Regularity results for a class of obstacle problems with generalized Orlicz growth

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Introduction

Let $\Omega \subset \mathbb{R}^n (n \geqslant 2)$ be open and bounded. We consider local minimizers of the functional

$$\mathcal{K}^{\varphi}_{\psi}(\Omega) \ni u \mapsto \int_{\Omega} \varphi(x, |\nabla u|)\, dx$$

where the class

$$\mathcal{K}^{\varphi}_{\psi}(\Omega) := \{u \in W^{1,\varphi}(\Omega) \,|\, u \geqslant \psi \text{ a.e. in } \Omega\}$$

for a fixed function $\psi : \Omega \to [-\infty, \infty)$ which is called the *obstacle*.

Introduction

- We consider the functional

$$\mathcal{F}(u,\Omega) := \int_\Omega F(x,\nabla u)\,dx.$$

  - (Standard) p-growth:

  $$F(x,z) \approx |z|^p,\ 1 < p < \infty.$$

  - General growth - the class of functions depending only on z:

  $$F(x,z) \approx \psi(|z|).$$

  - Non-autonomous functionals - the classes of functions whose growth or ellipticity depend on the x-variable:

    $p(x)$-growth : $F(x,z) \approx |z|^{p(x)}$

    Double phase : $F(x,z) \approx |z|^p + a(x)|z|^q, \ldots$

    Generalized Orlicz growth: $F(x,z) \approx \varphi(x,|z|)$

Standard growth

We consider the regularity of minimizers of the functional

$$\int_\Omega F(x, \nabla u)\, dx.$$

- Standard p-growth case (Giaquinta-Giusti, 1983)

  $F : \Omega \times \mathbb{R}^n \to [0, \infty)$ satisfies

$$\begin{cases} z \mapsto F(x,z) \text{ is } C^2, 1 < p < \infty, \\ \nu|z|^p \leqslant F(x,z) \leqslant L(1+|z|^p), \\ \nu|z|^{p-2}|\lambda|^2 \leqslant F_{zz}(x,z)\lambda\cdot\lambda \leqslant L|z|^{p-2}|\lambda|^2 \\ |F(x,z) - F(y,z)| \leqslant \omega(|x-y|)(1+|z|^p). \end{cases}$$

  - (De Giorgi's method) $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$.
  - $\lim_{r\to 0^+} \omega(r) = 0 \Longrightarrow u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for any $\alpha \in (0,1)$.
  - $\omega(r) \lesssim r^\beta$ for some $\beta > 0 \Longrightarrow u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$.

Non-standard growth

We consider the regularity of minimizers of the functional

$$\int_\Omega F(x, \nabla u)\, dx.$$

- Functionals with (p, q)-growth conditions introduced by Marcellini (1989):

  $F : \Omega \times \mathbb{R}^n \to [0, \infty)$ satisfies

$$\begin{cases} z \mapsto F(x,z) \text{ is } C^2, 1 < p \leqslant q \\ \nu|z|^p \leqslant F(x,z) \leqslant L(1+|z|^q), \\ \nu|z|^{p-2}|\lambda|^2 \leqslant F_{zz}(x,z)\lambda\cdot\lambda \leqslant L|z|^{q-2}|\lambda|^2. \end{cases}$$

  - When $F(x,z) \equiv \varphi(x,|z|)$,

$$\begin{cases} t \mapsto \varphi(x,t) \text{ is } C^2, 1 < p \leqslant q, \\ p-1 \leqslant \frac{t\varphi''(x,t)}{\varphi'(x,t)} \leqslant q-1 \ (\Longleftrightarrow t\varphi''(x,t) \approx \varphi'(x,t)). \end{cases}$$

Non-standard growth

We consider minimizers of the functionals with generalized Orlicz growth

$$W^{1,\varphi}(\Omega) \ni u \mapsto \int_\Omega F(x, \nabla u)\, dx,$$

where $F : \Omega \times \mathbb{R}^n \to [0, \infty)$ satisfies

$$\begin{cases} z \mapsto F(x,z) \text{ is } C^2, \\ \nu\varphi(x,|z|) \leqslant F(x,z) \leqslant L\varphi(x,|z|), \\ \nu \frac{\varphi'(x,|z|)}{|z|}|\lambda|^2 \leqslant F_{zz}(x,z)\lambda \cdot \lambda \leqslant L\frac{\varphi'(x,|z|)}{|z|}|\lambda|^2. \end{cases}$$

Generalized Orlicz function

- We say that $\varphi : \Omega \times [0, \infty) \to [0, \infty]$ is a Φ-*prefunction* if the following hold:
  (i) For every $t \in [0, \infty)$ the function $x \mapsto \varphi(x, t)$ is measurable.
  (ii) For every $x \in \Omega$ the function $t \mapsto \varphi(x, t)$ is non-decreasing.
  (iii) $\lim_{t \to 0^+} \varphi(x, t) = \varphi(x, 0) = 0$ and $\lim_{t \to \infty} \varphi(x, t) = \infty$ for every $x \in \Omega$.

- A Φ-prefunction is a weak Φ-function, denoted by $\varphi \in \Phi_w(\Omega)$, if the following hold:
  (iv) The function $t \mapsto \frac{\varphi(x,t)}{t}$ is almost increasing in $(0, \infty)$ for every $x \in \Omega$.
  (v) The function $t \mapsto \varphi(x, t)$ is left-continuous for every $x \in \Omega$.

Here, almost increasing means that $g(s) \leqslant Lg(t)$ when $t > s$ for some $L \geqslant 1$.
cf. almost decreasing means that $g(t) \leqslant Lg(s)$ when $t > s$ for some $L \geqslant 1$.

- Harjulehto, Hästö: Orlicz spaces and generalized Orlicz spaces, 2019.

Conditions on φ

Let $\varphi : \Omega \times [0, \infty) \to [0, \infty)$ and $\gamma > 0$. We say that φ satisfies

$(\mathrm{aInc})_\gamma$ if $t \mapsto \frac{\varphi(x,t)}{t^\gamma}$ is almost increasing with constant $L \geqslant 1$ uniformly in $x \in \Omega$.

$(\mathrm{aDec})_\gamma$ if $t \mapsto \frac{\varphi(x,t)}{t^\gamma}$ is almost decreasing with constant $L \geqslant 1$ uniformly in $x \in \Omega$.

When $L = 1$, $(\mathrm{aInc})_\gamma = (\mathrm{Inc})_\gamma$, $(\mathrm{aDec})_\gamma = (\mathrm{Dec})_\gamma$.

(A0) if there exists $L \geqslant 1$ such that $L^{-1} \leqslant \varphi(x, 1) \leqslant L$ for every $x \in \Omega$.

- Assumption

  $p - 1 \leqslant \frac{t\varphi''(x,t)}{\varphi'(x,t)} \leqslant q - 1$

  $\iff t\varphi''(x, t) \approx \varphi'(x, t)$

  $\iff \varphi'$ satisfies $(\mathrm{Inc})_{p-1}$ and $(\mathrm{Dec})_{q-1}$

  $\implies \varphi$ satisfies $(\mathrm{Inc})_p$ and $(\mathrm{Dec})_q$

Non-standard growth

- Let u be a minimizer of the functional

$$\int_\Omega \varphi(x, |\nabla u|)\, dx$$

where $\varphi(x,t) = a(x)\psi(t)$ satisfies Orlicz growth:

$$\begin{cases} \psi \text{ is } C^2, \\ 0 < \nu \leqslant a(\cdot) \leqslant L, \\ t\psi''(t) \approx \psi'(t) \end{cases}$$

  - $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$.
  - $\lim_{r\to 0^+} \omega(r) = 0$ ($a \in C^0$) $\Longrightarrow u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for any $\alpha \in (0,1)$.
  - $\omega(r) \lesssim r^\beta$ for some $\beta > 0$ ($a \in C^\beta$) $\Longrightarrow u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$.

- Lieberman (1991), ...

Conditions on φ

We say that φ satisfies

(A1) if there exists $L \geqslant 1$ such that, for every ball $B_r \Subset \Omega$

$$\varphi^+_{B_r}(t) \leqslant L\varphi^-_{B_r}(t) \quad \text{when} \quad \varphi^-_{B_r}(t) \in \left[1, \frac{1}{|B_r|}\right].$$

(A1-n) if there exists $L \geqslant 1$ such that, for every ball $B_r \Subset \Omega$

$$\varphi^+_{B_r}(t) \leqslant L\varphi^-_{B_r}(t) \quad \text{when} \quad t^n \in \left[1, \frac{1}{|B_r|}\right].$$

Here, $\varphi^+_{B_r}(t) := \sup_{x \in B_r} \varphi(x,t)$, $\varphi^-_{B_r}(t) := \inf_{x \in B_r} \varphi(x,t)$.

Non-standard growth

- Let u be a minimizer of the functional

$$\int_\Omega \varphi(x, |\nabla u|)\, dx.$$

  - Variable exponent case: $\varphi(x,t) = t^{p(x)} w(x)$

    (A0) $w \approx 1$

    (aInc) $\inf p > 1$

    (aDec) $\sup p < \infty$

    (A1) p is locally log-Hölder continuous.

    (A1-n) p is locally log-Hölder continuous.

  - Double phase case : $\varphi(x,t) = t^p + a(x)t^q$

    (A0) $a \in L^\infty$

    (aInc) $p > 1$

    (aDec) $q < \infty$

    (A1) $a \in C^{0,\beta}$, $q \leqslant p + \frac{p}{n}\beta$

    (A1-n) $a \in C^{0,\beta}$, $q \leqslant p + \beta$

Non-standard growth

- $p(x)$-growth case: $\varphi(x,t) = t^{p(x)}$
  - $\lim_{r\to 0} \omega_p(r)\log(\frac{1}{r}) = L < \infty \Longrightarrow u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$.
  - $\lim_{r\to 0} \omega_p(r)\log(\frac{1}{r}) = 0 \Longrightarrow u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for any $\alpha \in (0,1)$.
  - $\omega_p(r) \lesssim r^\beta$ for some $\beta > 0 \Longrightarrow u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$.
  - Acerbi-Coscia-Mingione (1999, 2001), ...
- Double phase case: $\varphi(x,t) = t^p + a(x)t^q$

  with $1 < p \leqslant q < \infty,\ a \in C^{0,\beta}(\Omega)$ for some $\beta \in (0,1]$.
  - Either $\frac{q}{p} \leqslant 1 + \frac{\beta}{n}$ or $u \in L^\infty$ and $q \leqslant p + \beta$
    $\Longrightarrow u \in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$.
  - $\frac{q}{p} \leqslant 1 + \frac{\beta}{n} \Longrightarrow u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$.
  - Baroni-Colombo-Mingione (2015, 2018), ...

Conditions on φ

We say that φ satisfies

(A1) if there exists $L \geqslant 1$ such that, for every ball $B_r \Subset \Omega$

$$\varphi^+_{B_r}(t) \leqslant L\varphi^-_{B_r}(t) \quad \text{when} \quad \varphi^-_{B_r}(t) \in \left[1, \frac{1}{|B_r|}\right].$$

(VA1) if there exists a non-decreasing continuous function $\omega : [0,\infty) \to [0,1]$ with $\omega(0) = 0$ such that for any small $B_r \Subset \Omega$,

$$\varphi^+_{B_r}(t) \leqslant (1+\omega(r))\varphi^-_{B_r}(t) \quad \text{when} \quad \varphi^-_{B_r}(t) \in \left[\omega(r), \frac{1}{|B_r|}\right].$$

(wVA1) if for any $\varepsilon > 0$, there exists a non-decreasing continuous function $\omega = \omega_\varepsilon : [0,\infty) \to [0,1]$ with $\omega(0) = 0$ such that for any small ball $B_r \Subset \Omega$,

$$\varphi^+_{B_r}(t) \leqslant (1+\omega(r))\varphi^-_{B_r}(t) + \omega(r) \quad \text{when} \quad \varphi^-_{B_r}(t) \in \left[\omega(r), \frac{1}{|B_r|^{1-\varepsilon}}\right].$$

- (VA1) $\Longrightarrow$ (wVA1) $\Longrightarrow$ (A1).
- Double phase case : $\varphi(x,t) = t^p + a(x)t^q, \quad a \in C^{0,\beta}$
  - ★ $\frac{q}{p} < 1 + \frac{\beta}{n} \Longrightarrow$ (VA1) holds with $\omega(r) \lesssim r^\gamma$ and $\gamma = \beta - \frac{n(q-p)}{p} > 0$.
  - ★ $\frac{q}{p} = 1 + \frac{\beta}{n} \Longrightarrow$ (VA1) does not hold. φ needs (wVA1).

Unconstrained case

Let u be a minimizer of the functional

$$\int_\Omega \varphi(x, |\nabla u|)\, dx.$$

- Generalized Orlicz case
  - Harjulehto-Hästö-Toivanen (2017), Harjulehto-Hästö-L (to appear)

    Either φ satisfies (A1), or that $u \in L^\infty$ and φ satisfies (A1-n)

    $\Longrightarrow u \in C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$.

  - Hästö-Ok (to appear)

    (i) φ satisfies (wVA1) $\Longrightarrow u \in C^{0,\alpha}_{loc}(\Omega)$ for all $\alpha \in (0,1)$.

    (ii) φ satisfies (wVA1) and $\omega(r) \lesssim r^\beta$ for some $\beta > 0$

    $\Longrightarrow u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$.

Obstacle problem

Let $\Omega \subset \mathbb{R}^n (n \geqslant 2)$ be open and bounded. We consider local minimizers of the functional

$$\mathcal{K}^{\varphi}_{\psi}(\Omega) \ni u \mapsto \int_{\Omega} \varphi(x, |\nabla u|)\, dx$$

where the class

$$\mathcal{K}^{\varphi}_{\psi}(\Omega) := \{u \in W^{1,\varphi}(\Omega) \,|\, u \geqslant \psi \text{ a.e. in } \Omega\}$$

for a fixed function $\psi : \Omega \to [-\infty, \infty)$ which is called the *obstacle*.

Hölder type regularity

- Standard p-growth case : Michael, Ziemer, Choe, Lewis, Fuchs, Mingione, Mu, Ok, Lindqvist, Benassi, Caselli . . . .
- $p(x)$-growth case : Eleuteri, Habermann, Hästö, Harjulehto, Lukkari, Marola, Byun, Oh, Ok, ...
- Double phase case : De Filippis, Chlebicka, ...

Solution

Let $\Omega \subset \mathbb{R}^n$ be open and bounded.

- We consider local minimizers of the functional

$$\mathcal{K}_\psi^\varphi(\Omega) \ni u \mapsto \int_\Omega \varphi(x, |\nabla u|)\, dx$$

 where the class $\mathcal{K}_\psi^\varphi(\Omega) := \{u \in W^{1,\varphi}(\Omega) \,|\, u \geqslant \psi \text{ a.e. in } \Omega\}$ for a function $\psi : \Omega \to [-\infty, \infty)$ called the *obstacle*.

- If $\varphi \in \Phi_w(\Omega) \cap C^1([0, \infty))$, we say that a function $u \in \mathcal{K}_\psi^\varphi(\Omega)$ is *a solution to the $\mathcal{K}_\psi^\varphi(\Omega)$-obstacle problem* if it satisfies

$$\int_\Omega \left(\frac{\partial_t \varphi(x, |\nabla u|)}{|\nabla u|} \nabla u \right) \cdot \nabla(\eta - u)\, dx \geqslant 0$$

 for all $\eta \in \mathcal{K}_\psi^\varphi(\Omega)$ with $\mathrm{supp}(\eta - u) \subset \Omega$, which is equivalent to

$$\int_\Omega \left(\frac{\partial_t \varphi(x, |\nabla u|)}{|\nabla u|} \nabla u \right) \cdot \nabla \eta\, dx \geqslant 0$$

 for all $\eta \in W^{1,\varphi}(\Omega)$ with a compact support and $\eta \geqslant \psi - u$ a.e. in Ω.

Main result

Theorem (Karppinen-L. submitted)

Let $\varphi \in \Phi_w(\Omega)$ satisfy (aInc), (aDec), (A0) and (A1). Suppose that the obstacle $\psi \in C^{0,\beta}_{\mathrm{loc}}(\Omega)$ for some $\beta \in (0,1)$. Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$.

- (A1) condition: There exists $L \geqslant 1$ such that, for every ball $B_r \Subset \Omega$

$$\varphi^+_{B_r}(t) \leqslant \varphi^-_{B_r}(t) \quad \text{when} \quad \varphi^-_{B_r}(t) \in \left[1, \frac{1}{|B_r|}\right].$$

- The proof is based on constructing classical Harnack's inequality. The supremum estimates of the solution are proved via Caccioppoli type energy estimate and unconstrained results. The standard arguments provide the weak Harnack inequality.

Sketch of Proof

Step1 Supremum bounds

- u satisfies the Caccioppoli type inequality

$$\int_{A(\ell,r)} \varphi(x, |\nabla(u-\ell)_+|)\, dx \lesssim \int_{A(\ell,2r)} \varphi\left(x, \frac{(u-\ell)_+}{r}\right) dx$$

for any $\ell \geqslant 0$, where $A(k,r) := Q_r \cap \{u > k\}$.

- Then if $\psi \in W^{1,\varphi}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ and $\theta \in \left[\frac{1}{2}, 1\right)$, we have

$$\operatorname*{ess\,sup}_{Q_{\theta r}} (u-\ell)_+ \lesssim (1-\theta)^{-4nq^2} \left[\left(\fint_{Q_r} (u-\ell)_+^q \, dx \right)^{1/q} + |(u-\ell)_{Q_{r/2}}| \right] + r \tag{1}$$

for any $Q_{2r} \subset \Omega$ and $\ell \geqslant \sup_{Q_{2r}} \psi$, provided that $\varrho_{L^\varphi(Q_{2r})}(|\nabla u|) \leqslant 1$. The term $|(u-\ell)_{Q_{r/2}}|$ can be omitted if $u - \ell$ is non-negative almost everywhere in Q_{2r}. Furthermore, if $u \in L^\infty(Q_r)$ satisfies (1) without the term $|(u-\ell)_{Q_{r/2}}|$, then

$$\operatorname*{ess\,sup}_{Q_{r/2}} (u-\ell)_+ \lesssim \left(\fint_{Q_r} (u-\ell)_+^h \, dx \right)^{\frac{1}{h}} + r$$

for any $h \in (0, \infty)$.

Sketch of Proof

Step2 Infimum bounds

- Since solution to an obstacle problem is also a superminimizer, the standard arguments provide the following weak Harnack inequality:

  Suppose that $u \in W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ is a non-negative solution of the $\mathcal{K}^{\varphi}_{\psi}(\Omega)$-obstacle problem. Then there exists $h_0 > 0$ such that

  $$\left(\fint_{Q_r} u^{h_0}\, dx \right)^{\frac{1}{h_0}} \lesssim \operatorname*{ess\,inf}_{Q_{r/2}} u + r$$

  when $Q_{2r} \subset \Omega$ and $\varrho_{L^{\varphi}(Q_{2r})}(|\nabla u|) \leqslant 1$.

Step3 Combining the supremum and infimum bounds, we obtain Harnack's inequality and then it implies the oscillation decay estimates

$$\mathrm{osc}(u, r/4) \leqslant \frac{C}{C+1}\,\mathrm{osc}(u, r) + cr^{\beta},$$

which means the Hölder continuity of u.

Main result

Theorem (Karppinen-L. submitted)

- *Let $\varphi(x,\cdot) \in C^1([0,\infty))$ for any $x \in \Omega$ with φ' satisfying (A0), (Inc)$_{p-1}$, (Dec)$_{q-1}$ for some $1 < p \leqslant q$. Suppose $\psi \in C^{1,\beta}_{\mathrm{loc}}(\Omega)$ for some $\beta \in (0,1)$.*
  - (i) *If φ satisfies (wVA1), then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for any $\alpha \in (0,1)$.*
  - (ii) *If φ satisfies (wVA1) and*

$$\omega(r) \lesssim r^{\delta_\varepsilon} \text{ for all } r \in (0,1] \text{ and for some } \delta_\varepsilon > 0,$$

then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$.

- (wVA1) condition: For any $\varepsilon > 0$, there exists a non-decreasing continuous function $\omega = \omega_\varepsilon : [0,\infty) \to [0,1]$ with $\omega(0) = 0$ such that for any small ball $B_r \Subset \Omega$,

$$\varphi^+_{B_r}(t) \leqslant (1+\omega(r))\varphi^-_{B_r}(t) + \omega(r) \quad \text{when} \quad \varphi^-_{B_r}(t) \in \left[\omega(r), \frac{1}{|B_r|^{1-\varepsilon}}\right].$$

cf Double phase case : $\varphi(x,t) = t^p + a(x)t^q, \quad a \in C^{0,\beta}$

* $\frac{q}{p} \leqslant 1 + \frac{\beta}{n} \Longrightarrow$ (wVA1) holds with $\omega(r) \lesssim r^{\delta_\varepsilon}$ and $\delta_\varepsilon = \beta - \frac{n(1-\varepsilon)(q-p)}{p} > 0$.

Sketch of Proof

Step1 Higher Integrability:

- There exists $\sigma_0 \in (0,1)$ such that $\varphi(\cdot, |\nabla u|) \in L^{1+\sigma_0}(B_r)$, and for any $\sigma \in (0, \sigma_0]$,

$$\begin{aligned} &\fint_{B_r} \varphi(x, |\nabla u|)^{1+\sigma}\, dx \\ &\leqslant c_1 \left[\left(\fint_{B_{2r}} \varphi(x, |\nabla u|)\, dx \right)^{1+\sigma} + \fint_{B_{2r}} \varphi(x, |\nabla \psi|)^{1+\sigma}\, dx + 1 \right] \end{aligned}$$

  for some $c_1 = c_1(n, p, q, L) > 0$.

- It is proved by Gehring's lemma with Sobolev–Poincaré inequality:
  - ⋆ For $1 \leqslant s < \frac{n}{n-1}$, there exists $\beta > 0$ such that

$$\left(\fint_{B_r} \varphi\left(x, \frac{\beta\, |u|}{r}\right)^s dx \right)^{\frac{1}{s}} \leqslant \int_{B_r} \varphi(x, |\nabla u|)\, dx + \frac{|\{\nabla u \neq 0\} \cap B_r|}{|B_r|}$$

    for any $u \in W_0^{1,1}(B_r)$ with $\|\nabla u\|_{\varphi} < 1$. If additionally $1 \leqslant s \leqslant p$, then

$$\fint_{B_r} \varphi\left(x, \beta\, \frac{|u - u_{B_r}|}{r}\right) dx \leqslant \left(\fint_{B_r} \varphi(x, |\nabla u|)^{\frac{1}{s}}\, dx \right)^s + 1$$

    for any $u \in W^{1,1}(B_r)$ with $\|\nabla u\|_{\varphi^{1/s}} < 1$.

Sketch of Proof

Step1

- From the higher Integrability, we obtain the reverse Hölder type estimates:

For every $t \in (0,1]$, there exists $c_t = c_t(c_2, t, q) > 0$ such that

$$\left(\fint_{B_r} \varphi(x, |\nabla u|)^{1+\sigma_0}\, dx\right)^{1/(1+\sigma_0)}$$
$$\leqslant c_t \left[\left(\fint_{B_{2r}} \varphi(x, |\nabla u|)^t\, dx\right)^{1/t} + \left(\fint_{B_{2r}} \varphi(x, |\nabla \psi|)^{1+\sigma_0}\, dx\right)^{1/(1+\sigma_0)} + 1\right].$$

Additionally,

$$\fint_{B_r} \varphi(x, |\nabla u|)\, dx \leqslant \left(\fint_{B_r} \varphi(x, |\nabla u|)^{1+\sigma_0}\, dx\right)^{1/(1+\sigma_0)}$$
$$\leqslant c \left[\varphi^-_{B_{2r}}\left(\fint_{B_{2r}} |\nabla u|\, dx\right) + \left(\fint_{B_{2r}} \varphi(x, |\nabla \psi|)^{1+\sigma_0}\, dx\right)^{1/(1+\sigma_0)} + 1\right],$$

where $c \geqslant 1$ depends on c_2, q and L.

Sketch of Proof

Step2 Comparison functions

- For comparison argument, we shall use the regularized Orlicz function $\tilde{\varphi}$ constructed from a generalized Orlicz function φ, which was introduced by Hästö-Ok (to appear in JEMS).

- We define the regularized Orlicz function $\tilde{\varphi}$ by

$$\tilde{\varphi}(t) := \int_0^\infty \varphi_{B_{2r}}(t\sigma)\eta_r(\sigma - 1)\, d\sigma = \int_0^\infty \varphi_{B_{2r}}(s)\eta_{rt}(s-t)\, ds,$$

with $\tilde{\varphi}(0) := 0$ for $\eta \in C_0^\infty(\mathbb{R})$ with $\eta \geqslant 0$, $\operatorname{supp} \eta \subset (0,1)$ and $\|\eta\|_1 = 1$ where $\eta_r(t) := \frac{1}{r}\eta\left(\frac{t}{r}\right)$.

Here,

$$\varphi_{B_{2r}}(t) := \int_0^t \tau_{B_{2r}}(s)\, ds.$$

with

$$\tau_{B_{2r}}(t) := \begin{cases} \varphi'(x_0, t_1)\left(\frac{t}{t_1}\right)^{p-1}, & \text{if } 0 \leqslant t < t_1, \\ \varphi'(x_0, t) & \text{if } t_1 \leqslant t \leqslant t_2, \\ \varphi'(x_0, t_2)\left(\frac{t}{t_2}\right)^{p-1}, & \text{if } t_2 < t < \infty, \end{cases}$$

where $t_1 := (\varphi^-_{B_{2r}})^{-1}(\omega(2r))$ and $t_2 := (\varphi^-_{B_{2r}})^{-1}(|B_{2r}|^{-1})$.

Sketch of Proof

Step2 Comparison functions

- The regularized Orlicz function $\tilde{\varphi}$ satisfies the following properties:

  (i) $\varphi_{B_{2r}}(t) \leqslant \tilde{\varphi}(t) \leqslant (1+cr)\varphi_{B_{2r}}(t)$ for all $t>0$ with $c>0$ depending only on q, and $0 \leqslant \tilde{\varphi}(t) - \varphi(x_0,t) \leqslant cr\varphi^-(t) + c\omega(2r) \leqslant c\varphi^-(t)$ for all $t \in [t_1,t_2]$;

  (ii) $\tilde{\varphi} \in C^1([0,\infty))$ and it satisfies $(\mathrm{Inc})_p$, $(\mathrm{Dec})_q$ and (A0), and $\tilde{\varphi}'$ satisfies $(\mathrm{Inc})_{p-1}$ and $(\mathrm{Dec})_{q-1}$ and (A0). In particular $\tilde{\varphi}'(t) \approx t\tilde{\varphi}''(t)$ for all $t>0$;

  (iii) $\tilde{\varphi}(t) \leqslant c\varphi(x,t)$ for all $(x,t) \in B_{2r} \times [1,\infty)$, and so $\tilde{\varphi}(t) \lesssim \varphi(x,t)+1$ for all $(x,t) \in B_{2r} \times [0,\infty)$.

Step3

- Consider the unique weak solution $w \in W^{1,\tilde{\varphi}}(B_r)$ of

$$\begin{cases} -\mathrm{div}\left(\frac{\tilde{\varphi}'(|\nabla w|)}{|\nabla w|}\nabla w\right) = -\mathrm{div}\left(\frac{\tilde{\varphi}'(|\nabla \psi|)}{|\nabla \psi|}\nabla \psi\right) & \text{in } B_r, \\ w = u & \text{on } \partial B_r, \end{cases}$$

and the unique weak solution $v \in W^{1,\tilde{\varphi}}(B_r)$ of

$$\begin{cases} -\mathrm{div}\left(\frac{\tilde{\varphi}'(|\nabla v|)}{|\nabla v|}\nabla v\right) = 0 & \text{in } B_r, \\ v = w & \text{on } \partial B_r. \end{cases}$$

Sketch of Proof

Step3

- Consider the unique weak solution $v \in W^{1,\tilde{\varphi}}(B_r)$ of

$$\begin{cases} -\mathrm{div}\left(\frac{\tilde{\varphi}'(|\nabla v|)}{|\nabla v|}\nabla v\right) = 0 & \text{in} \quad B_r, \\ \qquad\qquad\qquad\qquad v = w & \text{on} \quad \partial B_r. \end{cases}$$

Remark As v is the solution of the $\tilde{\varphi}$-Laplacian equation, it is known to have C^{1,α_0}-regularity for some $\alpha_0 \in (0,1)$ (Lieberman, 1991):
For any $B(x_0,\varrho) \subset B_r$ and $\tau \in (0,1)$, we have

$$\sup_{B(x_0,\varrho/2)} |\nabla v| \leqslant c \fint_{B(x_0,\varrho)} |\nabla v| \, dx,$$

$$\fint_{B(x_0,\tau\varrho)} |\nabla v - (\nabla v)_{B(x_0,\tau\varrho)}| \, dx \leqslant c\tau^{\alpha_0} \fint_{B(x_0,\varrho)} |\nabla v| \, dx.$$

- Energy estimates for w and v :

$$\fint_{B_r} \tilde{\varphi}(|\nabla w|) \, dx \leqslant c \left(\fint_{B_r} \tilde{\varphi}(|\nabla u|) \, dx + 1 \right),$$

$$\fint_{B_r} \tilde{\varphi}(|\nabla v|) \, dx \leqslant c \left(\fint_{B_r} \tilde{\varphi}(|\nabla u|) \, dx + 1 \right).$$

Sketch of Proof

Step3

- We have

$$\left(\fint_{B_r} \varphi(x, |\nabla u|)^{1+\sigma_0}\, dx\right)^{\frac{1}{1+\sigma_0}} \lesssim \tilde{\varphi}\left(\fint_{B_{2r}} |\nabla u|\, dx\right) + 1$$

by using the reverse Hölder type estimates, and

$$\fint_{B_r} \varphi(x, |\nabla w|)\, dx \lesssim \left(\fint_{B_r} \varphi(x, |\nabla u|)^{1+\frac{\sigma_0}{2}}\, dx + 1\right)^{\frac{2}{2+\sigma_0}},$$

by using Calderón-Zygmund type estimates:

★ for the solution $w \in W^{1,\tilde{\varphi}}(B_r)$ of

$$\begin{cases} -\mathrm{div}\left(\frac{\tilde{\varphi}'(|\nabla w|)}{|\nabla w|}\nabla w\right) = -\mathrm{div}\left(\frac{\tilde{\varphi}'(|\nabla \psi|)}{|\nabla \psi|}\nabla \psi\right) & \text{in } B_r, \\ w = u & \text{on } \partial B_r, \end{cases}$$

$$\|\varphi(|\nabla w|)\|_{L^\theta(B_r)} \leqslant c\left(\|\varphi(|\nabla \psi|)\|_{L^\theta(B_r)} + \|\varphi(|\nabla u|)\|_{L^\theta(B_r)}\right)$$

for any $\theta \in \Phi_w(B_r)$ satisfying (A0), (A1), $(\mathrm{aInc})_{p_1}$ and $(\mathrm{aDec})_{q_1}$ with constant $L \geqslant 1$ and $1 < p_1 \leqslant q_1$.

Sketch of Proof

Step4
- By using the previous estimates, we obtain
  - ★ $$\fint_{B_r} |\nabla u - \nabla w|\, dx \leqslant c(\omega(2r)^{p/q} + r^\gamma + r^\beta)^{\frac{1}{2q}} \left(\fint_{B_{2r}} |\nabla u|\, dx + 1 \right)$$
    for some $\gamma = (n, p, q, L) \in (0, 1)$.
  - ★ $$\fint_{B_r} |\nabla w - \nabla v|\, dx \leqslant c\, r^{\frac{\beta}{2q}} \left(\fint_{B_{2r}} |\nabla u|\, dx + 1 \right)$$
  - ★ $$\fint_{B_r} |\nabla v|\, dx \lesssim \fint_{B_{2r}} |\nabla u| + 1\, dx$$

Then combining above estimates,

- $$\begin{aligned} \int_{B_\varrho} |\nabla u|\, dx &\leqslant \int_{B_r} |\nabla u - \nabla w|\, dx + \int_{B_r} |\nabla w - \nabla v|\, dx + \int_{B_\varrho} |\nabla v|\, dx \\ &\lesssim \omega_0(r) \int_{B_{2r}} |\nabla u| + 1\, dx + \varrho^n \sup_{B_{r/2}} |\nabla v| \\ &\lesssim \omega_0(r_0) \int_{B_{2r}} |\nabla u| + 1\, dx + \varrho^n \fint_{B_r} |\nabla v|\, dx \\ &\lesssim \left(\omega_0(r_0) + \left(\frac{\varrho}{r} \right)^n \right) \int_{B_{2r}} |\nabla u|\, dx + r^n. \end{aligned}$$

Sketch of Proof

- Then we have that for any $\mu \in (0, n)$,

$$\int_{B_\varrho} |\nabla u| \, dx \lesssim \left(\frac{\varrho}{r_0} \right)^{n-\mu} \left(\int_{B_{r_0}} |\nabla u| \, dx + r_0^{n-\mu} \right)$$

for all $B_\varrho \subset \Omega'$ with $\varrho \in (0, r_0)$.

- Thus, we take $1 - \mu = \alpha$ to obtain that $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega')$ by Morrey type embedding.

Sketch of Proof

- $$\begin{aligned}
&\fint_{B_\varrho} |\nabla u - (\nabla u)_{B_\varrho}|\,dx \leqslant 2\fint_{B_\varrho} |\nabla u - (\nabla v)_{B_\varrho}|\,dx \\
&\leqslant 2\fint_{B_\varrho} |\nabla u - \nabla w|\,dx + 2\fint_{B_\varrho} |\nabla w - \nabla v|\,dx + 2\fint_{B_\varrho} |\nabla v - (\nabla v)_{B_\varrho}|\,dx \\
&\lesssim \left(\frac{r}{\varrho}\right)^n \fint_{B_r} |\nabla u - \nabla w|\,dx + \left(\frac{r}{\varrho}\right)^n \fint_{B_r} |\nabla w - \nabla v|\,dx + \left(\frac{\varrho}{r}\right)^{\alpha_0} \fint_{B_{r/2}} |\nabla v|\,dx \\
&\lesssim \left(\frac{r}{\varrho}\right)^n (\omega(2r)^{p/q} + r^\gamma + r^\beta)^{\frac{1}{2q}} \left(\fint_{B_{2r}} |\nabla u|\,dx + 1\right) + \left(\frac{\varrho}{r}\right)^{\alpha_0} \fint_{B_{r/2}} |\nabla v|\,dx \\
&\lesssim \left(r^{\delta_0}\left(\frac{r}{\varrho}\right)^n + \left(\frac{\varrho}{r}\right)^{\alpha_0}\right)\left(\fint_{B_{2r}} |\nabla u|\,dx + 1\right) \leqslant c_\mu r^{-\mu}\left(r^{\delta_0}\left(\frac{r}{\varrho}\right)^n + \left(\frac{\varrho}{r}\right)^{\alpha_0}\right),
\end{aligned}$$

  where $\delta_0 := \frac{1}{2q}\min\{\frac{\delta p}{q}, \gamma, \beta\}$.
- Thus, we obtain that $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega')$ with $\alpha = \frac{\alpha_0\delta_0}{4n+2\delta_0}$ by Campanato type embedding.

Thank you.