

Objective of this talk

In this talk I

1. Motivate the study of generalized Orlicz spaces.
2. Explain and motivate assumptions from our recent book.
3. Illustrate how the techniques can be applied to the study of PDE.



LNM 2236 (2019)

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Motivation/Background

Generalized Orlicz spaces cover almost everything

Lebesgue spaces	L^p	t^p
Zygmund spaces	$L^p \log L^q$	$t^p \log(e+t)^q$
Exponential spaces	$\exp L$	$e^t - t - 1$
Orlicz spaces	L^φ	$\varphi(t)$
Weighted Lebesgue spaces	L^p_w	$t^p w(x)$
Variable exponent spaces	$L^{p(\cdot)}$	$t^{p(x)}$
Double phase spaces (DPS)	L^H	$t^p + a(x)t^q$
Generalized Orlicz spaces	L^φ	$\varphi(x,t)$

Generalized Orlicz spaces were studied in since the 60s e.g. by H. Hudzik, A. Kamińska and J. Musielak

Harmonic analysis was studied by L. Diening (2005) and F.-Y. Maeda, Y. Mizuta, T. Ohno and T. Shimomura from 2013

Relationship to calculus of variations

M. Giaquinta and E. Giusti (1983, 1984) studied regularity of minimizers

$$\min_{u \in W^{1,p}} \int_\Omega F(x, \nabla u)\, dx$$

when $F : \Omega \times \mathbb{R}^n \to [0, \infty)$ has p-type growth:

$$\begin{cases} z \mapsto F(x,z) \text{ is } C^2, \\ \nu |z|^p \leqslant F(x,z) \leqslant L(1 + |z|^p), \\ \nu(\mu^2 + |z|)^{\frac{p-2}{2}} |\lambda|^2 \leqslant F_{zz}(x,z)\lambda \cdot \lambda \leqslant L(\mu^2 + |z|^2)^{\frac{p-2}{2}} |\lambda|^2, \\ |F(x,z) - F(y,z)| \leqslant \omega(|x-y|)(1 + |z|^p). \end{cases}$$

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when $F : \Omega \times \mathbb{R}^n \to [0, \infty)$ has p-type growth:

$$\begin{cases} z \mapsto F(x,z) \text{ is } C^2, \\ \nu|z|^p \leqslant F(x,z) \leqslant L(1+|z|^{q}), \\ \nu(\mu^2+|z|)^{\frac{p-2}{2}}|\lambda|^2 \leqslant F_{zz}(x,z)\lambda\cdot\lambda \leqslant L(\mu^2+|z|^2)^{\frac{q-2}{2}}|\lambda|^2, \\ |F(x,z)-F(y,z)| \leqslant \omega(|x-y|)(1+|z|^{q}). \end{cases}$$

P. Marcellini (1989, 1991) introduced **non-standard growth** assumptions with different exponents $p < q$ on the left- and right-hand sides.

(p,q)-growth is still an active field, cf. Bella–Hirsch–Schäffner (2020)
De Filippis–Mingione (2020), Eleuteri–Mascolo–Marcellini (2020)

Famous non-standard growth papers

V. Zhikov (1986, 1992) introduced three elasticity-related energies of the form

$$\inf \int_\Omega \varphi(x, |\nabla u|)\, dx.$$

I. Perturbed Orlicz: $a(x)\varphi_0(t)$, where $0 < \nu \leqslant a \leqslant L$.

II. Variable exponent: $t^{p(x)}$, where $1 < p^- \leqslant p \leqslant p^+$.

III. Double phase: $t^p + a(x)t^q$, where $1 < p \leqslant q$ and $a \geqslant 0$.

They have been very influential. G. Mingione (2006) stated:

> *[R]egularity results should be chased [in more general cases] by looking at special classes of functionals and thinking of relevant model examples, thereby limiting the degree of generality one wants to achieve.*

Non-standard growth special cases

Variable exponent spaces $t^{p(x)}$	1000s of papers...
Perturbed variable exponent spaces $t^{p(x)} \log(e+t)^{q(x)}$	Giannetti, Passarelli di Napoli, Liang, Cai, Zheng, Ok
Orlicz variable exponent $\psi(t)^{p(x)}$ or $\psi(t^{p(x)})$	Capone, Cruz-Uribe, Fiorenza, Giannetti, Passarelli di Napoli, Ragusa, Tachikawa
Double variable exponent $t^{p(x)} + t^{q(x)}$	Cencelj, Rădulescu, Repovš, Shi, Zhang
Double phase spaces (DPS) $t^p + a(x)t^q$	Zhikov, Baroni, Colombo, Mingione, ...
Degenerate DPS $t^p + a(x)t^p \log(e+t)$	Baroni, Colombo, Mingione, Byun, Oh, Coscia, Balci, Surnachev
Variable exponent DPS $t^{p(x)} + a(x)t^{q(x)}$	Maeda, Mizuta, Ohno, Shimomura, Ragusa, Tachikawa
Orlicz DPS $\psi(t) + a(x)\xi(t)$	Baasandorj, Byun, Oh, Lee
Triple phase spaces $t^p + a(x)t^q + b(x)t^r$	De Filippis, Oh, Fang, Zhang, Zhang

Generalized Orlicz growth

The "special cases"-doctrine has recently been complemented by studies of the generalized Orlicz case.

Harmonic analysis	Maeda, Mizuta, Ohno, Shimomura, S. Yang, D. Yang, W. Yuan, Ahmida, Fiorenza, Youssfi, Karaman, Baruah, Cruz-Uribe, Ferreira, Ribeiro
Existence	Chlebica, Gwiazda, Skrzypczak, Zatorska-Goldstein, Bulíček, Kalousek, Y. Wang, Khaled, Rhoudaf, Sabiki, Bourahma, Benkirane, Bennoura, El Moumi
Regularity	Shah, Skrypnik, Voitovych, Benyaiche, Khlifi, Ahn Bui, B. Wang, D. Liu, P. Zhao, Harjulehto, Juusti, Karppinen, Klén, M. Lee, J. Ok

It has become apparent that there are some new difficulties in the more general setting.

Difficulty 1: Key estimate a.k.a. Diening's trick

$$
\begin{aligned}
&\left(\fint_B |f|\,dx\right)^{p(z)} \\
&\leqslant \left(\fint_B |f|^{p_B^-}\,dx\right)^{\frac{p(z)}{p_B^-}} \\
&\leqslant \Bigg(\underbrace{\fint_B |f|^{p(x)} + 1\,dx}_{\leqslant c/|B|}\Bigg)^{\frac{p(z)}{p_B^-}} \\
&= \left(\tfrac{c}{|B|}\right)^{\frac{p(z)-p_B^-}{p_B^-}} \left(\fint_B |f|^{p(x)}\,dx + 1\right) \\
&\leqslant c\left(\fint_B |f|^{p(x)}\,dx + 1\right).
\end{aligned}
$$

Difficulty 1: Key estimate a.k.a. Diening's trick

$$\left(\fint_B |f|\,dx\right)^{p(z)} \leqslant \left(\fint_B |f|^{p_B^-}\,dx\right)^{\frac{p(z)}{p_B^-}} \leqslant \Bigg(\underbrace{\fint_B |f|^{p(x)} + 1\,dx}_{\leqslant c/|B|}\Bigg)^{\frac{p(z)}{p_B^-}} = \left(\tfrac{c}{|B|}\right)^{\frac{p(z)-p_B^-}{p_B^-}} \left(\fint_B |f|^{p(x)}\,dx + 1\right) \leqslant c\left(\fint_B |f|^{p(x)}\,dx + 1\right).$$

$$\varphi\left(z, \fint_B |f|\,dx\right) \leqslant \varphi\left(z, (\varphi_B^-)^{-1}\left(\fint_B \varphi_B^-(|f|)\,dx\right)\right) \leqslant \varphi\Bigg(z, (\varphi_B^-)^{-1}\Bigg(\underbrace{\fint_B \varphi(x,|f|)\,dx}_{\leqslant c/|B|}\Bigg)\Bigg) = F\left(\tfrac{c}{|B|}\right)\left(\fint_B \varphi(x,|f|)\,dx\right) \leqslant c\fint_B \varphi(x,|f|)\,dx.$$

Here $F(t) := \frac{\varphi(z,(\varphi_B^-)^{-1}(t))}{t}$.

But φ_B^- is not convex!

Difficulty 2: splicing

In the approximation method for PDE, we approximate the solution of

$$\operatorname{div}(\varphi'(x, |\nabla u|)\tfrac{\nabla u}{|\nabla u|}) = 0 \quad \text{in } \Omega$$

in a ball $B_r \subset \Omega$ by the solution of an autonomous PDE

$$\operatorname{div}(\varphi_B'(|\nabla v|)\tfrac{\nabla v}{|\nabla v|}) = 0 \quad \text{in } B_r, \quad v - u \in W_0^{1,\varphi_B}(B_r)$$

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The choices $\varphi_B = \varphi_{B_r}^-$ and $\varphi_B = \varphi(x_0,\cdot)$ do not work! We used

$$\varphi_B(t) := \int_0^t \psi_B(s)\,ds, \quad \psi_B(t) := \begin{cases} \varphi'(x_0,t_1)\,(\tfrac{t}{t_1})^{p-1} & t < t_1, \\ \varphi'(x_0,t) & t \in [t_1,t_2], \\ \varphi'(x_0,t_2)\,(\tfrac{t}{t_2})^{p-1} & t > t_2, \end{cases}$$

This approach requires robust properties of Φ-functions!

cf. Hästö & Ok: Maximal regularity for local minimizers of non-autonomous functionals, *J. Eur. Math. Soc.*, to appear, & Jihoon Ok's talk.

Novelties for Orlicz spaces (long)

Orlicz spaces

We want to replace the function t^p in the space L^p with something more general. Classically, one assumes e.g. the following:

- $\varphi : [0, \infty) \to [0, \infty)$.
- $\varphi(t) = 0$ if and only if $t = 0$.
- φ is increasing and convex.

For instance, we may take $\varphi(t) = t^p \log(1 + t)$ or $\varphi(t) = e^t - 1$.

We define a *modular* and a *norm* by

$$\varrho_\varphi(f) := \int_\Omega \varphi(f)\,dx \quad \text{and} \quad \|f\|_\varphi := \inf\{\lambda > 0 \,:\, \varrho_\varphi(f/\lambda) \leqslant 1\}.$$

Orlicz spaces 2

The previous assumptions do not cover L^∞. The assumptions may be relaxed to the semimodular case (class Φ_c):

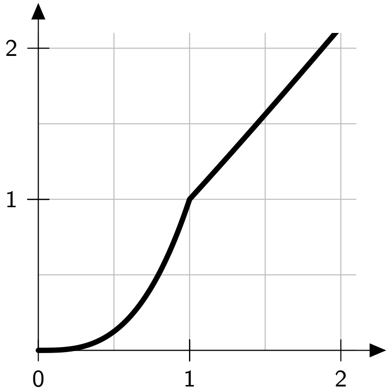
- $\varphi : [0, \infty) \to [0, \infty]$.
- $\varphi(0) = 0$, $\lim_{t\to 0} \varphi(t) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.
- φ is increasing and convex (on $[0, \infty]$).
- φ is left-continuous.

For instance, we may take $\varphi(t) = \infty\chi_{(1,\infty)}(t)$.

This approach covers all "normal" spaces, but it is not robust: the perturbation of a Φ-function need not be a Φ-function.

cf. Diening, Harjulehto, Hästö, Růžička: *Lebesgue and Sobolev spaces with variable exponents*, 2011.

Lack of robustness



The function $t \mapsto \min\{t^{1.1}, t^3\}$ is not convex.

Orlicz spaces 3

Robustness can be obtained by the following variant (class Φ_w):

- $\varphi : [0, \infty) \to [0, \infty]$.
- $\varphi(0) = 0$, $\lim_{t\to 0} \varphi(t) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.
- φ is increasing.
- $t \mapsto \frac{\varphi(t)}{t}$ is almost increasing.

(*Almost increasing* means that $f(s) \leqslant af(t)$ when $t > s$.)

Φ-functions are *equivalent*, $\varphi \simeq \psi$, if $\varphi(t/L) \leqslant \psi(t) \leqslant \varphi(Lt)$.

cf. Harjulehto, Hästö: *Orlicz spaces and generalized Orlicz spaces*, 2019.

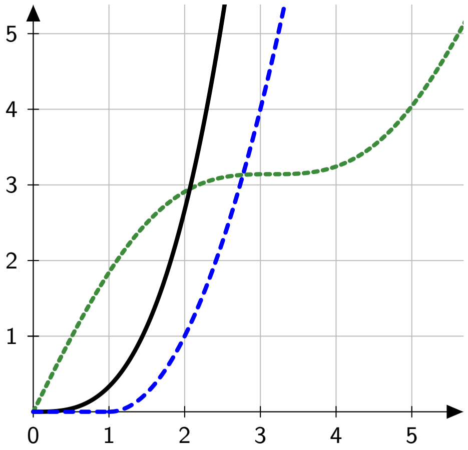
Examples

Define, for $t \geqslant 0$,

$$\begin{aligned}
\varphi^p(t) &:= \tfrac{1}{p} t^p, \quad p \in (0, \infty) \\
\varphi_{\max}(t) &:= \max\{0, (t-1)\}^2, \\
\varphi_{\sin}(t) &:= t + \sin(t), \\
\varphi_{\exp}(t) &:= e^t - 1, \\
\varphi_{\infty}(t) &:= \infty \chi_{(1,\infty)}(t)
\end{aligned}$$

See figure. We observe that $\varphi^p \in \Phi_c$ if and only if $p \geqslant 1$. Furthermore, $\varphi_{\max}, \varphi_{\exp} \in \Phi_c$ and $\varphi_{\sin} \in \Phi_w \setminus \Phi_c$.

Examples 2



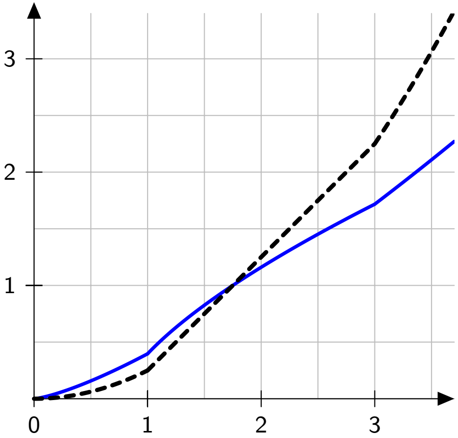
Functions φ^3 (solid black), φ_{max} (dashed blue) and φ_{sin} (dotted green) from the previous slide

Observations

- We observe that $\varphi^1 \simeq \varphi_{\mathsf{sin}}$. Therefore, Φ_c is not invariant under equivalence of Φ-prefunctions.
- Second, we observe that $t^p \to \varphi^\infty + \chi_{\{1\}}$. Therefore, Φ_c is not invariant under point-wise limits of Φ-prefunctions.
- Third, we note that $\min\{\varphi^1, \varphi^2\} \notin \Phi_c$, so Φ_c is not preserved under point-wise minimum.

More realistic examples: improved convexity

The function $\psi(t) := \frac{1}{2}\max\{\varphi^2(t), 2t - \frac{3}{2}\}$ is convex, but $\psi^{\frac{1}{p}}$ is not. The improved convexity is lost by an inconsequential change.



Functions ψ (dashed black) and $\psi^{\frac{1}{p}}$ (blue) with $p = 1.5$

Problems of convexity

The infimum φ_B^- of convex functions is not convex. (E.g. $\min\{t, t^2\}$.)

The function $\varphi(t)^{\frac{1}{p}}$ is not convex even when φ satisfies the ∇_2 condition. (E.g. piece-wise linear.)

Trick by Maeda–Mizuta–Ohno–Shimomura *et al.*: instead of convexity or p-convexity assume that

$$t \mapsto \frac{\varphi(t)}{t} \quad \text{or} \quad t \mapsto \frac{\varphi(t)}{t^p}$$

is (almost) increasing.

This condition is invariant under equivalence of Φ-functions. It allows us to easily regain convexity when necessary.

Robust convexity

In our approach, improved convexity is measured by the $(\mathrm{aInc})_p$ condition:

$$t \mapsto \frac{\varphi(t)}{t^p} \quad \text{is almost increasing.}$$

Similarly, we quantify the doubling behavior by the $(\mathrm{aDec})_q$ condition:

$$t \mapsto \frac{\varphi(t)}{t^q} \quad \text{is almost decreasing.}$$

These conditions are invariant under equivalence of Φ-functions.
NB! Matuszewska–Orlicz indices and

$$p \leqslant \frac{t\varphi'(t)}{\varphi(t)} \leqslant q.$$

Jensen's inequality

If $\varphi \in \Phi_w$ satisfies $(\mathrm{aInc})_p$, then there exists $\psi \in \Phi_c$ such that $\psi \simeq \varphi$ and $\psi^{\frac{1}{p}}$ is convex.

When $p = 1$, we get Jensen's inequality

$$\varphi\left(\beta \fint |f|\,dx\right) \leqslant \fint \varphi(|f|)\,dx.$$

If $\varphi \in \Phi_w$ satisfies $(\mathrm{aDec})_q$, we similarly obtain

$$\varphi^{-1}\left(\fint \varphi(|f|)\,dx\right) \lesssim \left(\fint |f|^q\,dx\right)^{\frac{1}{q}},$$

etc.

Harmonic analysis in generalized Orlicz spaces

Our assumptions

(A0) $\varphi^{-1}(x,1) \approx 1$ (un-weighted)

(A2) $\varphi^{-1}(x,t) \lesssim \varphi^{-1}(y,t)$ for $t \in [h(x)+h(y),1]$, where $h \in L^1 \cap L^\infty$ (decay at infinity)

(A1) $\varphi^{-1}(x,t) \lesssim \varphi^{-1}(y,t)$ for $t \in [1,|x-y|^{-n}]$ (local continuity)

(A1) $\varphi(x,t) \lesssim \varphi(y,t)$ for $\varphi(y,t) \in [1,|x-y|^{-n}]$

(A1-s) $\varphi(x,t) \lesssim \varphi(y,t)$ for $t^s \in [1,|x-y|^{-n}]$

To justify these assumptions we consider the following theorem and special cases (next slide).

Theorem

Let $\varphi \in \Phi_w((0,1))$ satisfy (A0) and (aInc) and be monotone in x. Then the Hardy–Littlewood maximal operator is bounded in $L^\varphi((0,1))$ if and only if φ satisfies (A1).

Special cases

(A0) $\varphi^{-1}(x,1) \approx 1$ (un-weighted)

(A1) $\varphi^{-1}(x,t) \lesssim \varphi^{-1}(y,t)$ for $t \in [1, |x-y|^{-n}]$ (local continuity)

(A2) $\varphi^{-1}(x,t) \lesssim \varphi^{-1}(y,t)$ for $t \in [h(x)+h(y), 1]$, where $h \in L^1 \cap L^\infty$ (decay at infinity)

$\varphi(x,t)$	(A0)	(A1)	(A2)	$(\mathrm{aInc})_p$
$t^{p(x)}a(x)$	$a \approx 1$	$\frac{1}{p} \in C^{\log}$	Nekv	$p^- > 1$
$t^{p(x)}\log(e+t)$	—	$\frac{1}{p} \in C^{\log}$	Nekv	$p^- > 1$
$t^p + a(x)t^q$	$a \in L^\infty$	$a \in C^{\frac{n}{p}(q-p)}$	$a \in L^\infty$	$p > 1$
$t^p + a(x)t^p\log(e+t)$	$a \in L^\infty$	$a \in C^{\log}$	$a \in L^\infty$	$p > 1$

NB! Sharpness of assumptions

Test case: double phase functional

Take $H(x,t) := t^p + a(x)t^q$, $p < q$. Then

$$\frac{H(x,t)}{H(y,t)} = 1 + \frac{a(x)-a(y)}{t^p + a(y)t^q}t^q \leqslant 1 + |a(x)-a(y)|t^{q-p}.$$

If $a \in C^\alpha$ and $t^s \lesssim |x-y|^{-n}$, then the RHS is bounded when

$$|x-y|^\alpha |x-y|^{-\frac{n}{s}(q-p)} \leqslant M \quad \Leftrightarrow \quad \alpha - \tfrac{n}{s}(q-p) \geqslant 0.$$

Thus H satisfies (A1-s) when $q-p \leqslant \frac{s}{n}\alpha$, in particular (A1) when $q-p \leqslant \frac{p}{n}\alpha$ and (A1-n) when $q-p \leqslant \alpha$, the conditions of BCM.

If we use the wrong exponent in the range condition, then the results will not be sharp!

cf. Benyaiche, Harjulehto, Hästö, Karppinen: The weak Harnack inequality for unbounded supersolutions of equations with generalized Orlicz growth, & Petteri Harjulehto's talk Nov 2nd.

Some of our publications

- A. Benyaiche, P. Harjulehto, P. Hästö and A. Karppinen: The weak Harnack inequality for unbounded supersolutions of equations with generalized Orlicz growth, arXiv:2006.06276.
- D. Cruz-Uribe and P. Hästö: Extrapolation and interpolation in generalized Orlicz spaces, *Trans. Amer. Math. Soc.* 370 (2018), no. 6, 4323–4349.
- P. Harjulehto, P. Hästö and M. Lee: Hölder continuity of quasiminimizers and ω-minimizers of functionals with generalized Orlicz growth, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, to appear.
- P. Hästö: The maximal operator on Musielak–Orlicz spaces, *J. Funct. Anal.* 269 (2015), no. 12, 4038–4048.
- P. Hästö and J. Ok: Maximal regularity for local minimizers of non-autonomous functionals, *J. Eur. Math. Soc.*, to appear.

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