Monday's Nonstandard Seminar 2020/21

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On the validity of variational inequalities for obstacle problems with non-standard growth

Monday February 15th, 2021

Outline of the seminar

The aim of the seminar is to show that the solutions to variational problems with non-standard growth conditions satisfy a corresponding variational inequality without any smallness assumptions on the gap between growth and coercitivity exponents

Our results rely on techniques based on Convex Analysis that consist in establishing duality formulas and pointwise relations between minimizers and corresponding dual maximizers, for suitable approximating problems, that are preserved passing to the limit

In this respect we are able to show that the right class of competitors are the functions with finite energy in agreement with the unconstrained results

Joint project with Prof. A. Passarelli di Napoli [1]

[1] M. E., A. Passarelli di Napoli, preprint arXiv:2010.02964 (2020)

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- The model problem
- Statement of the main results
- Dual formulation of the obstacle problem
- Proof of the main result

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More than 30 years ago, the celebrated papers by Marcellini [2] opened the way to the study of the regularity properties of minimizers of integral functionals with non-standard growth conditions

Since then, many contributions appeared in several directions and many problems have been solved; however not all the questions have been addressed in an exhaustive way, in particular for what concerns the obstacle problems

It is well known that, for both constrained and unconstrained minimization problems, the regularity of the solutions often comes from the fact that are also extremals, i.e. they solve a corresponding variational inequality or equality

[2] P. Marcellini: *Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions*, Arch. Ration. Mech. Anal., **105** (1989), no. 3, 267–284

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Actually, in the recent paper [3] the authors, concerning the question of Lipschitz continuity for minimizers of the obstacle problem, were forced to deal with the relation between minima and extremals, in the sense of solutions to a corresponding variational inequality

In that specific situation, this problem has been solved due to a suitable higher differentiability result and imposing a smallness condition on the gap between the coercivity and the growth exponent of the lagrangian

We decided to deal with the question in the full generality

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The unconstrained case

Already for unconstrained minimizers with non-standard growth, the relation between extremals and minima is an issue that required a careful investigation

Indeed, a direct derivation of such a relation can be obtained in a trivial way only if the gap between the growth and the ellipticity exponent satisfies a suitable smallness condition.

Otherwise, using a regularization procedure and convex duality theory, much stronger results have been obtained by Carozza, Kristensen and Passarelli di Napoli for unconstrained minimizers [4]

As far as we know, such investigation has not been carried out for constrained minimizers

We aim to find conditions so that the solutions to variational obstacle problems with non standard growth conditions satisfy a corresponding variational inequality

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The model problem

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More precisely, let us consider a class to variational obstacle problems of the form

$$\min \left\{ \int_\Omega F(Dz) : z \in \mathbb{K}^F_\psi(\Omega) \right\},$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 2$

The function $\psi : \Omega \to [-\infty, +\infty)$, called *obstacle*, is such that

$$F(D\psi) \in L^1(\Omega)$$

and the class $\mathbb{K}^F_\psi(\Omega)$ is defined as

$$\mathbb{K}^F_\psi(\Omega) := \left\{ z \in u_0 + W^{1,p}_0(\Omega) : z \geq \psi \text{ a.e. in } \Omega, \; F(Dz) \in L^1(\Omega) \right\},$$

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The assumptions

We shall consider integrands $F : \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^1$ and satisfying the following growth and strict convexity assumptions:

$$\ell|\xi|^p \leq F(\xi) \leq L\,(1+|\xi|^q) \tag{H1}$$

$$\nu\,|V_p(\xi)-V_p(\eta)|^2 \leq F(\xi)-F(\eta)-\langle F'(\eta),\xi-\eta\rangle \tag{H2}$$

for all $\xi,\eta \in \mathbb{R}^n$, for $0<\ell<L$, $\nu>0$ and $1<p\leq q<\infty$ and where we used the customary notation

$$V_p(\xi) = (1+|\xi|^2)^{\frac{p-2}{4}}\xi.$$

To simplify the statement of our main result, we shall assume that the integrand F satisfies a sort of Δ_2 condition, i.e.

$$F(\lambda\xi) \leq C(\lambda)\,F(\xi) \qquad \text{(H3)},$$

for every real positive $\lambda > 1$ and every $\xi \in \mathbb{R}^n$

Actually, without (H3), our result holds true supposing that $F(cDu_0) \in L^1(\Omega)$, for some constant $c > 1$

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A remark

Let us notice that, by replacing u_0 by $\tilde{u}_0 = \max\{u_0, \psi\}$, we may assume, without loss of generality, that the boundary value function u_0 is such that $u_0 \in \mathbb{K}^F_\psi(\Omega)$

Indeed $\tilde{u}_0 = (\psi - u_0)^+ + u_0$ so $\tilde{u}_0 \geq \psi$

Moreover, since

$$0 \leq (\psi - u_0)^+ \leq (u - u_0)^+ \in W^{1,p}_0(\Omega),$$

the function $(\psi - u_0)^+$, and hence $u - \tilde{u}_0$, belongs to $W^{1,p}_0(\Omega)$

Finally our assumptions on u_0 and ψ imply $F(D\tilde{u}_0) \in L^1(\Omega)$. Indeed

$$\begin{aligned} \int_\Omega F(D\tilde{u}_0)\,dx &= \int_{\Omega\cap\{u_0\geq\psi\}} F(Du_0)\,dx + \int_{\Omega\cap\{u_0<\psi\}} F(D\psi)\,dx \\ &\leq \int_\Omega \Big(F(Du_0) + F(D\psi)\Big)\,dx < +\infty, \end{aligned}$$

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It is worth mentioning that if G is a $\mathcal{C}^1$ function satisfying (H1) (growth) and (H2) (strict convexity) with $\boxed{p = q}$, i.e. G satisfies standard $p-$growth conditions, the minimization problem reduces to

$$\min \left\{ \int_\Omega G(Dz) : z \in \mathcal{K}_\psi(\Omega) \right\},$$

where

$$\mathcal{K}_\psi(\Omega) := \left\{ z \in u_0 + W_0^{1,p}(\Omega) : z \geq \psi \text{ a.e. in } \Omega \right\}$$

and the assumptions $F(D\psi), F(Du_0) \in L^1(\Omega)$ reduce in turn to $\psi, u_0 \in W^{1,p}(\Omega)$

In this case, because of the standard growth conditions, it is well known that, if $u \in u_0 + W_0^{1,p}(\Omega)$ is a solution to the minimization problem, then the corresponding variational inequality

$$\int_\Omega \langle G'(Du), Dz - Du \rangle \, dx \geq 0$$

holds true, for every $z \in \mathcal{K}_\psi(\Omega)$

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then also the following inequality holds

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for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi \geq 0$.

Our goal is to show that the solutions to obstacle problems with non standard growth conditions solve the corresponding variational inequalities, *without any restriction on the gap* $\frac{q}{p}$

Moreover we will show that the right class of competitors are the functions with finite energy and that, in case of standard growth conditions, this coincides with $\mathcal{K}_\psi(\Omega)$

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Moreover we will show that the right class of competitors are the functions with finite energy and that, in case of standard growth conditions, this coincides with $\mathcal{K}_\psi(\Omega)$

Statement of the results

The main result

Theorem

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$ function satisfying

$$\ell|\xi|^p \leq F(\xi) \leq L(1+|\xi|^q) \tag{H1}$$

$$\nu\,|V_p(\xi)-V_p(\eta)|^2 \leq F(\xi)-F(\eta)-\langle F'(\eta), \xi-\eta\rangle \tag{H2}$$

$$F(\lambda\xi) \leq C(\lambda)\,F(\xi) \tag{H3}$$

Assume moreover that

$$F(D\psi), F(u_0) \in L^1(\Omega)$$

Suppose finally that $u \in \mathbb{K}^F_\psi(\Omega)$ is the solution to the obstacle problem

$$\min\left\{\int_\Omega F(Dz) : z \in \mathbb{K}^F_\psi(\Omega)\right\},$$

where

$$\mathbb{K}^F_\psi(\Omega) := \left\{z \in u_0 + W^{1,p}_0(\Omega) : z \geq \psi \text{ a.e. in } \Omega,\ F(Dz) \in L^1(\Omega)\right\},$$

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Then

$$F^*(F'(Du)) \in L^1(\Omega) \qquad \langle F'(Du), Du\rangle \in L^1(\Omega)$$

and

$$\operatorname{div} F'(Du) \leq 0$$

in the distributional sense

Moreover the following variational inequality

$$\int_\Omega \langle F'(Du), Dz - Du\rangle \geq 0$$

also holds for all $z \in \mathbb{K}^F_\psi(\Omega)$ such that $F(\pm Dz) \in L^1(\Omega)$

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A key example

It is worth noticing that, if there exists $f : [0, +\infty) \to [0, +\infty)$ such that

$$F(\xi) = f(|\xi|),$$

then assumption $F(\pm Dz) \in L^1(\Omega)$ is trivially satisfied

On the other hand, in order to have $F(\pm Dz) \in L^1(\Omega)$ satisfied for every $z \in \mathbb{K}^F_\psi(\Omega)$, it suffices to assume that $F(\xi) = F(-\xi)$

Under this assumption $F(\xi)$ needs not to depend on the length of ξ nor to be the sum of its components ξ_i

Indeed, an example of $F(\xi)$ satisfying our assumptions is [5]

$$F(\xi) = |\xi_1 - \xi_2|^q + |\xi_1 + \xi_2|^p \log^\alpha(1 + |\xi_1|) \qquad \xi \in \mathbb{R}^2,$$

with $2 \leq p \leq q$

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A regularity result

In case the gap $\frac{q}{p}$ satisfies a suitable smallness assumption and if the gradient of the obstacle $D\psi \in W^{1,q}_{\mathrm{loc}}(\Omega)$, we are able to prove that the solution to problem

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solves the corresponding variational inequality *without any regularity on the boundary datum* u_0

Moreover, we can prove that the solution to this minimization problem locally belongs to $W^{1,q}_{\mathrm{loc}}(\Omega)$

This result is particularly important in order to prevent the Lavrentiev phenomenon that may occurr in the case of anisotropic growth conditions [6]

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Moreover $u \in W^{1,q}_{\text{loc}}(\Omega)$

Note that in case [7]

$$\frac{np}{n-1} \leq q < p^*$$

and $D\psi \in W^{1,q}_{\text{loc}}(\Omega)$, then u belongs to $W^{1,r}_{\text{loc}}(\Omega)$ for all $r < \bar{p}$ being

$$\bar{p} := \frac{np}{n - \frac{p}{p-1}\left(1 - n\left(\frac{1}{p} - \frac{1}{q}\right)\right)}$$

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A few words about the techniques

Let's mention a few words about the techniques employed

Our Lagrangian F is suitably approximated by strictly convex and uniformly elliptic integrands F_k

The minimizers of F_k, say u_k, strongly converge in $W^{1,p}$ to the minimizer u of original obstacle problem

To every such minimizer u_k we can associate the solutions of certain dual maximization problems in the sense of Convex Analysis, for divergence-measure fields

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These duality formulas and pointwise relations between minimizers and dual maximizers are preserved in passing to the limit

Such estimates will provide conditions in order for the variational inequality to hold for a constrained minimizer

The statement and the proofs of our results are the natural counterpart of those in the unconstrained setting

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Dual formulation of the obstacle problem

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Let us establish the dual formulation of obstacle problems with standard growth conditions, extending classical ideas of Kohn and Temam and Anzellotti [8]

Given a convex continuous function $F : \mathbb{R}^n \to \mathbb{R}$, its polar (or Fenchel conjugate) is defined by

$$F^*(\zeta) := \sup_{\xi \in \mathbb{R}^n} (\langle \zeta, \xi \rangle - F(\xi)) \qquad \forall \zeta \in \mathbb{R}^n. \tag{1}$$

The function $F^* : \mathbb{R}^n \to \mathbb{R}$ is convex and, if F satisfies a $p, q-$growth condition, then F^* has $q', p'-$ growth, i.e. there exist constants $c(L), c(\ell)$ such that

$$c(L)|\zeta|^{q'} \leq F^*(\zeta) \leq c(\ell)|\zeta|^{p'} \qquad \forall \zeta \in \mathbb{R}^n.$$

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One can check that the bipolar integrand $F^{**} := (F^*)^*$ equals F at ξ if and only if F is lower semicontinuous and convex at ξ, as it is the case here

From the definition of polar function directly follows the Young-type (or Fenchel) inequality

$$\langle \zeta, \xi \rangle \leq F^*(\zeta) + F^{**}(\xi)$$

for all $\zeta, \xi \in \mathbb{R}^n$

Notice that, for a given ξ, we have equality in the Fenchel inequality precisely for $\zeta \in \partial F^{**}(\xi)$, the subgradient of F^{**} at ξ

In particular, when F is C^1, for every $\xi \in \mathbb{R}^n$, we have equality in the Fenchel inequality precisely for $\zeta = F'(\xi)$

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Now, we consider for any $p > 1$

$$S_{-}^{p'}(\Omega) = \{\sigma \in L^{p'}(\Omega) : \ \mathrm{div}\sigma \leq 0 \quad \text{in} \quad \mathcal{D}'(\Omega)\},$$

where as usual $p' = \frac{p}{p-1}$ and, for $u_0 \in W^{1,p}(\Omega)$, $U \in L^p(\Omega)$ we introduce a measure $[\![\sigma, DU]\!]_{u_0}$ on Ω by setting

$$[\![\sigma, DU]\!]_{u_0} = \int_\Omega (U - u_0)\, d(-\mathrm{div}\sigma) + \int_\Omega \langle \sigma, Du_0 \rangle \, dx.$$

For $U \in u_0 + W_0^{1,p}(\Omega)$, the measure $[\![\sigma, DU]\!]_{u_0}$ corresponds to the function $\langle \sigma, DU \rangle \in L^1(\Omega)$ as it follows from the well known integration by parts formula

$$\int_\Omega \varphi d(-\mathrm{div}\sigma) = \int_\Omega \langle \sigma, D\varphi \rangle \, dx,$$

for every $\varphi \in W_0^{1,p}(\Omega)$

Dual formulation of the obstacle problem

Now, we consider for any $p > 1$

$$S_-^{p'}(\Omega) = \{\sigma \in L^{p'}(\Omega) : \ \mathrm{div}\sigma \leq 0 \quad \text{in} \quad \mathcal{D}'(\Omega)\},$$

where as usual $p' = \frac{p}{p-1}$ and, for $u_0 \in W^{1,p}(\Omega)$, $U \in L^p(\Omega)$ we introduce a measure $[\![\sigma, DU]\!]_{u_0}$ on Ω by setting

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Dual formulation of the obstacle problem

Theorem

Let $G : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$, strictly convex function satisfying

$$\ell_p\,(|\xi|^p - 1) \leq \; G(\xi) \leq \; L_p(1 + |\xi|^p),$$

for all $\xi \in \mathbb{R}^n$ and an exponent $p > 1$. Then

$$\min_{v \in \mathcal{K}_\psi(\Omega)} \int_\Omega G(Dv)\,dx = \max_{\sigma \in S^{p'}_-(\Omega)} \left([\![\sigma, D\psi]\!]_{u_0} - \int_\Omega G^*(\sigma)\,dx \right)$$

where we recall that

$$\mathcal{K}_\psi(\Omega) := \left\{ z \in u_0 + W^{1,p}_0(\Omega) : z \geq \psi \text{ a.e. in } \Omega \right\}$$

If moreover $u \in \mathcal{K}_\psi(\Omega)$ is the solution to

$$\min \left\{ \int_\Omega G(Dz) : z \in \mathcal{K}_\psi(\Omega) \right\},$$

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$$\int_\Omega G(Du)\,dx = [\![G'(Du), D\psi]\!]_{u_0} - \int_\Omega G^*(G'(Du))\,dx.$$

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Dual formulation of the obstacle problem: idea of the proof

Step 1:

$$\min_{v\in\mathcal{K}_\psi(\Omega)}\int_\Omega G(Dv)\,dx \boxed{\geq} \max_{\sigma\in S_-^{p'}(\Omega)}\left([\![\sigma, D\psi]\!]_{u_0} - \int_\Omega G^*(\sigma)\,dx\right)$$

• we use the fact that if $\sigma \in S_-^{p'}(\Omega)$ then $-\text{div}\sigma$ is a non-negative Radon measure, so for every $v \in \mathcal{K}_\psi(\Omega)$

$$\int_\Omega (v-\psi)d(-\text{div}\sigma) \geq 0$$

Step 2:

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• We use

$$G(Du) + G^*(G'(Du)) = \langle G'(Du), Du\rangle,$$

and exploit the fact that u solution to the minimization problem, satisfies the corresponding variational inequality

Dual formulation of the obstacle problem: idea of the proof

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Proof of the main result

The main result

Theorem

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$ function satisfying

$$\ell|\xi|^p \le F(\xi) \le L\,(1+|\xi|^q) \tag{H1}$$

$$\nu\,|V_p(\xi)-V_p(\eta)|^2 \le F(\xi)-F(\eta)-\langle F'(\eta),\xi-\eta\rangle \tag{H2}$$

$$F(\lambda\,\xi) \le C(\lambda)\,F(\xi) \tag{H3}$$

Assume moreover that

$$F(D\psi), F(u_0) \in L^1(\Omega)$$

Suppose finally that $u \in \mathbb{K}^F_\psi(\Omega)$ is the solution to the obstacle problem

$$\min\left\{\int_\Omega F(Dz) : z \in \mathbb{K}^F_\psi(\Omega)\right\},$$

where

$$\mathbb{K}^F_\psi(\Omega) := \left\{z \in u_0 + W^{1,p}_0(\Omega) : z \ge \psi \text{ a.e. in } \Omega,\ \ F(Dz) \in L^1(\Omega)\right\},$$

The main result

Theorem

Then

$$F^*(F'(Du)) \in L^1(\Omega) \qquad \langle F'(Du), Du\rangle \in L^1(\Omega)$$

and

$$\operatorname{div} F'(Du) \leq 0$$

in the distributional sense

Moreover the following variational inequality

$$\int_\Omega \langle F'(Du), Dz - Du\rangle \geq 0$$

also holds for all $z \in \mathbb{K}^F_\psi(\Omega)$ such that $F(\pm Dz) \in L^1(\Omega)$

Strategy of the proof

Step 1: we construct a sequence of obstacle problems with standard growth condition for which the dual problem is given by the previous theorem

Step 2: we prove that the sequence of approximating minimizers converges to the solution to the original problem

Step 3: we prove that the sequence of dual maximizers converges to a field whose divergence is a non positive Radon measure

Step 4: we establish the validity of the variational inequality

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Step 1: approximation

Let F_k be the sequence of Lagrangians obtained applying a suitable approximation lemma to the integrand F and let $u_k \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem

$$\min_{w \in \mathcal{K}_\psi(\Omega)} \int_\Omega F_k(Dw)\,dx$$

and let

$$\sigma_k := F_k'(Du_k) \in \mathcal{S}^{p'}_-(\Omega)$$

be the solution to the dual problem, i.e. σ_k is such that

$$\max_{\sigma \in S^{p'}_-(\Omega)} \left\{ [\![\sigma, D\psi]\!]_{u_0} - \int_\Omega F_k^*(\sigma)\,dx \right\} = [\![\sigma_k, D\psi]\!]_{u_0} - \int_\Omega F_k^*(\sigma_k)\,dx,$$

where F_k^* denotes the polar function of F_k. Then we have

$$\int_\Omega F_k(Du_k)\,dx = [\![\sigma_k, D\psi]\!]_{u_0} - \int_\Omega F_k^*(\sigma_k)\,dx$$

holds for all $k \in \mathbb{N}$ and

$$\int_\Omega \langle \sigma_k, D\varphi - Du_k \rangle\,dx \geq 0 \qquad \forall\, \varphi \in \mathcal{K}_\psi(\Omega) \quad \text{and } \forall\, k \in \mathbb{N}.$$

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Step 2: passage to the limit (minimizers)

Our next purpose is to prove that $u_k \to u$ strongly in $W^{1,p}(\Omega)$, where u is the solution to the obstacle problem to our original problem

To this aim we exploit:

• the growth condition on F_k, the minimality of u_k and the reflexivity of $W^{1,p}$ to show that u_k weakly converge to some v

• the fact that the set $\mathcal{K}_\psi(\Omega)$ is closed and convex to state that $v \in \mathcal{K}_\psi(\Omega)$

• the lower semicontinuity of some F_{k_0}, the monotonicity of F_k, the monotone convergence theorem to show that actually $v \in \mathbb{K}_\psi(\Omega)$

• the minimality of u in the class $\mathbb{K}_\psi(\Omega)$ as well as the monotone convergence theorem to prove that $u = v$

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Step 3: passage to the limit (dual maximizers)

At this point the assumptions given from the approximation lemma yield

$$\sigma_k = F_k'(Du_k) \to F'(Du) \qquad \text{locally uniformly as } k \to \infty$$

It follows in particular that $F_k'(Du_k) \to F'(Du)$ in measure on Ω and so passing to the limit in the equality

$$\langle \sigma_k, Du_k \rangle = F_k^*(\sigma_k) + F_k(Du_k),$$

we recover, with $\sigma = F'(Du)$, the pointwise extremality relation

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Step 3: pointwise extremality relation

Once from the previous passage we established the pointwise extremality relation

$$\langle F'(Du), Du\rangle = F^*(F'(Du)) + F(Du)$$

we want to prove that $\langle F'(Du), Du\rangle \in L^1(\Omega)$ and $F^*(F'(Du)) \in L^1(\Omega)$

To this aim we derived the bound

$$\int_\Omega F^*(\sigma_k)\, dx \leq C \int_\Omega F(Du_0)\, dx$$

Since we already observed that $\sigma_k \to F'(Du)$ locally uniformly and $F^*(\sigma_k) \geq 0$ for every k, by Fatou's lemma and by previous estimate

$$\int_\Omega F^*(F'(Du))\, dx \leq \liminf_{k\to+\infty} \int_\Omega F^*(\sigma_k)\, dx \leq C \int_\Omega F(Du_0)\, dx.$$

Thus

$$F^*(F'(Du)) \in L^1(\Omega).$$

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Step 3: the limit field is a non positive Radon measure

We prove that $\boxed{\operatorname{div}\sigma \leq 0}$ in the distributional sense

To this aim we exploit:

- the fact that in view of the $(q', p')-$growth of $F^*(\sigma)$ and of $F_k^*(\sigma_k)$ we are able to deduce a bound for the term $\int_\Omega |F'(Du)|^{q'}\,dx$ and thus the fact $\sigma_k \to \sigma$ a.e. up to a subsequence

- The minimality of u_k yields the validity of the following variational inequality

$$\int_\Omega \langle \sigma_k, D\eta \rangle \, dx \geq 0 \qquad \text{for all } \eta \in C_0^\infty(\Omega), \ \ \eta \geq 0,$$

and so, by the weak convergence of σ_k to σ in $L^{q'}(\Omega)$, passing to the limit as $k \to \infty$ in previous inequality, also

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Step 4: the validity of the variational inequality

We have that

$$|\langle \sigma_k, Dz \rangle| \leq 2F_k^*(\sigma_k) + F(Dz) + F(-Dz),$$

Moreover

$$\begin{aligned}\int_\Omega |\langle \sigma_k, Dz\rangle|\, dx &\leq \int_\Omega F_k^*(\sigma_k) + \int_\Omega F(Dz)\, dx + \int_\Omega F(-Dz)\, dx \\ &\leq C\left(\int_\Omega F(Du_0)\, dx + \int_\Omega F(Dz)\, dx + \int_\Omega F(-Dz)\, dx\right).\end{aligned}$$

thus, by our assumptions, the sequence $\langle \sigma_k, Dz\rangle$ is equi-integrable

Using that $\langle \sigma_k, Dz\rangle \to \langle \sigma, Dz\rangle$ a.e., Vitali's convergence Theorem implies

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Step 4: the validity of the variational inequality

At this point, we start from the variational inequality

$$\int_\Omega \langle \sigma_k, Dz - Du_k \rangle \, dx \geq 0 \qquad \text{for all } z \in \mathbb{K}_\psi^F(\Omega),$$

since $\mathbb{K}_\psi^F(\Omega) \subset \mathcal{K}_\psi(\Omega)$, writing it as

$$\int_\Omega \langle \sigma_k, Dz \rangle \, dx \geq \int_\Omega \langle \sigma_k, Du_k \rangle \, dx \qquad \text{for all } z \in \mathbb{K}_\psi^F(\Omega)$$

and taking the liminf as $k \to +\infty$ in previous equality, we get

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Thank you very much for the attention!