Global higher integrability for minimisers of convex functionals with (p,q)-growth

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Functionals with (p,q)-growth

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$$\min_{u \in W_g^{1,p}(\Omega,\mathbb{R}^m)} \mathscr{F}(u) \qquad \text{where } \mathscr{F}(u) = \int_\Omega F(x, Du)\,\mathrm{d}x.$$

Here $F(x,z)$ is a convex functional with controlled (p,q)-growth in z satisfying a natural uniform α-Hölder condition in x.

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Here $F(x,z)$ is a convex functional with controlled (p,q)-growth in z satisfying a natural uniform α-Hölder condition in x.

Goal: Obtain conditions on the gap $q-p$ that guarantee $W^{1,q}(\Omega)$-regularity of minimisers of $\mathscr{F}(\cdot)$.

many results due to Acerbi, Baroni, Colombo, Diening, Fusco, Giaquinta, Harjulehto, Hästö, Marcellini, Mingione, Ruzicka, ...
In particular
Carozza, Kristensen, Passarelli di Napoli ('13): natural growth conditions,$q < \frac{np}{n-1}$

Schäffner ('19): controlled growth conditions, $q < p\left(1 + \frac{2}{n-1}\right)$

de Filippis, K., Kristensen ('20): controlled duality growth conditions, $q < \frac{np}{n-2}$

Growth conditions

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- natural growth: $|z|^p \lesssim F(x,z) \lesssim |z|^q$
- controlled growth: $|z|^{p-2}|\xi|^2 \lesssim F_{zz}(x,z)\xi\cdot\xi \lesssim |z|^{q-2}|\xi|^2$
- controlled duality growth:
  $|z|^{p-2}|\xi|^2 \lesssim F_{zz}(x,z)\xi\cdot\xi \lesssim |\partial_z F(x,z)|^{\frac{q-2}{q-1}}|\xi|^2$

Related results in the local non-autonomous case

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Esposito, Leonetti, Mingione ('04): $q \leq \frac{(n+\alpha)p}{n}$ is necessary condition, see also Fonseca, Malý, Mingione ('04), Balci, Diening, Surnachev ('20)

Esposito, Leonetti, Mingione ('04): controlled growth, $q < \frac{(n+\alpha)p}{n}$ for many examples including double-phase, $p(x)$

Esposito, Leonetti, Petricca ('19): extension to functionals satisfying additional assumption on the x-dependence

few global results both in the autonomous and non-autonomous case (Byun, Oh ('17), Bulíček, Maringová, Stroffolini, Verde ('18))

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Let $0 < \alpha \leq 1$. $F(x,z)$ is measurable in x, continuously differentiable in z and moreover satisfies

$$\nu(\mu^2 + |z|^2 + |w|^2)^{\frac{p-2}{2}} \leq \frac{F(x,z) - F(x,w) - \langle \partial_z F(x,w), z - w\rangle}{|z-w|^2} \tag{H1}$$

$$|F(x,z)| \lesssim (1+|z|^2)^{\frac{q}{2}} \tag{H2}$$

$$|F(x,z) - F(y,z)| \leq \Lambda |x-y|^\alpha \left(1 + |z|^2\right)^{\frac{q}{2}}. \tag{H3}$$

for some $\mu, \nu, \Lambda > 0$, all $z, w \in \mathbb{R}^{n\times m}$ and almost every $x, y \in \Omega$, where $1 \leq p \leq q$.

There is $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $x \in \Omega$ there is $\hat{y} \in \overline{B_\varepsilon(x) \cap \Omega}$ such that

$$F(\hat{y}, z) \leq F(y, z) \qquad \forall y \in \overline{B_\varepsilon(x) \cap \Omega}, \qquad z \in \mathbb{R}^{n\times m}. \tag{H4}$$

previously considered by Esposito, Leonetti, Petricca ('19) and similar to conditions of Zhikov ('95)

Examples

(I) $\mathscr{F}_3(u) = \int_\Omega |Du|^p + a(x)|Du|^q \,\mathrm{d}x$ where $0 \leq a(x) \in C^{0,\alpha}(\Omega)$,

(II) $\mathscr{F}_3(u) = \int_\Omega |Du|^{p(x)} \,\mathrm{d}x$ where $p \leq p(x) \leq q$,

(III) $\mathscr{F}_4(u) = \int_\Omega |Du|^p + |a^{i,j}_{\alpha,\beta}(x) D_i u^\alpha D_j u^\beta|^{\frac{q}{2}} \,\mathrm{d}x$ where $a^{i,j}_{\alpha,\beta}(\cdot) \in C^{0,\alpha}(\Omega)$ and for all $x \in \Omega$ and $\xi \in \mathbb{R}^{n\times m}$,

(IV) $\mathscr{F}_8(u) = \int_\Omega F(x, Du) \,\mathrm{d}x$ where $F(x,z) = h(a(x), z)$ where

- (i) $t \to h(t,z)$ is increasing
- (ii) $h(x,z)$ is convex in the second argument
- (iii) $a(x) \in C(\overline{\Omega})$
- (iv) $F(x,z)$ satisfies (H1)-(H3)

Pointwise minimisers

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$u \in W^{1,1}_g(\Omega)$ is a **(pointwise) minimiser** of $\mathscr{F}(\cdot)$ in the class $W^{1,p}_g(\Omega)$ if it holds that $F(x, Du) \in L^1(\Omega)$ and

$$\int_\Omega F(x, Du)\,\mathrm{d}x \le \int_\Omega F(x, Du + D\phi)\,\mathrm{d}x$$

for all $\phi \in W^{1,1}_0(\Omega)$.

$u \in W^{1,p}_g(\Omega)$ is a $W^{1,q}$-**relaxed minimiser** of $\mathscr{F}(\cdot)$ in the class $W^{1,p}_g(\Omega)$ if u minimises the **relaxed functional**

$$\overline{\mathscr{F}}(v) = \inf_{(v_j)\subset Y} \left\{ \liminf_{j\to\infty} \int_\Omega F(Dv_j) : v_j \rightharpoonup v \text{ weakly in } X \right\}$$

amongst all $v \in X = W^{1,p}_g(\Omega)$ where $Y = W^{1,q}_g(\Omega)$.

Serrin ('61),Marcellini ('86), Zhikov ('95), Buttazzo, Belloni ('95), Fonseca, Malý ('97), Foss ('01), Acerbi, Bouchitté, Fonseca ('03), Schmidt ('08), . . .

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Consider a topological space X of weakly differentiable functions and a dense subspace $Y \subset X$.

$$\overline{\mathscr{F}}_X = \sup\{\mathscr{G} : X \to [0,\infty] : \mathscr{G} \text{ slsc }, \mathscr{G} \le \mathscr{F} \text{ on } X\}$$
$$\overline{\mathscr{F}}_Y = \sup\{\mathscr{G} : X \to [0,\infty] : \mathscr{G} \text{ slsc }, \mathscr{G} \le \mathscr{F} \text{ on } Y\}$$

Define the Lavrentiev gap functional for $u \in X$ as

$$\mathscr{L}(u, X, Y) = \begin{cases} \overline{\mathscr{F}}_Y(u) - \overline{\mathscr{F}}_X(u) & \text{if } \overline{\mathscr{F}}_X(u) < \infty \\ 0 & \text{else.} \end{cases}$$

In this talk: $X = W^{1,p}_g(\Omega)$, $Y = W^{1,q}_g(\Omega)$.

A regularity result

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Theorem (L.K. '20)

Suppose Ω is a Lipschitz domain. Let $g \in W^{1+\alpha,q}(\Omega)$. Suppose $F(x,\cdot)$ satisfies (H1)-(H3) with $1 < p \leq q < \frac{(n+\alpha)p}{n}$. If u is a relaxed minimiser of $\mathscr{F}(\cdot)$ in the class $W^{1,p}_g(\Omega)$, then $u \in W^{1,q}(\Omega)$. Moreover for any $0 \leq \beta < \alpha$ there is $\gamma > 0$ such that

$$\|u\|_{W^{1,\frac{np}{n-\beta}}(\Omega)} \lesssim \left(1 + \overline{\mathscr{F}}(u) + \|g\|_{W^{1+\alpha,q}(\Omega)}\right)^{\gamma}. \tag{1}$$

Suppose that in fact Ω is a $C^{1,\alpha}$-domain and $g \in W^{1+\max\left(\alpha,\frac{1}{q}\right),q}(\Omega)$. Suppose $F(x,\cdot)$ satisfies in addition (H4) and let u be a pointwise minimiser of $\mathscr{F}(\cdot)$ in the class $W^{1,p}_g(\Omega)$. Then $u \in W^{1,q}(\Omega)$ and the estimate (1) holds.

Outline of the proof: an a-priori estimate

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$$\mathscr{F}_\varepsilon(u) = \int_\Omega F(x, Du) + \varepsilon |Du|^q \,\mathrm{d}x$$

Lemma (L.K. '20)

Suppose Ω is a Lipschitz domain. Let $1 \leq p \leq q < \frac{(n+\alpha)p}{n}$, $g \in W^{1+\alpha,q}(\Omega)$. Let v_ε be the minimiser of $\mathscr{F}_\varepsilon(\cdot)$ in the class $W^{1,q}_g(\Omega)$. Then for any $0 \leq \beta < \alpha$ there is $\gamma > 0$ such that the estimate

$$\|v_\varepsilon\|_{W^{1,\frac{np}{n-\beta}}(\Omega)} \lesssim \left(1 + \mathscr{F}_\varepsilon(v_\varepsilon) + \|g\|_{W^{1+\alpha,q}(\Omega)}\right)^\gamma$$

holds, with the implicit constant independent of ε and γ.

Lemma (Marcellini ('89))

Let $g \in W^{1,q}(\Omega)$. Suppose $F(x, z)$ satisfies (H1) *and* (H2)*. Suppose u is a relaxed minimiser of $\mathscr{F}(\cdot)$ in the class $W^{1,p}_g(\Omega)$ and u_ε is the pointwise minimiser of $\mathscr{F}_\varepsilon(\cdot)$ in the class $W^{1,q}_g(\Omega)$. Then $\mathscr{F}_\varepsilon(u_\varepsilon) \to \overline{\mathscr{F}}(u)$ as $\varepsilon \to 0$. Moreover up to passing to a subsequence $u_\varepsilon \to u$ in $W^{1,p}(\Omega)$.*

allows to pass to the limit in the a-priori estimate for relaxed minimisers.

A remark on Besov-spaces

$$B_q^{s,p}(\Omega) = B_q^{s,p}(\Omega, \mathbb{R}^m) = [W^{1,p}(\Omega, \mathbb{R}^m), L^p(\Omega, \mathbb{R}^m)]_{s,q}$$

Let D be a set generating $\mathbb{R}^n$, star-shaped with respect to 0. For $s \in (0,1)$, $p \in [1,\infty]$, consider

$$[v]_{s,p,\Omega}^p = \sup_{h \in D \setminus \{0\}} \int_{\Omega_h} \left| \frac{v_h(x) - v(x)}{|h|^s} \right|^p \mathrm{d}x.$$

$$C_1 \|v\|_{B_\infty^{s,p}(\Omega)} \leq \|v\|_{L^p(\Omega)} + [v]_{s,p,\Omega} \leq C_2 \|v\|_{B_\infty^{s,p}(\Omega)}.$$

When $D = C_\rho(\theta, \boldsymbol{n})$ is a cone, C_1, C_2 are independent of the choice of $\boldsymbol{n}$.

Outline of the proof: a difference quotient argument I

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As Ω is Lipschitz, it satisfies the **uniform cone property**: there are $\rho_0, \theta_0 > 0$ and a map $\boldsymbol{n}\colon \mathbb{R}^n \to S^{n-1}$ such that for every $x \in \mathbb{R}^n$

$$C_{\rho_0}(\theta_0, \boldsymbol{n}(x)) \subset \{\, h \in \mathbb{R}^n : |h| \leq \rho_0, (B_{3\rho_0}(x) \setminus \Omega) + h \subset \mathbb{R}^n \setminus \Omega \,\}.$$

Based on an argument by Savaré introduce

$$T_h v = \phi v_h + (1 - \phi) v.$$

Outline of the proof: a difference quotient argument II

Supppose $u_\varepsilon = v_\varepsilon + g$ is a minimiser of $\mathscr{F}_\varepsilon(\cdot)$. Consider

$$\sup_{h \in C_{\rho_0}(\theta_0, \boldsymbol{n}(x_0))} \frac{\mathscr{F}_\varepsilon(T_h \tilde{v}_\varepsilon + g) - \mathscr{F}_\varepsilon(v_\varepsilon + g)}{|h|^\alpha}.$$

Outline of the proof: a difference quotient argument II

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Supppose $u_\varepsilon = v_\varepsilon + g$ is a minimiser of $\mathscr{F}_\varepsilon(\cdot)$. Consider

$$\sup_{h \in C_{\rho_0}(\theta_0, \boldsymbol{n}(x_0))} \frac{\mathscr{F}_\varepsilon(T_h \tilde{v}_\varepsilon + g) - \mathscr{F}_\varepsilon(v_\varepsilon + g)}{|h|^\alpha}.$$

lower bound: use the Euler-Lagrange equation for v_ε

The upper bound in the difference quotient argument I

Assume $g = 0$. Set $\Delta_h v = v - v_h$.

$$
\begin{aligned}
&\mathscr{F}_\varepsilon(T_h \tilde{v}_\varepsilon) - F_\varepsilon(v_\varepsilon) \\
&= \int_\Omega F_\varepsilon(x, T_h D\tilde{v} + D\phi(\tilde{v}_h - v)) - F_\varepsilon(x, T_h D\tilde{v})\,\mathrm{d}x \\
&\qquad + \int_\Omega F_\varepsilon(x, T_h D\tilde{v}) - F_\varepsilon(x, D\tilde{v})\,\mathrm{d}x = A_1 + A_2.
\end{aligned}
$$

$$
\begin{aligned}
|A_1| \leq& (\Lambda + \varepsilon) \int_\Omega |D\phi \Delta_h \tilde{v}| \left(1 + |T_h D\tilde{v}|^2 + |D\phi \Delta_h \tilde{v}|^2\right)^{\frac{q-1}{2}}\,\mathrm{d}x \\
\lesssim& |h| \left(1 + \|Dv\|_{L^q(\Omega)}^q + \|g\|_{W^{1+\alpha,q}(\Omega)}^q\right).
\end{aligned}
$$

The upper bound in the difference quotient argument II

$$
\begin{aligned}
|A_2| &\leq \int_{B_{2\rho_0}(x_0)} \phi(F_\varepsilon(x, D\tilde{v}_h) - F_\varepsilon(x, D\tilde{v})\,\mathrm{d}x \\
&= \int_{B_{3\rho_0}(x_0)} \phi(x-h)F_\varepsilon(x-h, D\tilde{v}) - \phi(x)F_\varepsilon(x-h, D\tilde{v})\,\mathrm{d}x \\
&\qquad\qquad + \int_{B_{2\rho_0}(x_0)} \phi(x)\big(F_\varepsilon(x-h, D\tilde{v}) - F_\varepsilon(x, D\tilde{v})\big)\,\mathrm{d}x \\
&\lesssim |h|^\alpha \left(1 + \|Dv\|_{L^q(\Omega)}^q + \|g\|_{W^{1+\alpha,q}(\Omega)}^q\right).
\end{aligned}
$$

A family of cubes $\{ K_i \}_{i\in I}$ is called a Whitney-Besicovitch-covering (WB-covering) of Ω if there is a triple (δ, M, ε) of positive numbers such that

$$\bigcup_{i\in I} \frac{1}{1+\delta} K_i = \bigcup_{i\in I} K_i = \Omega$$

$$\sum_{i\in I} \chi_{K_i} \leq M$$

$$K_i \cap K_j \neq \emptyset \Rightarrow |K_i \cap K_j| \geq \varepsilon \max(|K_i|, |K_j|).$$

Kislyakov, Kruglyak ('05): if Ω has non-empty complement a WB-covering exists. Moreover there is a partition of unity $\{\psi_i\}$ adapted to this cover.

Outline of the proof: constructing approximating sequences

Lemma (L.K. '20)

Let Ω be a domain. Suppose $1 < p \leq q < \frac{(n+\alpha)p}{n}$. Given $u \in W^{1,p}(\Omega)$, $\varepsilon \in (0, 1/(1+1/6)^n)$ and $m \geq 1$ write

$$u_\varepsilon = \sum_{i \in I} u \star \phi_{\varepsilon\delta_i}\psi_i \qquad \text{with } \delta_i = |K_i|^{\frac{m}{n}}.$$

Assume that $F(x,\cdot)$ satisfies (H1)-(H3) and (H4). Then if m is sufficiently large, up to passing to a subsequence if necessary, $u_\varepsilon \in W^{1,p}_u(\Omega) \cap W^{1,q}_{\mathrm{loc}}(\Omega)$, $u_\varepsilon \to u$ in $W^{1,p}(\Omega)$ and

$$\int_\Omega F(x, Du_\varepsilon)\,\mathrm{d}x \to \int_\Omega F(x, Du)\,\mathrm{d}x \qquad \text{as } \varepsilon \searrow 0.$$

Some details of the proof of Lemma 4 I

$$Du_\varepsilon = \sum_{i \in I} Du \star \phi_{\varepsilon\delta_i} \psi_i + \sum_{i \in I} u \star \phi_{\varepsilon\delta_i} \otimes D\psi_i = A_1 + A_2.$$

Some details of the proof of Lemma 4 I

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$$Du_\varepsilon = \sum_{i\in I} Du \star \phi_{\varepsilon\delta_i}\psi_i + \sum_{i\in I} u \star \phi_{\varepsilon\delta_i} \otimes D\psi_i = A_1 + A_2.$$

$$\begin{aligned}
&\int_\Omega F(x, Du_\varepsilon)\,\mathrm{d}x \\
&\lesssim \int_\Omega F(x, A_1)\,\mathrm{d}x + \int_\Omega |A_2|(1 + |A_1|^{q-1} + |A_2|^{q-1}) \\
&\lesssim \sum_{i\in I}\int_\Omega F(x, Du \star \phi_{\varepsilon\delta_i})\psi_i\,\mathrm{d}x + \int_\Omega |A_2|(1 + |A_1|^{q-1} + |A_2|^{q-1})
\end{aligned}$$

Set $G_\varepsilon(x,z) = \min_{y\in\overline{B_\varepsilon(x)\cap\Omega}} F(y,z)$. Let $|z| \le C\varepsilon^{-\frac{n}{p}}$ and $\varepsilon \in (0,\varepsilon_0)$.

$$
\begin{aligned}
G_\varepsilon(x,z) \geq &\delta F(x,z) - \delta\Lambda(1+|z|^2)^{\frac{q}{2}} + (1-\delta)|z|^p\\
\gtrsim &\delta F(x,z) - \delta\Lambda C^{q-p}\varepsilon^{\alpha+\frac{n(p-q)}{p}}|z|^p + (1-\delta)|z|^p - \delta(\Lambda\varepsilon^\alpha+1)\\
\geq &\delta F(x,z) + \left(1-\delta-\Lambda C^{q-p}\delta\varepsilon_0^{\alpha+\frac{n(p-q)}{p}}\right)|z|^p - \Lambda(\varepsilon_0^\alpha+1).
\end{aligned}
$$

$$
\begin{aligned}
G_\varepsilon(x, Du\star\phi_\varepsilon) = &F(\hat{y}, Du\star\phi_\varepsilon) \leq \int_{B_1} F(\hat{y}, Du(y))\phi_\varepsilon(x-y)\,\mathrm{d}y\\
\leq &(F(\cdot, Du(\cdot))\star\phi_\varepsilon)(x)
\end{aligned}
$$

Some details of the proof of Lemma 4 III

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Set $\Theta = 1 + n\left(\frac{1}{q} - \frac{1}{p}\right)$ if $p < n$, $\Theta = \frac{n}{q}$ if $p \geq n$.

$$\int_\Omega |A_1|^{q-1}|A_2|\,\mathrm{d}x$$

$$\lesssim C^q \sum_{j\in I}(\varepsilon \|D\psi_j\|_{L^q(\Omega_j)}^{-m})^{-\frac{n}{p}\left(1-\frac{p}{q}\right)} \sum_{I_j} \varepsilon^{\Theta}\|D\psi_i\|_{L^\infty(\Omega_i)}^{-m\Theta+1}\|1_{\psi_i}\|_{L^q(\Omega)}$$

$$\lesssim C^q \varepsilon^{\tau} \sum_{i\in I} \|D\psi_i\|_{L^\infty(\Omega_i)}^{-m\tau+1}\|1_{\psi_i}\|_{L^q(\Omega)}$$

with $\tau = \Theta - \frac{n(q-1)}{p}\left(1 - \frac{p}{q}\right) > 0$

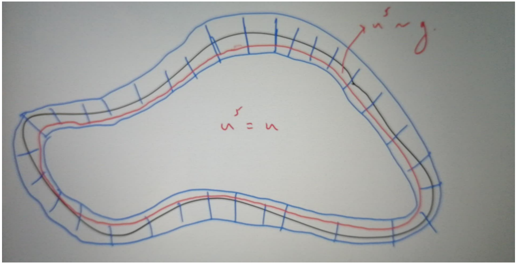
Let $\frac{1}{2} < s < 1$ and assume $F(x,z) = F(z)$.

$$u^s(x) = \begin{cases} su(x/s) & \text{in } sB_1 \\ |x|g(x/|x|) & \text{in } B_1 \setminus sB_1. \end{cases}$$

Then $u^s \to u$ in $W^{1,p}(B_1)$, $u^s \in W^{1,q}$ near ∂B_1, $u^s = g$ on ∂B_1 and

$$\mathscr{F}(u^s) \to \mathscr{F}(u) \qquad \text{as } s \to 1.$$

Outline of the proof: Regularising near the boundary



$$F^s(x,z) = F\left(\Psi_s^{-1}(x), zD\Psi_s(\Psi_s^{-1}(x))\right)$$

$$\mathscr{F}^s(u) = \int_\Omega F^s(x, Du)\,\mathrm{d}x$$

$$u^s(x) = u(\Psi_s^{-1}(x)).$$

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Proposition (L.K. '20)

Suppose Ω is a Lipschitz domain and $g \in W^{1+\frac{1}{q},q}(\Omega)$. Let $X = W^{1,p}_g(\Omega)$ endowed with the weak topology. Suppose that $1 < p \leq q < \frac{(n+\alpha)p}{n}$ and that $F(x,\cdot)$ satisfies (H1)-(H3) and (H4). Then with the choice $Y = W^{1,q}_{\mathrm{loc}}(\Omega)$,

$$\mathscr{L}(\cdot, X, Y) = 0 \text{ on } X.$$

If in fact $1 < p \leq q < min\left(p+1, \left(1+\frac{\alpha}{(\alpha+1)n}\right)p\right)$, then $\mathscr{L}(\cdot, X, Y) = 0$ on X with the choice $Y = W^{1,q}_g(\Omega)$.

Idea: Decompose Ω into star-shaped Lipschitz domains and apply the regularisation from the case of the ball.

$$\int_\Omega F(x, Du_\varepsilon^s)\,\mathrm{d}x$$
$$= \int_\Omega F(x/s, Du_\varepsilon^s) + \int_\Omega F(x, Du_\varepsilon^s) - F(x/s, Du_\varepsilon^s)\,\mathrm{d}x.$$

Now change coordinates in the first term and estimate the second term similar to before.

Thanks for your attention!