Regularity for non-uniformly elliptic problems

Mathias Schäffner

02.11.2020 Nonstandard Seminar

Two non-uniformly elliptic problems.

- Linear (with P. Bella (TU Dortmund))

$$\nabla \cdot \omega \nabla u = 0$$

where

$$\omega \in L^p, \ \frac{1}{\omega} \in L^q$$

Two non-uniformly elliptic problems.

- Linear (with P. Bella (TU Dortmund))

$$\nabla \cdot \omega \nabla u = 0$$

where

$$\omega \in L^p, \ \tfrac{1}{\omega} \in L^q$$

- nonlinear (with P. Bella (TUD) & J. Hirsch (U Leipzig))

$$v \mapsto \int f(x, \nabla v)\, dx$$

with

$$|z|^p \lesssim f(x,z) \lesssim 1 + |z|^q$$

- divergence form elliptic equation ($\mathbb{R}^d \ni x \mapsto \mathbf{a}(x) \in \mathbb{R}^{d\times d}$, $d \geq 2$)

$$\nabla \cdot \mathbf{a} \nabla u = 0$$

- divergence form elliptic equation $(\mathbb{R}^d \ni x \mapsto \mathbf{a}(x) \in \mathbb{R}^{d\times d},\ d \geq 2)$

$$\nabla \cdot \mathbf{a}\nabla u = 0$$

- $\mathbf{a}$ uniformly elliptic: $\lambda|\xi|^2 \leq \mathbf{a}\xi \cdot \xi$, $|\mathbf{a}\xi| \leq \mu|\xi|$.
- [De Giorgi, Nash, Moser]:

$\Rightarrow$ u locally bounded and $\exists \alpha > 0$: $u \in C^{0,\alpha}_{\mathrm{loc}}$

$\Rightarrow$ if $u \geq 0$ in $B_1 \Rightarrow \sup_{B_{\frac{1}{2}}} u \lesssim \inf_{B_{\frac{1}{2}}} u$ (Harnack inequality)

- divergence form elliptic equation $(\mathbb{R}^d \ni x \mapsto \mathbf{a}(x) \in \mathbb{R}^{d\times d},\ d \geq 2)$

$$\nabla \cdot \mathbf{a}\nabla u = 0$$

- $\mathbf{a}$ uniformly elliptic: $\lambda|\xi|^2 \leq \mathbf{a}\xi\cdot\xi$, $|\mathbf{a}\xi| \leq \mu|\xi|$.
- [De Giorgi, Nash, Moser]:

$\Rightarrow$ u locally bounded and $\exists\alpha > 0$: $u \in C^{0,\alpha}_{\mathrm{loc}}$

$\Rightarrow$ if $u \geq 0$ in $B_1 \Rightarrow \sup_{B_{\frac{1}{2}}} u \lesssim \inf_{B_{\frac{1}{2}}} u$ (Harnack inequality)

Here:

- *pointwise* ellipticity replaced by *averaged* ellipticity

$\Rightarrow$ u locally bounded and Harnack inequality

Motivation:

- stochastic homogenization ('Invariance principle')

a non-uniformly elliptic:

- $0 < \lambda, \mu < \infty$ a.e. with $\lambda := \inf_{\xi \in \mathbb{R}^d} \frac{\xi \cdot \mathbf{a}\xi}{|\xi|^2}$, $\mu := \sup_{\xi \in \mathbb{R}^d} \frac{|\mathbf{a}\xi|^2}{\xi \cdot \mathbf{a}\xi}$
- $\exists\ p, q \in (1, \infty]$ s.t. $\frac{1}{\lambda} \in L^q_{\mathrm{loc}}$, $\mu \in L^p_{\mathrm{loc}}$.

a non-uniformly elliptic:

- $0 < \lambda, \mu < \infty$ a.e. with $\lambda := \inf_{\xi \in \mathbb{R}^d} \frac{\xi \cdot \mathbf{a}\xi}{|\xi|^2}$, $\mu := \sup_{\xi \in \mathbb{R}^d} \frac{|\mathbf{a}\xi|^2}{\xi \cdot \mathbf{a}\xi}$
- $\exists\ p, q \in (1, \infty]$ s.t. $\frac{1}{\lambda} \in L^q_{\mathrm{loc}}$, $\mu \in L^p_{\mathrm{loc}}$.

Theorem: [Trudinger ARMA'71]
$\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$. $\exists\ c \in [1, \infty)$ s.t. $\nabla \cdot \mathbf{a} \nabla u = 0$ in B_1 implies

(i) (local boundedness) $\|u\|_{L^\infty(B_{\frac{1}{2}})} \leq c \|u\|_{L^1(B_1)}$.

(ii) (Harnack inequality) If $u \geq 0$ in $B_1 \Rightarrow \sup_{B_{\frac{1}{2}}} u \leq c \inf_{B_{\frac{1}{2}}} u$

- $\frac{1}{p} + \frac{1}{q} < \frac{2}{d} \Rightarrow$ weighted Sobolev inequality
  - PDE: [Murthy, Stampacchia'68, Cupini, Marcellini, Mascolo '18,...]
  - Probability: [Fannjiang, Komorowski '97, Andres, Deuschel, Slowik 15, ...],

a non-uniformly elliptic:

- $0 < \lambda, \mu < \infty$ a.e. with $\lambda := \inf_{\xi \in \mathbb{R}^d} \frac{\xi \cdot \mathbf{a}\xi}{|\xi|^2}$, $\mu := \sup_{\xi \in \mathbb{R}^d} \frac{|\mathbf{a}\xi|^2}{\xi \cdot \mathbf{a}\xi}$
- $\exists\ p, q \in (1, \infty]$ s.t. $\frac{1}{\lambda} \in L^q_{\mathrm{loc}}$, $\mu \in L^p_{\mathrm{loc}}$.

Theorem: [Trudinger ARMA'71]
$\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$. $\exists\ c \in [1, \infty)$ s.t. $\nabla \cdot \mathbf{a}\nabla u = 0$ in B_1 implies

(i) (local boundedness) $\|u\|_{L^\infty(B_{\frac{1}{2}})} \leq c\|u\|_{L^1(B_1)}$.

(ii) (Harnack inequality) If $u \geq 0$ in $B_1 \Rightarrow \sup_{B_{\frac{1}{2}}} u \leq c \inf_{B_{\frac{1}{2}}} u$

- $\frac{1}{p} + \frac{1}{q} < \frac{2}{d} \Rightarrow$ weighted Sobolev inequality
  - PDE: [Murthy, Stampacchia'68, Cupini, Marcellini, Mascolo '18,...]
  - Probability: [Fannjiang, Komorowski '97, Andres, Deuschel, Slowik 15, ...],
- $\frac{1}{p} + \frac{1}{q} > \frac{2}{d-1} \Rightarrow$ Counterexample $d \geq 4$
  - PDE: [Franchi, Serapioni, Serra Cassano '98] $\exists \omega \in L^p(B_1)$ with $\frac{1}{\omega} \in L^q(B_1)$ and **unbounded** solutions $\nabla \cdot \omega \nabla u = 0$ in B_1.
  - Probability: [Biskup, Kumagai '14] $\nexists\ L^\infty$-sublinear corrector

a non-uniformly elliptic:

- $0 < \lambda, \mu < \infty$ a.e. with $\lambda := \inf_{\xi \in \mathbb{R}^d} \frac{\xi \cdot \mathbf{a}\xi}{|\xi|^2}$, $\mu := \sup_{\xi \in \mathbb{R}^d} \frac{|\mathbf{a}\xi|^2}{\xi \cdot \mathbf{a}\xi}$
- $\exists\ p, q \in (1, \infty]$ s.d. $\frac{1}{\lambda} \in L^q$, $\mu \in L^p$.

Theorem: [with Bella, CPAM to appear]
$\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}$. $\exists\ c \in [1, \infty)$ s.t. $\nabla \cdot \mathbf{a} \nabla u = 0$ in B_1 implies

(i) (local boundedness) $\|u\|_{L^\infty(B_{\frac{1}{2}})} \leq c \|u\|_{L^1(B_1)}$.

(ii) (Harnack inequality) If $u \geq 0$ in $B_1 \Rightarrow \sup_{B_{\frac{1}{2}}} u \leq c \inf_{B_{\frac{1}{2}}} u$

- $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$: [Trudinger '71] $\Rightarrow$ (i),(ii)
- $\frac{1}{p} + \frac{1}{q} > \frac{2}{d-1} \Rightarrow$ Counterexample $d \geq 4$

a non-uniformly elliptic:

- $0 < \lambda, \mu < \infty$ a.e. with $\lambda := \inf_{\xi \in \mathbb{R}^d} \frac{\xi \cdot \mathbf{a}\xi}{|\xi|^2}$, $\mu := \sup_{\xi \in \mathbb{R}^d} \frac{|\mathbf{a}\xi|^2}{\xi \cdot \mathbf{a}\xi}$
- $\exists\ p, q \in (1, \infty]$ s.d. $\frac{1}{\lambda} \in L^q$, $\mu \in L^p$.

Theorem: [with Bella, CPAM to appear]
$\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}$. $\exists\ c \in [1, \infty)$ s.t. $\nabla \cdot \mathbf{a}\nabla u = 0$ in B_1 implies

(i) (local boundedness) $\|u\|_{L^\infty(B_{\frac{1}{2}})} \leq c\|u\|_{L^1(B_1)}$.

(ii) (Harnack inequality) If $u \geq 0$ in $B_1 \Rightarrow \sup_{B_{\frac{1}{2}}} u \leq c \inf_{B_{\frac{1}{2}}} u$

- $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$: [Trudinger '71] $\Rightarrow$ (i),(ii)
- $\frac{1}{p} + \frac{1}{q} > \frac{2}{d-1}$ $\Rightarrow$ Counterexample $d \geq 4$
- $d = 2$ local boundedness assuming $p = q = 1$ [with Bella]
- **Note:** global integrability compared to local $\lambda \sim \mu \in A_2$ (cf. [Fabes, Kenig, Serapioni '82], [talk Diening]) $|B|^{-2}\|\lambda^{-1}\|_{L^1(B)}\|\mu\|_{L^1(B)} \lesssim 1\ \forall B$

a non-uniformly elliptic:

- $0 < \lambda, \mu < \infty$ a.e. with $\lambda := \inf_{\xi \in \mathbb{R}^d} \frac{\xi \cdot \mathbf{a}\xi}{|\xi|^2}$, $\mu := \sup_{\xi \in \mathbb{R}^d} \frac{|\mathbf{a}\xi|^2}{\xi \cdot \mathbf{a}\xi}$
- $\exists\ p, q \in (1, \infty]$ s.d. $\frac{1}{\lambda} \in L^q$, $\mu \in L^p$.

Theorem: [with Bella, CPAM to appear]
$\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}$. $\exists\ c \in [1, \infty)$ s.t. $\nabla \cdot \mathbf{a} \nabla u = 0$ in B_1 implies

(i) (local boundedness) $\|u\|_{L^\infty(B_{\frac{1}{2}})} \leq c \|u\|_{L^1(B_1)}$.

(ii) (Harnack inequality) If $u \geq 0$ in $B_1 \Rightarrow \sup_{B_{\frac{1}{2}}} u \leq c \inf_{B_{\frac{1}{2}}} u$

- $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$: [Trudinger '71] $\Rightarrow$ (i),(ii)
- $\frac{1}{p} + \frac{1}{q} > \frac{2}{d-1} \Rightarrow$ Counterexample $d \geq 4$
- $d = 2$ local boundedness assuming $p = q = 1$ [with Bella]
- **Note:** global integrability compared to local $\lambda \sim \mu \in A_2$ (cf. [Fabes, Kenig, Serapioni '82], [talk Diening]) $|B|^{-2} \|\lambda^{-1}\|_{L^1(B)} \|\mu\|_{L^1(B)} \lesssim 1\ \forall B$
- **Note:** 'Lavrentiev phenomena' possible - we consider 'H-solutions' in closure of C^1 wrt $\int a \nabla u \cdot \nabla u + \mu u^2$ (cf. [Zhikov '01], [talk Balci])

Idea of proof - local boundedness.

Variation of Moser's iteration method. Suppose

$$\lambda \geq 1 \quad \text{(i.e. } q = \infty\text{) and } u > 0 \text{ smooth}$$

Test $\nabla \cdot \mathbf{a} \nabla u = 0$ with $\phi := u^{2\alpha-1}\eta^2$, $\eta \in C_0^1(B_1)$

$$\int |\nabla(u^\alpha)|^2 \eta^2 \lesssim \int \mu |\nabla \eta|^2 u^{2\alpha}$$

Idea of proof - local boundedness.

Variation of Moser's iteration method. Suppose

$$\lambda \geq 1 \quad (\text{i.e. } q = \infty) \text{ and } u > 0 \text{ smooth}$$

Test $\nabla \cdot \mathbf{a}\nabla u = 0$ with $\phi := u^{2\alpha-1}\eta^2$, $\eta \in C_0^1(B_1)$

$$\int |\nabla(u^\alpha)|^2\eta^2 \lesssim \int \mu|\nabla\eta|^2 u^{2\alpha}$$

Fix $\frac{1}{2} \leq \varrho < \sigma \leq 1$ and $\eta \in C_0^1(B_\sigma)$ with $\eta = 1$ in B_ϱ

- Suppose $\mu \in L^\infty$. Estimate LHS (from below) via Sobolev

$$\|u^\alpha\|_{L^{2\chi}(B_\varrho)}^2 \lesssim \|\mu\|_{L^\infty}\|\nabla\eta\|_{L^\infty}^2\|u^\alpha\|_{L^2(B_\sigma)}^2, \qquad \chi := \tfrac{d}{d-2} > 1$$

Idea of proof - local boundedness.

Variation of Moser's iteration method. Suppose

$$\lambda \geq 1 \quad (\text{i.e. } q = \infty) \text{ and } u > 0 \text{ smooth}$$

Test $\nabla \cdot \mathbf{a} \nabla u = 0$ with $\phi := u^{2\alpha-1}\eta^2$, $\eta \in C_0^1(B_1)$

$$\int |\nabla(u^\alpha)|^2 \eta^2 \lesssim \int \mu |\nabla \eta|^2 u^{2\alpha}$$

Fix $\frac{1}{2} \leq \varrho < \sigma \leq 1$ and $\eta \in C_0^1(B_\sigma)$ with $\eta = 1$ in B_ϱ

- Suppose $\mu \in L^\infty$. Estimate LHS (from below) via Sobolev

$$\|u^\alpha\|_{L^{2\chi}(B_\varrho)}^2 \lesssim \|\mu\|_{L^\infty} \|\nabla \eta\|_{L^\infty}^2 \|u^\alpha\|_{L^2(B_\sigma)}^2, \qquad \chi := \frac{d}{d-2} > 1$$

- Suppose $\mu \in L^p$. Estimate via Sobolev & Hölder

$$\|u^\alpha\|_{L^{2\chi}(B_\varrho)}^2 \lesssim \|\mu\|_{L^p(B_1)} \|\nabla \eta\|_{L^\infty}^2 \|u^\alpha\|_{L^{\frac{2p}{p-1}}(B_\sigma)}^2,$$

with $\chi = \frac{d}{d-2}$. In order to iterate, need $\chi > \frac{p}{p-1} \Leftrightarrow \frac{1}{p} < \frac{2}{d}$.

Variation of Moser's iteration method.

$$\int |\nabla(u^{\alpha})|^2\eta^2 \lesssim \int \mu|\nabla\eta|^2 u^{2\alpha} \qquad \forall \eta \in C_0^1(B_1)$$

Fix $\frac{1}{2} \leq \varrho < \sigma \leq 1$ and suppose $\eta \in C_0^1(B_\sigma)$ with $\eta = 1$ in B_ϱ.

- Sobolev on spheres instead balls ($d \to d-1$)

Variation of Moser's iteration method.

$$\int |\nabla(u^\alpha)|^2\eta^2 \lesssim \int \mu|\nabla\eta|^2 u^{2\alpha} \qquad \forall \eta \in C_0^1(B_1)$$

Fix $\frac{1}{2} \le \varrho < \sigma \le 1$ and suppose $\eta \in C_0^1(B_\sigma)$ with $\eta = 1$ in B_ϱ.

- Sobolev on spheres instead balls ($d \to d-1$)

$$\int_{B_\sigma \setminus B_\varrho} \mu u^{2\alpha} = \int_\varrho^\sigma \int_{S_r} \mu u^{2\alpha} \le \int_\varrho^\sigma \|\mu\|_{L^p(S_r)} \|u^\alpha\|^2_{L^{\frac{2p}{p-1}}(S_r)}$$

$$\lesssim \int_\varrho^\sigma \|\mu\|_{L^p(S_r)} \|u^\alpha\|^2_{W^{1,p_*}(S_r)}, \qquad \frac{1}{p_*} = \frac{1}{2} - \frac{1}{2p} + \frac{1}{d-1}$$

☑ $2 > p_* \Leftrightarrow \frac{1}{p} < \frac{2}{d-1}$

☒ $\frac{1}{p} + \frac{2}{p_*} = 1 + \frac{2}{d-1} > 1$

Variation of Moser's iteration method.

$$\int |\nabla(u^\alpha)|^2\eta^2 \lesssim \int \mu|\nabla\eta|^2 u^{2\alpha} \qquad \forall \eta \in C_0^1(B_1)$$

Fix $\frac{1}{2} \leq \varrho < \sigma \leq 1$ and suppose $\eta \in C_0^1(B_\sigma)$ with $\eta = 1$ in B_ϱ.

- Sobolev on spheres instead balls and optimize $\eta = \eta(|x|)$

$$\int \mu|\nabla\eta|^2 u^{2\alpha} \lesssim \int_\varrho^\sigma \eta'(r)^2 \|\mu\|_{L^p(S_r)} \|u^\alpha\|^2_{W^{1,p_*}(S_r)}$$

$$\text{[1d minimization]} \quad \leq \left(\int_\varrho^\sigma (\|\mu\|_{L^p(S_r)} \|u^\alpha\|^2_{W^{1,p_*}(S_r)})^{-1}\right)^{-1}$$

$$\text{['harm. m.} \leq \text{arithm. m']} \leq (\sigma-\varrho)^{-\frac{\gamma+1}{\gamma}} \left(\int_\varrho^\sigma (\|\mu\|_{L^p(S_r)} \|u^\alpha\|^2_{W^{1,p_*}(S_r)})^{\gamma}\right)^{\frac{1}{\gamma}}$$

$$\leq (\sigma-\varrho)^{-\frac{2d}{d-1}} \|\mu\|_{L^p(B_1)} \|u^\alpha\|^2_{W^{1,p_*}(B_\sigma)}$$

$2 > p_* \Leftrightarrow \frac{1}{p} < \frac{2}{d-1}$

Variation of Moser's iteration method.

$$\int |\nabla(u^\alpha)|^2\eta^2 \lesssim \int \mu|\nabla\eta|^2 u^{2\alpha} \qquad \forall \eta \in C_0^1(B_1)$$

Fix $\frac{1}{2} \leq \varrho < \sigma \leq 1$ and suppose $\eta \in C_0^1(B_\sigma)$ with $\eta = 1$ in B_ϱ.

- Sobolev on spheres instead balls and optimize $\eta = \eta(|x|)$

$$\int |\nabla\eta|^2\mu u^{2\alpha} \lesssim (\sigma - \varrho)^{-\frac{2d}{d-1}} \|\mu\|_{L^p(B_1)} \|u^\alpha\|^2_{W^{1,p_*}(B_\sigma)}$$

Altogether,

$$\|u^{\alpha\chi}\|^{\frac{1}{\alpha\chi}}_{W^{1,p_*}(B_\varrho)} \lesssim \left(\frac{\|\mu\|^{\frac{1}{2}}_{L^p(B_1)}\alpha\chi}{(\sigma-\varrho)^{\frac{d}{d-1}}}\right)^{\frac{1}{\alpha\chi}} \|u^\alpha\|^{\frac{1}{\alpha}}_{W^{1,p_*}(B_\sigma)}$$

where $\chi = 1 + \frac{1}{d-1} - \frac{1}{2p} > 1$. Iterate $\Rightarrow$ local boundedness $\Box$

Summary

- Local boundedness & Harnack inequality under optimal integrability assumptions on $\mathbf{a}$, $\mathbf{a}^{-1}$
- **Trick:** optimize cut-off in Caccioppoli & Sobolev on spheres
  - [Manfredi JGA'94]
  - [Briane, Diaz JDE'16...]

(??) (...)

Summary

- Local boundedness & Harnack inequality under optimal integrability assumptions on $\mathbf{a}$, $\mathbf{a}^{-1}$
- **Trick:** optimize cut-off in Caccioppoli & Sobolev on spheres
  - [Manfredi JGA'94]
  - [Briane, Diaz JDE'16...]

(??) (...)

Further results:

☑ Parabolic equations

$$\partial_t u - \nabla \cdot \mathbf{a} \nabla u = 0$$

[with Bella, arXiv] (discrete)

$$q > \frac{d}{2},\ p > 1 \qquad \frac{1}{q} + \frac{1}{p} < \frac{2}{d-1}$$

($q > \frac{d}{2}$ necessary)

☑ right-hand side & "logarithmic improvement" of (p, q) condition [with Bella, Hirsch, in preparation]

Nonlinear problems:

Integral functionals with (p, q)-growth

Consider

$$v \mapsto \int_\Omega f(\nabla v)\,dx \tag{1}$$

where $z \mapsto f(z)$ is C^2 and

$$\begin{cases} |z|^p \lesssim f(z) \lesssim |z|^q \\ (1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \lesssim \langle D^2 f(z)\xi,\xi\rangle \lesssim (1+|z|^2)^{\frac{q-2}{2}}|\xi|^2 \end{cases}$$

Consider

$$v \mapsto \int_\Omega f(\nabla v)\,dx \tag{1}$$

where $z \mapsto f(z)$ is C^2 and

$$\begin{cases} |z|^p \lesssim f(z) \lesssim |z|^q \\ (1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \lesssim \langle D^2 f(z)\xi,\xi\rangle \lesssim (1+|z|^2)^{\frac{q-2}{2}}|\xi|^2 \end{cases}$$

Euler-Lagrange equation for (1) reads

$$\nabla \cdot Df(\nabla u) = 0$$

Linearization non-uniformly elliptic

$$\frac{\text{largest eigenvalue of } D^2 f(z)}{\text{smallest eigenvalue of } D^2 f(z)} \sim (1+|z|^2)^{\frac{q-p}{2}}$$

Consider

$$v \mapsto \int_\Omega f(\nabla v)\, dx$$

where $z \mapsto f(z)$ is C^2 and

$$\begin{cases} |z|^p \lesssim f(z) \lesssim |z|^q \\ (1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \lesssim \langle D^2 f(z)\xi, \xi\rangle \lesssim (1+|z|^2)^{\frac{q-2}{2}}|\xi|^2 \end{cases}$$

- Gradient regularity ([Marcellini '91, Esposito, Mingione, Leonetti '99])

$$\frac{q}{p} < 1 + \frac{2}{d} \quad \Rightarrow \qquad \nabla v \in \begin{cases} L^\infty_{\mathrm{loc}} & \text{(scalar)} \\ L^q_{\mathrm{loc}} & \text{(vectorial)} \end{cases}$$

Consider

$$v \mapsto \int_\Omega f(\nabla v)\,dx$$

where $z \mapsto f(z)$ is C^2 and

$$\begin{cases} |z|^p \lesssim f(z) \lesssim |z|^q \\ (1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \lesssim \langle D^2 f(z)\xi,\xi\rangle \lesssim (1+|z|^2)^{\frac{q-2}{2}}|\xi|^2 \end{cases}$$

- Gradient regularity ([Marcellini '91, Esposito, Mingione, Leonetti '99])

$$\frac{q}{p} < 1 + \frac{2}{d} \quad \Rightarrow \quad \nabla v \in \begin{cases} L^\infty_{\mathrm{loc}} & \text{(scalar)} \\ L^q_{\mathrm{loc}} & \text{(vectorial)} \end{cases}$$

- Local boundedness [Fusco,Sbordonne '90][Cupini,Marcellini,Mascolo '15]

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{d} \quad \Rightarrow \quad v \in L^\infty_{\mathrm{loc}}$$

(+ sharp results for anisotropic growth)

Consider

$$v \mapsto \int_\Omega f(\nabla v)\, dx$$

where $z \mapsto f(z)$ is C^2 and

$$\begin{cases} |z|^p \lesssim f(z) \lesssim |z|^q \\ (1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \lesssim \langle D^2 f(z)\xi,\xi\rangle \lesssim (1+|z|^2)^{\frac{q-2}{2}}|\xi|^2 \end{cases}$$

- Gradient regularity ([Marcellini '91, Esposito, Mingione, Leonetti '99])

$$\frac{q}{p} < 1 + \frac{2}{d} \quad \Rightarrow \quad \nabla v \in \begin{cases} L^\infty_{\mathrm{loc}} & \text{(scalar)} \\ L^q_{\mathrm{loc}} & \text{(vectorial)} \end{cases}$$

- Local boundedness [Fusco,Sbordonne '90][Cupini,Marcellini,Mascolo '15]

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{d} \quad \Rightarrow \quad v \in L^\infty_{\mathrm{loc}}$$

(+ sharp results for anisotropic growth)

- Counterexamples to regularity [Marcellini '91],[Giaquinta '87]

$$\frac{1}{p} - \frac{1}{d-1} > \frac{1}{q} \quad \text{local boundedness fails}$$

Consider

$$v \mapsto \int_\Omega f(\nabla v)\, dx$$

where $z \mapsto f(z)$ is C^2 and

$$\begin{cases} |z|^p \lesssim f(z) \lesssim |z|^q \\ (1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \lesssim \langle D^2 f(z)\xi,\xi\rangle \lesssim (1+|z|^2)^{\frac{q-2}{2}}|\xi|^2 \end{cases}$$

Theorem: [Marcellini JOTA'96]
Let $2 \leq p < q < \infty$ and let u be a local minimizer. Then

$$\frac{q}{p} < 1 + \frac{2}{d} \quad \Rightarrow \quad \nabla u \in L^\infty_{\mathrm{loc}}$$

Related results by

[Acerbi, Beck, Byun, Brasco, Carozza, Chlebicka, Cianchi, Cupini, Colombo, De Filippis, Esposito, Koch, Leonetti, Fusco, Lieberman, Fonseca, Hästö, Harjulehto, Kristensen, Maly, Mascolo, Mingione, Oh, Ok, Passarelli di Napoli, Sbordone,...]

Consider

$$v \mapsto \int_\Omega f(\nabla v)\,dx$$

where $z \mapsto f(z)$ is C^2 and

$$\begin{cases} |z|^p \lesssim f(z) \lesssim |z|^q \\ (1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \lesssim \langle D^2 f(z)\xi,\xi\rangle \lesssim (1+|z|^2)^{\frac{q-2}{2}}|\xi|^2 \end{cases}$$

Theorem: [Marcellini JOTA'96]
Let $2 \leq p < q < \infty$ and let u be a local minimizer. Then

$$\frac{q}{p} < 1 + \frac{2}{d} \quad \Rightarrow \quad \nabla u \in L^\infty_{\mathrm{loc}}$$

Relation to linear equation (!formal!):

Suppose $p = 2$ & $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \Rightarrow$

$$\nabla \cdot \mathbf{a}\nabla\partial_i u = 0 \qquad \mathbf{a} := D^2 f(\nabla u) \in L^{\frac{2}{q-2}=:P}$$

$$\frac{q}{2} < 1 + \frac{2}{d} \quad \Leftrightarrow \quad \frac{1}{P} = \frac{q-2}{q} < \frac{2}{d} \quad \sim \quad \text{('Trudinger condition')}$$

Consider

$$v \mapsto \int_\Omega f(\nabla u)\,dx \qquad u : \Omega \to \mathbb{R}$$

where $z \mapsto f(z)$ is C^2 and

$$\begin{cases} |z|^p \lesssim f(z) \lesssim |z|^q \\ (1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \lesssim \langle D^2 f(z)\xi,\xi\rangle \lesssim (1+|z|^2)^{\frac{q-2}{2}}|\xi|^2 \end{cases}$$

Theorem: [with Bella, Analysis & PDE, to appear]
Let $2 \le p < q < \infty$, $d \ge 3$ and let u be a local minimizer. Then

$$\frac{q}{p} < 1 + \frac{2}{d-1} \qquad \Rightarrow \qquad \nabla u \in L^\infty_{\mathrm{loc}}$$

- a priori estimate (for $d \ge 4$)

$$\|\nabla u\|_{L^\infty(\frac12 B)} \lesssim \left(\fint_B f(\nabla u)\,dx + 1 \right)^{\frac{2}{(d+1)p-(d-1)q}}$$

Consider

$$v \mapsto \int_\Omega f(\nabla u)\,dx \qquad u:\Omega\to\mathbb{R}^N \quad (\textit{vectorial})$$

where $z \mapsto f(z)$ is C^2 and

$$\begin{cases} |z|^p \lesssim f(z) \lesssim |z|^q \\ (1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \lesssim \langle D^2 f(z)\xi,\xi\rangle \lesssim (1+|z|^2)^{\frac{q-2}{2}}|\xi|^2 \end{cases}$$

Theorem: [S., arXiv]
Let $2 \le p < q < \infty$, $d \ge 3$ and let u be a local minimizer. Then

$$\frac{q}{p} < 1 + \frac{2}{d-1} \qquad \Rightarrow \qquad \nabla u \in L^q_{\mathrm{loc}}$$

- Improves [Esposito, Mingione, Leonetti JDE'99] for $d \ge 3$.

Consider

$$v \mapsto \int_\Omega f(\nabla u)\,dx \qquad u:\Omega \to \mathbb{R}^N \quad (\textit{vectorial})$$

where $z \mapsto f(z)$ is C^2 and

$$\begin{cases} |z|^p \lesssim f(z) \lesssim |z|^q \\ (1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \lesssim \langle D^2 f(z)\xi,\xi\rangle \lesssim (1+|z|^2)^{\frac{q-2}{2}}|\xi|^2 \end{cases}$$

Theorem: [S., arXiv]
Let $2 \le p < q < \infty$, $d \ge 3$ and let u be a local minimizer. Then

$$\frac{q}{p} < 1 + \frac{2}{d-1} \qquad \Rightarrow \qquad \nabla u \in L^q_{\mathrm{loc}}$$

- Improves [Esposito, Mingione, Leonetti JDE'99] for $d \ge 3$.
- Direct corollary: Higher differentiability integrability

$$|\nabla u|^{\frac{p-2}{2}}\nabla u \in W^{1,2}_{\mathrm{loc}}, \qquad \nabla u \in L^{\chi p}_{\mathrm{loc}}, \quad \chi := \tfrac{d}{d-2}$$

Consider

$$v \mapsto \int_\Omega f(\nabla u)\,dx \qquad u:\Omega \to \mathbb{R}^N \quad (\textit{vectorial})$$

where $z \mapsto f(z)$ is C^2 and

$$\begin{cases} |z|^p \lesssim f(z) \lesssim |z|^q \\ (1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \lesssim \langle D^2 f(z)\xi,\xi\rangle \lesssim (1+|z|^2)^{\frac{q-2}{2}}|\xi|^2 \end{cases}$$

Theorem: [S., arXiv]
Let $2 \le p < q < \infty$, $d \ge 3$ and let u be a local minimizer. Then

$$\frac{q}{p} < 1 + \frac{2}{d-1} \qquad \Rightarrow \qquad \nabla u \in L^q_{\mathrm{loc}}$$

- Improves [Esposito, Mingione, Leonetti JDE'99] for $d \ge 3$.
- Direct corollary: Higher differentiability integrability

$$|\nabla u|^{\frac{p-2}{2}}\nabla u \in W^{1,2}_{\mathrm{loc}}, \qquad \nabla u \in L^{\chi p}_{\mathrm{loc}}, \quad \chi := \tfrac{d}{d-2}$$

- Less direct consequence: partial regularity (following & improving (in some aspects) [Bildhauer, Fuchs CVPDE'01])

Consider

$$v \mapsto \int_\Omega f(x, \nabla u)\, dx \qquad u : \Omega \to \mathbb{R}$$

where $z \mapsto f(x,z)$ is convex

$$|z|^p \lesssim f(x,z) \lesssim |z|^q, \qquad f(2z) \lesssim f(z) + 1$$

Theorem: [with Hirsch, Comm. Cont. Math., to appear]
Let $1 < p \leq q < \infty$, and let u be a local minimizer. Then

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{d-1} \qquad \Rightarrow \qquad u \in L^\infty_{\mathrm{loc}}$$

- $\frac{1}{q} \geq \frac{1}{p} - \frac{1}{d-1}$ optimal in view of counterexample [Marcellini JDE'91].

Consider

$$v \mapsto \int_\Omega f(x, \nabla u)\, dx \qquad u : \Omega \to \mathbb{R}$$

where $z \mapsto f(x, z)$ is convex

$$|z|^p \lesssim f(x, z) \lesssim |z|^q, \qquad f(2z) \lesssim f(z) + 1$$

Theorem: [with Hirsch, Comm. Cont. Math., to appear]
Let $1 < p \leq q < \infty$, and let u be a local minimizer. Then

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{d-1} \qquad \Rightarrow \qquad u \in L^\infty_{\mathrm{loc}}$$

- $\frac{1}{q} \geq \frac{1}{p} - \frac{1}{d-1}$ optimal in view of counterexample [Marcellini JDE'91].
- De Giorgi type argument: Caccioppoli inequality

$$\int_{A_{k,\sigma}} f(x, \nabla u) \lesssim \int_{A_{k,\sigma} \setminus B_\varrho} f(x, \nabla u) + \int_{A_{k,\sigma}} |\nabla \eta|^q ((u-k)_+)^q$$

$A_{\ell,r} := B_r \cap \{u > \ell\}$.

Consider

$$v \mapsto \int_\Omega f(x,\nabla u)\,dx \qquad u:\Omega\to\mathbb{R}$$

where $z\mapsto f(x,z)$ is convex

$$|z|^p \lesssim f(x,z)\lesssim |z|^q, \qquad f(2z)\lesssim f(z)+1$$

Theorem: [with Hirsch, Comm. Cont. Math., to appear]
Let $1<p\le q<\infty$, and let u be a local minimizer. Then

$$\frac{1}{q}\ge\frac{1}{p}-\frac{1}{d-1} \qquad \Rightarrow \qquad u\in L^\infty_{\mathrm{loc}}$$

- $\frac{1}{q}\ge\frac{1}{p}-\frac{1}{d-1}$ optimal in view of counterexample [Marcellini JDE'91].
- De Giorgi type argument: Caccioppoli inequality

$$\|(u-k)_+\|^p_{W^{1,p}(B_\varrho)}\lesssim(\sigma-\varrho)^{-\gamma}\|(u-h)_+\|^q_{W^{1,p}(B_\varrho)}+\dots$$

$k>h$ Optimize η, Sobolev, iterate,...

Consider

$$v \mapsto \int_\Omega f(x, \nabla u)\, dx \qquad u : \Omega \to \mathbb{R}$$

where $z \mapsto f(x,z)$ is convex

$$|z|^p \lesssim f(x,z) \lesssim |z|^q, \qquad f(2z) \lesssim f(z)$$

Theorem: [with Hirsch, Comm. Cont. Math., to appear]
Let $1 < p \leq q < \infty$, and let u be a local minimizer. Then

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{d-1} \qquad \Rightarrow \qquad u \in L^\infty_{\mathrm{loc}}$$

Back to autonomous functionals with

$$(1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 \lesssim \langle D^2 f(z)\xi, \xi\rangle \lesssim (1+|z|^2)^{\frac{q-2}{2}}|\xi|^2$$

[Carozza, Kristensen, Passarelli di Napoli AIHP'11]: $p < d$

$$q < p+2 \quad \& \quad u \in W^{1,p}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} \qquad \Rightarrow \qquad \nabla u \in L^q_{\mathrm{loc}}$$

(compare with $q < p + 2\frac{p}{d-1}$)

Consider

$$v \mapsto \int_\Omega f(x, \nabla u)\, dx \qquad u : \Omega \to \mathbb{R}$$

where $z \mapsto f(x, z)$ is convex

$$|z|^p \lesssim f(x, z) \lesssim |z|^q, \qquad f(2z) \lesssim f(z)$$

Theorem: [with Hirsch, Comm. Cont. Math., to appear]
Let $1 < p \leq q < \infty$, and let u be a local minimizer. Then

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{d-1} \qquad \Rightarrow \qquad u \in L^\infty_{\mathrm{loc}}$$

[Colombo, Mingione ARMA'15]:

Double phase: $$\int |\nabla u|^p + a(x)|\nabla u|^q \qquad a \in C^{0,\alpha}$$

$q < p + p\frac{\alpha}{d} \Rightarrow \nabla u \in C^{0,\beta}_{\mathrm{loc}}$
$q < p + \alpha \;\&\; u \in L^\infty_{\mathrm{loc}} \Rightarrow \nabla u \in C^{0,\beta}_{\mathrm{loc}}$

Consider

$$v \mapsto \int_\Omega f(x, \nabla u)\, dx \qquad u : \Omega \to \mathbb{R}$$

where $z \mapsto f(x, z)$ is convex

$$|z|^p \lesssim f(x, z) \lesssim |z|^q, \qquad f(2z) \lesssim f(z)$$

Theorem: [with Hirsch, Comm. Cont. Math., to appear]
Let $1 < p \leq q < \infty$, and let u be a local minimizer. Then

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{d-1} \qquad \Rightarrow \qquad u \in L^\infty_{\mathrm{loc}}$$

Thank you for the attention!