On the existence of integrable solutions to elliptic and parabolic systems with linear growth - applications in (visco)-elasticity theory

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Monday's Nonstandard Seminar

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The talk is based on the following results

- M. Bulíček, J. Málek, K. R. Rajagopal and J. R. Walton: **Existence of solutions for the anti-plane stress for a new class of "strain-limiting" elastic bodies**, Calc. Var. Partial Differential Equations, 2015
- M. Bulíček, J. Málek and E. Süli: **Analysis and approximation of a strain-limiting nonlinear elastic model**, Mathematics and Mechanics of Solids, 2014
- M. Bulíček, J. Málek, K. R. Rajagopal and E. Süli: **On elastic solids with limiting small strain: modelling and analysis**, EMS Surveys in Mathematical Sciences, 2014.
- L. Beck, M. Bulíček, J. Málek and E. Süli: **On the existence of integrable solutions to nonlinear elliptic systems and variational problems with linear growth**, ARMA 2017
- L. Beck, M. Bulíček, E. Maringová: **On regularity up to the boundary for variational problems with linear growth**, ESAIM Control Optim. Calc. Var. 2018
- L. Beck, M. Bulíček, F. Gmeineder: **On existence of $W^{1,1}$ solutions to variational problems with linear growth**, to appear at Annali della Scuola Normale Superiore (Pisa) 2020
- M. Bulíček, V. Patel, Y.Şengül, E. Süli:**Existence of large-data global weak solutions to a model of a strain-limiting viscoelastic body**, arXiv, 2020
- M. Bulíček, V. Patel, Y.Şengül, E. Süli:**Existence and uniqueness of global weak solutions to strain-limiting viscoelasticity with Dirichlet boundary**, arXiv, 2020

Minimal surface equation

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Answers:

- If Ω is convex (or generally has nonnegative mean curvature) then there always exists (smooth) solution.
- If Ω has negative mean curvature (or is nonconvex in 2D) then there always exists U_0 for which the solution does not exist.

Minimal surface equation - BV setting - relaxed formulation

- Minimize the relaxed functional over the space $BV(\Omega)$, i.e., find $U \in BV(\tilde{\Omega})$ with $\overline{\Omega} \subset \tilde{\Omega}$ that minimizes

$$\min_{U \in BV(\tilde{\Omega}); U = U_0 \text{ in } \tilde{\Omega} \setminus \overline{\Omega}} \int (1 + |(\nabla U)^r|^2)^{\frac{1}{2}} + |(\nabla U)^s|(\overline{\Omega}),$$

where $(\nabla U)^r$ is the regular (absolutely continuous w.r.t. Lebesgue measure) part of ∇U (which is a Radon measure), and $(\nabla U)^s$ is the singular part.

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- Weak lower semicontinuity $\implies$ minimizer always exists.
- De Giorgi $\implies$ $U \in \mathcal{C}^{1,\alpha}_{loc}(\Omega)$ and in fact we have the "half" relaxed formulation:

$$\min_{U \in W^{1,1}(\Omega)} \int (1 + |\nabla U|^2)^{\frac{1}{2}} + \int_{\partial\Omega} |U - U_0|$$

Generalized problem

DATA: $\Omega \subset \mathbb{R}^d$ open smooth bounded (connected), $a \in (0, \infty)$, $U_0 \in \mathcal{C}^\infty(\overline{\Omega})$, $G \in \mathcal{C}^\infty(\overline{\Omega}; \mathbb{R}^d)$, and $\Gamma_D, \Gamma_N \subset \partial\Omega$ are smooth open (in (d-1) sense) disjoint parts of the boundary whose union is of the full measure of $\partial\Omega$

GOAL: To find for $U \in W^{1,1}(\Omega)$ such that

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- Necessary compatibility condition

$$\|G\|_\infty \le 1$$

- Safe data condition

$$\|G\|_\infty < 1.$$

Concepts of solutions

- Weak solution: we look for U, such that $U - U_0 \in W^{1,1}_{\Gamma_D}(\Omega)$ and for all $\varphi \in W^{1,1}_{\Gamma_D}(\Omega)$

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- Relaxed formulation: to find $U_0 \in BV(\tilde{\Omega})$ being equal to U_0 outside Ω that minimizes

$$\min_{U \in BV} \int_\Omega F(|(\nabla U)^r|) + |\nabla U^s|(\overline{\Omega} \setminus \Gamma_N) - \langle G, \nabla U \rangle.$$

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- Problem 3: Can we find a range of a's and a class of nonconvex domains for which the weak solution always exists?

Problems (partially) solved:)

Theorem

- *Problem 3:* $a \in (0,2)$ *and* $\Gamma_D = \bigcup_{i=1}^{N} \Gamma_i$ *such that either* Γ_i *is uniformly convex and* U_0 *is smooth on* Γ_i *or* Γ_i *is flat and* U_0 *is constant there* $\implies$ *there exists a weak solution (B, Málek, Rajagopal, Walton).*
- *Problem 2:* $a \in (0,1]$ *and* $\Gamma_N = \emptyset$ $\implies$ *there is always a weak solution (Beck, B, Maringová).*
- *Problem 1:* $a > 0$, Ω **simply connected** *and* $\Gamma_D = \emptyset$ $\implies$ *there is always a weak solution. (Beck, B, Gmeineder)*

General result I - regularity up to the boundary

Theorem (Beck, Bulíček, Maringová)

Let $F \in \mathcal{C}^2(0,\infty)$ be increasing strictly convex fulfilling

$$\lim_{s\to\infty} \frac{F(s)}{s} = \lim_{s\to\infty} F'(s) = K, \qquad 0 < \lim_{s\to\infty} \frac{F''(2s)}{F''(s)} < \infty.$$

Then the following is equivalent

- *For any $\Omega \in \mathcal{C}^{1,1}$ and any $u_0 \in \mathcal{C}^{1,1}(\overline{\Omega})$ there exists unique $u \in W^{1,\infty}(\Omega)$ fulfilling*

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The second condition is equivalent to the fact that

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If $\int_1^\infty sF''(s) < \infty$ *then for any smooth domain satisfying the inner ball condition (in 2d any nonconvex domain) there exists a smooth function* U_0 *such that the minimizer does not belong to* $W^{1,1}(\Omega)\cap\mathcal{C}(\overline{\Omega})$*.*

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Let Ω be simply connected domain, $\Gamma_D = \emptyset$ and G satisfy the safe data condition. Then there exists a unique (up to a constant) weak solution.

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There is no improvement of integrability of ∇u!!!.

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If you are not interested in calculus of variations but you are interested in continuum mechanics, please wake up!

Linearized nonlinear elasticity

We consider the elastic deformation of the body $\Omega \subset \mathbb{R}^d$ with $\Gamma_D \cap \Gamma_N = \emptyset$ and $\overline{\Gamma_D \cup \Gamma_N} = \partial\Omega$ described by

$$\begin{aligned} -\operatorname{div} \mathbf{T} &= \boldsymbol{f} && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{u}_0 && \text{on } \Gamma_D, \qquad \text{and} \qquad \mathbf{T}\boldsymbol{n} = \boldsymbol{g} \qquad \text{on } \Gamma_N. \end{aligned} \tag{El}$$

where $\boldsymbol{u}$ is displacement, $\mathbf{T}$ the Cauchy stress, $\boldsymbol{f}$ the external body forces, $\boldsymbol{g}$ the external surface forces and $\boldsymbol{\varepsilon}$ is the linearized strain tensor, i.e.,

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- The key assumption in linearized elasticity

$$\boxed{|\boldsymbol{\varepsilon}| \ll 1}. \tag{A}$$

Motivation for symmetric p-Laplace like operator for $p = 1$ or $p = \infty$

- $p = 1$: the plasticity model, i.e.,

$$\mathbf{T} \sim \frac{\varepsilon}{|\varepsilon|} \qquad \text{for } |\varepsilon| \gg 1$$

- $p = \infty$: the limiting strain model, i.e.,

$$\varepsilon \sim \frac{\mathbf{T}}{|\mathbf{T}|} \qquad \text{for } |\mathbf{T}| \gg 1.$$

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$\Longrightarrow$ contradicts the assumption of the model (A) $\Longrightarrow$ not valid model at least in the neighborhood of x_0.

Simplified setting - potential structure

We look for $(\boldsymbol{u}, \mathbf{T})$ such that $\boldsymbol{u} = \boldsymbol{u}_0$ on Γ_D and $\mathbf{T}\boldsymbol{n} = \boldsymbol{g}$ on Γ_N and fulfilling

$$\left.\begin{aligned} -\operatorname{div}\mathbf{T} &= \boldsymbol{f}, \\ \varepsilon(\boldsymbol{u}) &= \varepsilon^*(\mathbf{T}). \end{aligned}\right\} \quad \Leftrightarrow \quad \left\{ -\operatorname{div}\mathbf{T}^*(\varepsilon(\boldsymbol{u})) = \boldsymbol{f}. \right.$$

in Ω with

$$\boxed{\varepsilon^*(\mathbf{T}) := \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} \qquad \text{and} \qquad T^*(\mathbf{W}) := (\varepsilon^*)^{-1}(\mathbf{W}) := \frac{\mathbf{W}}{(1-|\mathbf{W}|^a)^{\frac{1}{a}}}}$$

for all $\mathbf{T} \in \mathbb{R}^{d\times d}_{sym}$ and $\mathbf{W} \in \mathbb{R}^{d\times d}_{sym}$ satisfying $|\mathbf{W}| < 1$.

Simplified setting - potential structure

First, we introduce the space of functions having bounded the symmetric gradient

$$E := \{\boldsymbol{u} \in W^{1,1}(\Omega)^d;\ \boldsymbol{\varepsilon}(\boldsymbol{u}) \in L^\infty(\Omega)^{d\times d}\}.$$

and assume at least $\boldsymbol{u}_0 \in E$, $\boldsymbol{f} \in L^2(\Omega)^d$ and $\boldsymbol{g} \in L^1(\Gamma_N)^d$.

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- the set of admissible displacements

$$\mathcal{V} := \{\boldsymbol{u} \in E :\ \boldsymbol{u} - \boldsymbol{u}_0 \in W^{1,1}_{\Gamma_D}(\Omega)^d\}$$

- the set of admissible stresses

$$\mathcal{S} := \left\{\mathbf{T} \in L^1(\Omega)^{d\times d}_{sym} :\ \forall \boldsymbol{v} \in E \cap W^{1,1}_{\Gamma_D}\ \int_\Omega \mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{v}\right\}$$

Weak solution: Find $(\boldsymbol{u}, \mathbf{T}) \in \mathcal{V}\times\mathcal{S}$ such that $\varepsilon(\boldsymbol{u}) = \varepsilon^*(\mathbf{T})$ a.e. in Ω.

Potential structure - primary formulation

Find potential $F : \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}_+$ such that $F(0) = 0$ and

$$\frac{\partial F(\mathbf{W})}{\partial \mathbf{W}} = \mathbf{T}^*(\mathbf{W}) \qquad \text{if } |\mathbf{W}| < 1,$$
$$F(\mathbf{W}) = \infty \qquad \text{if } |\mathbf{W}| > 1.$$

Primary (variational) formulation: Find $\boldsymbol{u} \in \mathcal{V}$ such that for all $\boldsymbol{v} \in \mathcal{V}$

$$\boxed{\int_\Omega F(\varepsilon(\boldsymbol{u})) - f \cdot \boldsymbol{u} - \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{u} \le \int_\Omega F(\varepsilon(\boldsymbol{v})) - f \cdot \boldsymbol{v} - \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v}}$$

Lemma

Let $\|\varepsilon(\boldsymbol{u}_0)\|_\infty < 1$ (the safety strain condition). Then there exists a unique $\boldsymbol{u}$ solving the primary formulation. Moreover there exists $\mathbf{T} \in L^1(\Omega)^{d\times d}$ such that $\varepsilon(\boldsymbol{u}) = \varepsilon^*(\mathbf{T})$ and for all $\boldsymbol{v} \in \mathcal{V}$ such that $\mathbf{T}^*(\varepsilon(\boldsymbol{v})) \in L^1$ there holds

$$\int_\Omega \mathbf{T} \cdot \varepsilon(\boldsymbol{u} - \boldsymbol{v}) \le \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u} - \boldsymbol{v}) + \int_{\Gamma_N} \boldsymbol{g} \cdot (\boldsymbol{u} - \boldsymbol{v})$$

In addition, if there is a weak solution then it also solves the primary formulation. Similarly, if $\boldsymbol{u}$ satisfies the safety strain condition, then $(\boldsymbol{u}, \mathbf{T})$ is a weak solution.

Potential structure - dual formulation

Find potential $F^*: \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}_+$ such that $F(0) = 0$ and (note here that $F(\mathbf{W}) \sim |\mathbf{W}|$ at infinity

$$\frac{\partial F^*(\mathbf{W})}{\partial \mathbf{W}} = \boldsymbol{\varepsilon}^*(\mathbf{W}).$$

Dual (variational) formulation: Find $\mathbf{T} \in \mathcal{S}$ such that for all $\mathbf{W} \in \mathcal{S}$

$$\boxed{\int_\Omega F^*(\mathbf{T}) - \mathbf{T}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \le \int_\Omega F^*(\mathbf{W}) - \mathbf{W}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_0)}$$

Lemma

The existence of a weak solution is equivalent to the existence of a minimizer to the dual problem. Moreover, if $\|\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\|_\infty < 1$ (the safety strain condition) then there exists a finite infimum of the dual formulation which may be attained by $\overline{\mathbf{T}} \in \mathcal{M}(\overline{\Omega})^{d\times d}_{sym}$.

Potential structure - relaxed dual formulation

- the relaxed set of admissible stresses

$$\mathcal{S}^m := \left\{ \mathbf{T} \in \mathcal{M}(\overline{\Omega})^{d\times d}_{sym} : \forall \boldsymbol{v} \in \mathcal{C}^1_{\Gamma_D}(\Omega)^d \int_\Omega \mathbf{T} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v} \right\}$$

Dual (variational) relaxed formulation: For $\boldsymbol{u}_0 \in \mathcal{C}^1(\Omega)^d$, find $\mathbf{T} \in \mathcal{S}^m$ such that for all $\mathbf{W} \in \mathcal{S}^m$

$$\boxed{\int_\Omega F^*(\mathbf{T}^r) + (\mathbf{W}^r - \mathbf{T}^r) \cdot \varepsilon(\boldsymbol{u}_0) + |\mathbf{T}^s|(\overline{\Omega}) + \langle \mathbf{W}^s - \mathbf{T}^s, \varepsilon(\boldsymbol{u}_0)\rangle \leq \int_\Omega F^*(\mathbf{W}^r) + |\mathbf{W}^s|(\overline{\Omega})}$$

where $\mathbf{T} = \mathbf{T}^r + \mathbf{T}^s$ and $\mathbf{T}^r$ is a regular part (i.e., absolutely continuous w.r.t. Lebesgue measure) and $\mathbf{T}^s$ is a singular part (i.e., supported on the set of zero Lebesgue measure).

Lemma

Let $\|\varepsilon(\boldsymbol{u}_0)\|_\infty < 1$. Then there exists a minimizer to relaxed dual formulation. Moreover, the regular part $\mathbf{T}^r$ is unique and satisfies $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^*(\mathbf{T}^r)$, where $\boldsymbol{u}$ is the (unique) minimizer to primary formulation. In addition, if $\mathbf{T}^s_1$ and $\mathbf{T}^s_2$ are two singular parts then for all $\boldsymbol{v} \in \mathcal{C}^1_{\Gamma_D}(\Omega)^d$

$$|\mathbf{T}^s_1|(\overline{\Omega}) - \langle \mathbf{T}^s_1, \boldsymbol{\varepsilon}(\boldsymbol{u}_0)\rangle = |\mathbf{T}^s_2|(\overline{\Omega}) - \langle \mathbf{T}^s_2, \boldsymbol{\varepsilon}(\boldsymbol{u}_0)\rangle \text{ and } \langle \mathbf{T}^s_1 - \mathbf{T}^s_2, \nabla \boldsymbol{v}\rangle = 0$$

Conclusion

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- We solved the problem completely. Natural setting is the relaxed dual formulation. The displacement is unique. The regular part of the Cauchy stress is unique. There is non-uniquely given singular part of the Cauchy stress.

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- We solved the problem completely. Natural setting is the relaxed dual formulation. The displacement is unique. The regular part of the Cauchy stress is unique. There is non-uniquely given singular part of the Cauchy stress.
- Where is the singular measure supported? Is it really there? How do you explain that the regular part did not solve the balance equation? Is there some crack/damage possible region? Is there any influence of the shape of Ω or the parameter a? etc. etc.

Limiting strain model - anti-plane stress

We consider the following special geometry

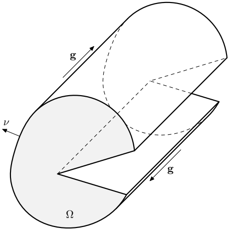

Figure: Anti-plane stress geometry.

and we look for the solution in the following from:

$$\boldsymbol{u} = \boldsymbol{u}(x_1, x_2) = (0, 0, u(x_1, x_2)), \quad \boldsymbol{g}(x) = (0, 0, g(x_1, x_2)),$$

and

$$\mathbf{T}(x) = \begin{pmatrix} 0 & 0 & T_{13}(x_1, x_2) \\ 0 & 0 & T_{23}(x_1, x_2) \\ T_{13}(x_1, x_2) & T_{23}(x_1, x_2) & 0 \end{pmatrix}. \tag{1}$$

Equivalent reformulation-simply connected domain

- Find $U : \Omega \to \mathbb{R}$ - the Airy stress function such that

$$T_{13} = \frac{1}{\sqrt{2}} U_{x_2} \quad \text{and} \quad T_{23} = -\frac{1}{\sqrt{2}} U_{x_1}.$$

$\implies \operatorname{div} \mathbf{T} = \mathbf{0}$ is fulfilled.

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$\implies \operatorname{div} \mathbf{T} = \mathbf{0}$ is fulfilled.

- U must satisfy $(\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}})$

$$\begin{aligned} \operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^a)^{\frac{1}{a}}}\right) &= 0 && \text{in } \Omega, \\ U_{x_2}\boldsymbol{n}_1 - U_{x_1}\boldsymbol{n}_2 &= \sqrt{2}g && \text{on } \partial\Omega. \end{aligned}$$

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$$U_{x_2}\boldsymbol{n}_1-U_{x_1}\boldsymbol{n}_2=\sqrt{2}g \qquad \text{on } \partial\Omega.$$

- Dirichlet problem, indeed assume that $\partial\Omega$ is parameterized by $\gamma(s)=(\gamma_1(s),\gamma_2(s))$. Then

$$U(\gamma(s_0))=a_0+\sqrt{2}\int_0^{s_0} g(\boldsymbol{\gamma}(s))\sqrt{(\gamma_1'(s))^2+(\gamma_2'(s))^2}ds=:U_0(x).$$

Consequences for U

- We look for $U \in W^{1,1}(\Omega)$

$$\operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^a)^{\frac{1}{a}}}\right) = 0 \quad \text{in } \Omega, \qquad U = U_0 \quad \text{on } \partial\Omega.$$

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- Calculus of variations and BV setting are back! Please wake up!

General setting & general geometry

- All presented results were based either on the scalar structure or on the *radial* structure
- There is NO theory for nonlinear systems having the *radial*-like structure with respect to the symmetric gradient, which would be better than the theory for general elliptic systems having no structure.

Result for particular model and general geometry

Consider $\varepsilon^*(\mathbf{T}) = \mathbf{T}/(1+|\mathbf{T}|^a)^{\frac{1}{a}}$:

Theorem (General result for $a>0$)

Let $a>0$ and $\boldsymbol{u}_0$ satisfy the safety strain condition. Then there exists a unique triple $(\boldsymbol{u}, \mathbf{T}, \tilde{\boldsymbol{g}}) \in \mathcal{V} \times L^1(\Omega)^{d\times d}_{sym} \times (\mathcal{C}^1_0(\Gamma_N))^*$ such that for all $\boldsymbol{v} \in \mathcal{C}^1_{\Gamma_D}(\overline{\Omega})$

$$\varepsilon(\boldsymbol{u}) = \varepsilon^*(\mathbf{T})$$

$$\int_\Omega \mathbf{T}\cdot\varepsilon(\boldsymbol{u}-\boldsymbol{w}) \le \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{u}-\boldsymbol{w}) + \int_{\Gamma_N} \boldsymbol{g}\cdot(\boldsymbol{u}-\boldsymbol{w})$$

$$\boldsymbol{u} = \boldsymbol{u}_0 \ \ on\ \Gamma_D,$$

where $\boldsymbol{w} \in \mathcal{V}$ is arbitrary such that there exists $\tilde{\mathbf{T}} \in L^1$ fulfilling $\varepsilon(\boldsymbol{w}) = \varepsilon^*(\tilde{\mathbf{T}})$.

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$$\boldsymbol{u} = \boldsymbol{u}_0 \ \ on \ \Gamma_D,$$

where $\boldsymbol{w} \in \mathcal{V}$ is arbitrary such that there exists $\tilde{\mathbf{T}} \in L^1$ fulfilling $\varepsilon(\boldsymbol{w}) = \varepsilon^*(\tilde{\mathbf{T}})$. Moreover,

$$\boxed{\int_\Omega \mathbf{T}\cdot\nabla\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + \langle \boldsymbol{g}-\tilde{\boldsymbol{g}}, \boldsymbol{v}\rangle_{\Gamma_N}}$$

Assumptions for general model

Assumptions on ε^*: Denote $\mathbf{A}(\mathbf{T}) := \frac{\partial \boldsymbol{\varepsilon}^*(\mathbf{T})}{\partial \mathbf{T}}$.

- ε^* is coercive, i.e.,
$$\varepsilon^*(\mathbf{T}) \cdot \mathbf{T} \geq C_1|\mathbf{T}| - C_2$$
- ε^* is h-elliptic, i.e., there exists nonincreasing function h such that for all $\mathbf{W} \neq 0$
$$0 < h(|\mathbf{T}|)|\mathbf{W}|^2 \leq (\mathbf{W},\mathbf{W})_{\mathbf{A}(\mathbf{T})} \leq \frac{|\mathbf{W}|^2}{1+|\mathbf{T}|},$$
where
$$(\mathbf{W},\mathbf{W})_{\mathbf{A}(\mathbf{T})} := \sum \mathbf{A}^{\nu i}_{\mu j}(\mathbf{T})\mathbf{W}^{\nu i}\mathbf{W}^{\mu j}, \qquad \mathbf{A}^{\nu i}_{\mu j}(\mathbf{T}) := \frac{\partial (\boldsymbol{\varepsilon}^*)^{\nu i}(\mathbf{T})}{\partial \mathbf{T}^{\mu j}}.$$
- $\mathbf{A}$ is asymptotically symmetric, i.e.,
$$\frac{|\mathbf{A}^s(\mathbf{T}) - \mathbf{A}(\mathbf{T})|^2}{h(|\mathbf{T}|)} \leq \frac{C_2}{1+|\mathbf{T}|}.$$
- either h does not decrease faster than $|\mathbf{T}|^{-1-2/d}$ or ε^* is asymptotically *radial*, i.e., there exists a function g such that $g(|\mathbf{T}|) \leq C(1+|\mathbf{T}|)$ fulfilling
$$\frac{|g(|\mathbf{T}|)\varepsilon^*(\mathbf{T}) - \mathbf{T}|^2}{h(|\mathbf{T}|)} \leq C_2(1+|\mathbf{T}|^3).$$

Assumptions for general models

Assumptions on data:

- $\boldsymbol{f} \in L^2$
- $\boldsymbol{g} \in L^1$
- $\boldsymbol{u}_0$ satisfies safety strain condition, i.e., there exists a compact set $K \subset \varepsilon^*(\mathbb{R}^{d\times d}_{sym})$ such that for almost all $x \in \Omega$
$$\boldsymbol{\varepsilon}(\boldsymbol{u}_0(x)) \in K$$
.

Result for limiting strain models

Theorem (General result)

There exists a unique triple $(\boldsymbol{u}, \mathbf{T}, \tilde{\boldsymbol{g}}) \in W^{1,1}(\Omega)^d \times L^1(\Omega)^{d\times d}_{sym} \times (\mathcal{C}^1_0(\Gamma_d))^*$ *such that*

$$\int_\Omega \mathbf{T} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \langle \boldsymbol{g} - \tilde{\boldsymbol{g}}, \boldsymbol{v} \rangle_{\Gamma_N}$$

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathbf{D}(\mathbf{T}) \in L^\infty(\Omega; \mathbb{R}^{d\times d})$$

$$\int_\Omega \mathbf{T} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{w}) \le \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u} - \boldsymbol{w}) + \int_{\Gamma_N} \boldsymbol{g} \cdot (\boldsymbol{u} - \boldsymbol{w})$$

$$\boldsymbol{u} = \boldsymbol{u}_0 \;\; on \; \Gamma_D,$$

are satisfied for all $\boldsymbol{v} \in \mathcal{C}^1_{\Gamma_D}(\overline{\Omega})$ *and all* $\boldsymbol{w} \in W^{1,\infty}(\Omega)$*, where* $\boldsymbol{w}$ *is an arbitrary function being equal to* $\boldsymbol{u}_0$ *on* Γ_D *and for which there exists* $\tilde{\mathbf{T}} \in L^1(\Omega)^{d\times d}_{sym}$ *fulfilling* $\varepsilon(\boldsymbol{w}) = \varepsilon^*(\tilde{\mathbf{T}})$*.*

Conclusion II

- The first result for the symmetric gradient, where the *radial* setting plays the crucial role.
- The same result obviously holds also for the full gradient case.
- For any $\mathcal{C}^1$ strictly monotone operator being asymptotically symmetric and radial we avoided the presence of the singular part in the interior!
- At least in 2D and simply connected domains, we can convert this setting to the minimal surface-like problems and get the same result.
- The method does not use the improved integrability result (which even may not be true)!
- The same theory for minimal surface-like problems and general geometries. Sharp identification of the cases when the theory can be built up to the boundary.

Scheme of the proof

We find a mollified problem for which we have a solution and then go to the limit. The approximation is of the form

$$\varepsilon_n^*(\mathbf{T}) := \varepsilon^*(\mathbf{T}) + n^{-1}\frac{\mathbf{T}}{(1+|\mathbf{T}|)^{1-\frac{1}{n}}}.$$

- The first a priori estimate

$$\int_\Omega |\mathbf{T}^n| \le C, \qquad \|\varepsilon(\boldsymbol{u}^n)\|_n \le C.$$

- $$\begin{aligned} \mathbf{T}^n &\rightharpoonup^* \overline{\mathbf{T}} && \text{in } \mathcal{M}(\overline{\Omega})^{d\times d}_{sym}, \\ \varepsilon(\boldsymbol{u}^n) &\rightharpoonup \varepsilon(\boldsymbol{u}) && \text{in } L^q(\Omega)^{d\times d}_{sym}, \text{ for all } q<\infty. \end{aligned}$$

  and $\overline{\mathbf{T}}$ solves the equation but we do not know that $\varepsilon(\boldsymbol{u}) = \varepsilon^*(\overline{\mathbf{T}})$

Scheme

- First we show that
$$\mathbf{T}^n \to \mathbf{T} \qquad \text{a.e. in } \Omega,$$
where $\mathbf{T} \in L^1(\Omega)^{d\times d}_{sym}$ but we do not know that $\mathbf{T} = \overline{\mathbf{T}}$.
- Then due to the continuity of ε^* we have
$$\varepsilon(\boldsymbol{u}) = \varepsilon^*(\mathbf{T}) \text{ a.e. in } \Omega.$$
- The Fatou lemma and monotonicity justify the limit passage in
$$\int_\Omega \mathbf{T} \cdot \varepsilon(\boldsymbol{u} - \boldsymbol{w}) \le \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{u} - \boldsymbol{w}) + \int_{\Gamma_N} \boldsymbol{g} \cdot (\boldsymbol{u} - \boldsymbol{w}).$$
- The final step is to show that
$$\boxed{-\operatorname{div} \mathbf{T} = \boldsymbol{f}.}$$

Point-wise convergence of $\mathbf{T}^n$

- The final uniform bound

$$\boxed{\int_\Omega \frac{\tau^2|\nabla \mathbf{T}^n|^2}{(1+|\mathbf{T}^n|)^{a+1}} \le C\sum_k \int_\Omega (\tau\partial_k \mathbf{T}^n, \tau\partial_k \mathbf{T}^n)_{\mathbf{A}^n(\mathbf{T}^n)} \le C.}$$

- we are able to deduce that

$$\mathbf{T}^n \to \mathbf{T} \text{ a.e. } \text{ in } \Omega$$

- Renormalized solution - for any $g \in \mathcal{D}(\mathbb{R})$ and any $\boldsymbol{v} \in \mathcal{D}(\Omega)^d$

$$\int_\Omega \mathbf{T}\cdot(g(|\mathbf{T}|)\nabla \boldsymbol{v}) - \int_\Omega g(|\mathbf{T}|)\boldsymbol{f}\cdot\boldsymbol{v} = -\int_\Omega \mathbf{T}\cdot(\boldsymbol{v}\otimes\nabla g(|\mathbf{T}|))$$

- Our goal is to let $g \nearrow 1$. In the first two terms it is easy. The last term causes troubles.

Time dependent models - elasticity

We look for $(\boldsymbol{u}, \mathbf{T})$ fulfilling $(Q := (0,T) \times \Omega)$

$$
\begin{aligned}
\partial_{tt}^2 \boldsymbol{u} - \operatorname{div} \mathbf{T} &= \boldsymbol{f} && \text{in } Q,\\
\boldsymbol{\varepsilon}(\boldsymbol{u}) &= \varepsilon^*(\mathbf{T}) && \text{in } Q,\\
\boldsymbol{u} &= \boldsymbol{u}_0 && \text{on } \Gamma_D \subset (0,T) \times \partial\Omega \cap \{0\} \times \Omega,\\
\mathbf{T}\boldsymbol{n} &= \boldsymbol{g} && \text{on } \Gamma_N := (0,T) \times \partial\Omega \setminus \Gamma_D
\end{aligned}
$$

with

$$
\boxed{\varepsilon^*(\mathbf{T}) := \frac{\mathbf{T}}{(1 + |\mathbf{T}|^a)^{\frac{1}{a}}}}
$$

- nonlinear hyperbolic system of second order
- case is lost - except one dimensional setting

Time dependent models - viscoelasticity

We look for $(\boldsymbol{u}, \mathbf{T})$ fulfilling $(Q := (0,T)\times\Omega)$

$$\begin{aligned}
\partial^2_{tt}\boldsymbol{u} - \operatorname{div}\mathbf{T} &= \boldsymbol{f} && \text{in } Q,\\
\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\varepsilon}(\partial_t \boldsymbol{u}) &= \boldsymbol{\varepsilon}^*(\mathbf{T}) && \text{in } Q,\\
\boldsymbol{u} &= \boldsymbol{u}_0 && \text{on } \Gamma_D \subset (0,T)\times\partial\Omega \cap \{0\}\times\Omega,\\
\mathbf{T}\boldsymbol{n} &= \boldsymbol{g} && \text{on } \Gamma_N := (0,T)\times\partial\Omega \setminus \Gamma_D
\end{aligned}$$

with

$$\boxed{\varepsilon^*(\mathbf{T}) := \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}}}$$

- nonlinear parabolic - hyperbolic system of second order
- a hope for the existence of solution

Limiting strain - viscoelastic

- Consider

$$\boxed{\varepsilon^*(\mathbf{T}) \sim \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}}.}$$

- Assume that the existence of ψ^* fulfilling (it has **linear** growth)

$$\boxed{\frac{\partial \psi^*(\mathbf{T})}{\partial \mathbf{T}} = \varepsilon^*(\mathbf{T})}$$

- convex conjugate (corresponds to the Helmholtz free energy)

$$\psi(\varepsilon) := \sup_{\mathbf{W}}(\varepsilon \cdot \mathbf{W} - \psi^*(\mathbf{W}))$$

$\psi(\varepsilon) = \infty$ if $|\varepsilon| > 1$.

Theorem (Bulíček, Patel, Şengül, Süli (2020))

Let $\Gamma_N = \emptyset$. Then for any reasonable data there exists a weak solution.

A priori estimates

- Multiply

$$\partial_{tt}^2 \boldsymbol{u} - \operatorname{div} \mathbf{T} = \boldsymbol{f}$$

by $\partial_t \boldsymbol{u}$ and integrate over Ω (e.g. periodic data).

- Using that $\partial_{\mathbf{T}} \psi^* = (\partial_{\boldsymbol{\varepsilon}} \psi)^{-1}$

$$\begin{aligned}
\int_\Omega \boldsymbol{f} \cdot \partial_t \boldsymbol{u} &= \frac{d}{dt} \int_\Omega \frac{|\partial_t \boldsymbol{u}|^2}{2} + \int_\Omega \mathbf{T} \cdot \partial_t \boldsymbol{\varepsilon}(\boldsymbol{u}) \\
&= \frac{d}{dt} \left(\int_\Omega \frac{|\partial_t \boldsymbol{u}|^2}{2} + \psi(\boldsymbol{\varepsilon}(\boldsymbol{u})) \right) + \int_\Omega (\mathbf{T} - \partial_{\boldsymbol{\varepsilon}} \psi(\boldsymbol{\varepsilon}(\boldsymbol{u}))) \cdot \partial_t \boldsymbol{\varepsilon}(\boldsymbol{u}) \\
&= \frac{d}{dt} \left(\int_\Omega \frac{|\partial_t \boldsymbol{u}|^2}{2} + \psi(\boldsymbol{\varepsilon}(\boldsymbol{u})) \right) + \int_\Omega (\mathbf{T} - \partial_{\boldsymbol{\varepsilon}} \psi(\boldsymbol{\varepsilon}(\boldsymbol{u}))) \cdot (\partial_{\mathbf{T}} \psi^*(\mathbf{T}) - \boldsymbol{\varepsilon}(\boldsymbol{u})) \\
&= \frac{d}{dt} \underbrace{\left(\int_\Omega \frac{|\partial_t \boldsymbol{u}|^2}{2} + \psi(\boldsymbol{\varepsilon}(\boldsymbol{u})) \right)}_{\text{energy}} + \underbrace{\int_\Omega (\mathbf{T} - \partial_{\boldsymbol{\varepsilon}} \psi(\boldsymbol{\varepsilon}(\boldsymbol{u}))) \cdot (\partial_{\mathbf{T}} \psi^*(\mathbf{T}) - \partial_{\mathbf{T}} \psi^*(\partial_{\boldsymbol{\varepsilon}} \psi(\boldsymbol{\varepsilon}(\boldsymbol{u})))}_{\text{dissipation } \geq 0}
\end{aligned}$$

A priori estimates

- First a priori estimate
$$|\boldsymbol{\varepsilon}(\boldsymbol{u})| \le 1,\ \partial_t \boldsymbol{u} \in L^\infty(L^2),\ \mathbf{T} \in L^1(Q)$$
- Second a priori estimates - test by $\Delta \boldsymbol{u}$, $\Delta\partial_t \boldsymbol{u}$ and $\partial^2_{tt}\boldsymbol{u}$
$$\boldsymbol{\varepsilon}(\boldsymbol{u}) \in L^\infty(L^2),\ \mathbf{T} \in L^\infty(L^1),\ \partial^2_{tt}\boldsymbol{u} \in L^2(Q),\ \int_Q (\nabla \mathbf{T}, \nabla \mathbf{T})_{\mathbf{A}(\mathbf{T})} < \infty$$
- Starting point done - time for renormalization, etc...