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Elliptic Equations with Degenerate Weights

Lars Diening

 

A. Balci   A. Passarelli di Napoli   R. Giova

Main Focus of this Talk

Poisson equation: Find u with

$$-\Delta u = -\operatorname{div}(\nabla u) = -\operatorname{div} F.$$

The mapping $F \mapsto \nabla u$ is a Calderón-Zygmund operator, i.e.

Regularity of F transfers to ∇u

$$\|\nabla u\|_{L^q} \leq C\|F\|_{L^q} \quad 1 < q < \infty.$$

We want to extend this to weighted equations:

$$-\operatorname{div}\left(\mathbb{A}(x)\nabla u\right) = -\operatorname{div} F,$$

$$-\operatorname{div}\left(|\mathbb{M}(x)\nabla u|^{p-2}\mathbb{M}^2(x)\nabla u\right) = -\operatorname{div} F \qquad \text{for } 1 < p < \infty.$$

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$$\begin{aligned} -\operatorname{div}\big(\mathbb{A}(x)\nabla u\big) &= -\operatorname{div} F, \\ -\operatorname{div}\big(|\mathbb{M}(x)\nabla u|^{p-2}\mathbb{M}^2(x)\nabla u\big) &= -\operatorname{div} F \qquad \text{for } 1 < p < \infty. \end{aligned}$$

Our focus is on degenerate weights.

Alternative Formulation

Weighted Laplacian: $-\operatorname{div}(\mathbb{A}(x)\nabla u) = -\operatorname{div} F.$

We want to show results like $\|\nabla u\|_{\mathbb{X}} \lesssim \|F\|_{\mathbb{X}}$ for some space $\mathbb{X}$.

It is better to replace F by $\mathbb{A}(x)G$ and show $\|\nabla u\|_{\mathbb{X}} \lesssim \|G\|_{\mathbb{X}}$.

Weighted Laplacian: $-\operatorname{div}(\mathbb{A}(x)\nabla u) = -\operatorname{div}(\mathbb{A}(x)G).$

Weighted p-Laplacian:

$$-\operatorname{div}\left(|\mathbb{M}(x)\nabla u|^{p-2}\mathbb{M}^2(x)\nabla u\right) = -\operatorname{div}\left(|\mathbb{M}(x)G|^{p-2}\mathbb{M}^2(x)G\right) \qquad \text{for } 1 < p < \infty.$$

Case $p = 2$ and $\mathbb{M}(x) = \mathbb{A}^{\frac{1}{2}}(x)$ recovers the weighted Laplacian.

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Meyers Limiting Example (1963)

Meyers Example

Let $0 < \varepsilon < \frac{1}{2}$. Then $u : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbb{A} : \mathbb{R} \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ with

$$u(x) := |x|^{1-\varepsilon} \frac{x_1}{|x|} \quad \text{and} \quad \mathbb{A}(x) := (1-\varepsilon)^2 \mathrm{Id} + \varepsilon^2 \frac{x}{|x|} \otimes \frac{x}{|x|}$$

satisfies $\quad -\operatorname{div}(\mathbb{A}(x)\nabla u) = 0.$

Uniform ellipticity: $(1-\varepsilon)^2|\xi|^2 \leq \langle \mathbb{A}(x)\xi, \xi\rangle \leq |\xi|^2.$

Weak solution: $u \in W^{1,2}.$

Hölder continuity: $u \in C^{0,1-\varepsilon}$ (Moser-Nash-De Giorgi)

Limited regularity: $\nabla u \in L^{\frac{2}{\varepsilon},\infty}_{\mathrm{loc}}$ (Marcinkiewicz space), $\nabla u \in L^q_{\mathrm{loc}}$ iff $q < \frac{2}{\varepsilon}$.

History: Weighted Laplacian

Weighted Laplacian

$$-\operatorname{div}(\mathbb{A}(x)\nabla u) = -\operatorname{div}(\mathbb{A}(x)G) = -\operatorname{div} F$$

Uniform ellipticity: $(1-\varepsilon)^2|\xi|^2 \leq \langle \mathbb{A}(x)\xi, \xi\rangle \leq |\xi|^2$.

Conditions on $\mathbb{A}$ that allow transfer of regularity $\|\nabla u\|_{\mathbb{X}} \lesssim \|G\|_{\mathbb{X}}$:

$\nabla u \in L^{2+\delta}$	for $\mathbb{A}$ uniformly elliptic	Meyers 1963
$\nabla u \in L^q$	for $\mathbb{A}$ uniformly elliptic + continuous	Classical: Gilbarg&Trudinger
$\nabla u \in L^q$	for $\mathbb{A}$ uniformly elliptic + VMO	Di Fazio 1996

We want to include degenerate weights like $\mathbb{A}(x) = |x|^{\pm\delta}\mathrm{Id}$ for $\delta > 0$.

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Degenerate Elliptic Weights

Weighted Laplacian: $-\operatorname{div}(\mathbb{A}(x)\nabla u) = -\operatorname{div}(\mathbb{A}(x)G) = -\operatorname{div} F$.

We replace uniform ellipticity $\lambda_1|\xi|^2 \leq \langle \mathbb{A}(x)\xi, \xi\rangle \leq \Lambda_1|\xi|^2$ by ...

Degenerate Ellipticity

$\Lambda^{-2}\mu(x)|\xi|^2 \leq \langle \mathbb{A}(x)\xi, \xi\rangle \leq \mu(x)|\xi|^2$ for some scalar weight μ.

Best possible upper bound for $\mu(x) := |\mathbb{A}(x)|$.

This is equivalent to: $|\mathbb{A}(x)|\,|\mathbb{A}^{-1}(x)| \leq \Lambda^2$ (uniformly bounded condition number).

Weighted Sobolev Spaces: $\nabla u \in L^2(\mu\,dx)$.

Fabes, Kenig, Serapioni (1982)

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If $\mu \in A_2$, and F is nice enough, then $u \in C^{0,\alpha}$ for some $\alpha > 0$.

Muckenhoupt: $\mu \in A_2$ if $[\mu]_{A_2} := \sup_B ⨍_B \mu\, dx ⨍_B \mu^{-1}\, dx < \infty.$

Includes $\mathbb{A}(x) = |x|^{\pm\varepsilon}\mathrm{Id}$ for small $\varepsilon > 0$.

We aim for gradient regularity $\|\nabla u\|_{\mathbb{X}} \lesssim \|G\|_{\mathbb{X}}$.

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Assume that $\mu \in A_2$ (Muckenhoupt class). If $\|\mathbb{A}\|_{\mathrm{BMO}^2_\mu} < \delta(q)$, then

$$\|\nabla u\|_{L^q(\mu\,dx)} \lesssim \|G\|_{L^q(\mu\,dx)}$$

Weighted BMO^1_μ: $\|\mathbb{A}\|_{\mathrm{BMO}^1_\mu} = \sup_B \frac{1}{\mu(B)} \int_B |\mathbb{A}(y) - \langle \mathbb{A}\rangle_B|\,dy.$

Includes $\mathbb{A}(x) = |x|^{\pm\varepsilon}\mathrm{Id}$ for small $\varepsilon > 0$.

No explicit dependence of δ on q.

Main Result – Linear Case

Let u solve $\quad -\operatorname{div}\big(\mathbb{A}(x)\nabla u\big) = -\operatorname{div}\big(\mathbb{A}(x)G\big)$.

Degenerate elliptic weight, i.e. $|\mathbb{A}(x)|\,|\mathbb{A}^{-1}(x)| \leq \Lambda^2$

Let $\omega(x) := |\mathbb{M}(x)| = |\mathbb{A}^{\frac{1}{2}}(x)|$.

Theorem [Balci, Diening, Giova, Passarelli di Napoli '20]

If $1 < q < \infty$ and $|\log \mathbb{A}|_{\mathrm{BMO}} \leq \kappa_\Lambda \min\big\{\frac{1}{q}, 1-\frac{1}{q}\big\}$, then $\|\omega\,\nabla u\|_q \lesssim \|\omega\, G\|_q$.

Standard BMO: $\quad |f|_{\mathrm{BMO}} = \sup_B ⨍_B |f - \langle f\rangle_B|\,dx$.

Comparison:

Cao, Mengesha, Pan (2018): $\quad \|\nabla u\|_{L^q(\omega^2\,dx)} \lesssim \|G\|_{L^q(\omega^2\,dx)}$

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Reformulation

Let $\mathbb{M}(x) := \mathbb{A}^{\frac{1}{2}}(x)$, then $\langle \mathbb{A}(x)\xi, \xi\rangle = |\mathbb{M}(x)\xi|^2$.

Weighted Laplacian: $-\operatorname{div}(\mathbb{M}^2(x)\nabla u) = -\operatorname{div}(\mathbb{M}^2(x)G)$.

Degenerate ellipticity: $\Lambda^{-1}\omega(x)|\xi| \leq |\mathbb{M}(x)\xi| \leq \omega(x)|\xi|$ for $\omega(x) := |\mathbb{M}(x)|$.

equivalent to: $|\mathbb{M}(x)|\,|\mathbb{M}^{-1}(x)| \leq \Lambda$ (uniformly bounded condition number).

Multiplicative weight: $\nabla u \in L^2_\omega$ with norm $\|f\omega\|_2$. Then $L^2_\omega = L^2(\omega^2\,dx)$.

Simple dual space: $(L^p_\omega)^* = L^{p'}_{1/\omega}$.

Multiplicative form for Muckenhoupt weights $\omega^p \in A_p$

$$[\omega^p]^{\frac{1}{p}}_{A_p} = \sup_B \left(\fint_B \omega^p\,dx\right)^{\frac{1}{p}} \left(\fint_B \omega^{-p'}\,dx\right)^{\frac{1}{p'}} < \infty.$$

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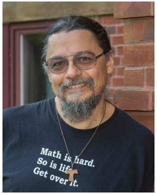

Thank you David

Logarithms of Weights

Our condition on $\mathbb{A}$ or $\mathbb{M} = \mathbb{A}^{\frac{1}{2}}$ is expressed by its logarithm $\log \mathbb{M} = \frac{1}{2} \log \mathbb{A}$.

Matrix functions: $\exp \mathbb{M} : \mathbb{R}^{n\times n}_{\text{sym}} \to \mathbb{R}^{n\times n}_{>0}$ and $\log \mathbb{M} : \mathbb{R}^{n\times n}_{>0} \to \mathbb{R}^{n\times n}_{\text{sym}}$.

Question: Why logarithm?

Define logarithmic mean $\langle \omega \rangle_B^{\log} := \exp(\langle \log \omega \rangle_B)$ and $\langle \mathbb{M} \rangle_B^{\log} := \exp(\langle \log \mathbb{M} \rangle_B)$

- Compatible with duality $(L^p_\omega)^* = L^{p'}_{1/\omega}$, since $\langle \omega^{-1} \rangle_B^{\log} = \dfrac{1}{\langle \omega \rangle_B^{\log}}$.
- ω^p is A_p-Muckenhoupt weight iff

$$\left(\fint_B \omega^p \, dx \right)^{\frac{1}{p}} \leq c_1 \langle \omega \rangle_B^{\log}, \qquad \left(\fint_B \omega^{-p'} \, dx \right)^{\frac{1}{p'}} \leq c_2 \frac{1}{\langle \omega \rangle_B^{\log}}.$$

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Sharpness of our Condition

Meyers Example

Let $0 < \varepsilon < \frac{1}{2}$. Then $u : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbb{A} : \mathbb{R} \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ with

$$u(x) := |x|^{1-\varepsilon} \frac{x_1}{|x|} \quad \text{and} \quad \mathbb{M}(x) := \varepsilon \mathrm{Id} + (1-\varepsilon)\frac{x}{|x|} \otimes \frac{x}{|x|}$$

satisfies $-\operatorname{div}(\mathbb{A}(x)\nabla u) = -\operatorname{div}(\mathbb{M}^2(x)\nabla u) = 0$.

Degenerate elliptic: $|\mathbb{M}(x)|\,|\mathbb{M}^{-1}(x)| \leq 2$.

Moreover, $|\log \mathbb{M}|_{\mathrm{BMO}} = \left|\log(1-\varepsilon)\left(\mathrm{Id} - \frac{x}{|x|} \otimes \frac{x}{|x|}\right)\right|_{\mathrm{BMO}} \leq \varepsilon.$

Now, $\nabla u \in L^q$ for $q > 2$ iff $|\log \mathbb{M}|_{\mathrm{BMO}} \leq \frac{2}{q}$.

Hence, our condition $|\log \mathbb{M}|_{\mathrm{BMO}} \leq \frac{\kappa_\Lambda}{q}$ is sharp (even for non-degenerate weights).

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Invariance Under the Scaling

Weighted Laplacian: $-\operatorname{div}(\mathbb{A}(x)\nabla u) = -\operatorname{div}(\mathbb{A}(x)G)$.

Degenerate ellipticity: $|\mathbb{A}(x)||\mathbb{A}^{-1}(x)| \leq \Lambda^2$.

Both are scaling invariant to $\mathbb{A} \to t\mathbb{A}$, $\nabla u \to \frac{1}{t}\nabla u$, $G \to \frac{1}{t}G$.

- Muckenhoupt condition of Fabes-Kenig-Serapioni is scaling invariant.
- Small BMO condition on $\mathbb{A}$ is not scaling invariant (Di Fazio).

$$\|\mathbb{A}\|_{\mathrm{BMO}} = \sup_B \fint_B |\mathbb{A}(y) - \langle\mathbb{A}\rangle_B|\, dy.$$

- Cao, Mengesha, Phan condition is scaling invariant (recall $\mu(x) = |\mathbb{A}(x)|$)

$$\|\mathbb{A}\|_{\mathrm{BMO}^1_\mu} = \sup_B \frac{1}{\mu(B)} \int_B |\mathbb{A}(y) - \langle\mathbb{A}\rangle_B|\, dy.$$

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Degenerate ellipticity: $\Lambda^{-1}\omega(x)|\xi| \leq |\mathbb{M}(x)\xi| \leq \omega(x)|\xi|$ or $|\mathbb{M}(x)|\,|\mathbb{M}^{-1}(x)| \leq \Lambda.$

Minimizer of $\mathcal{J}(v) = \int_\Omega \frac{1}{p}|\mathbb{M}\nabla v|^p\,dx - \int_\Omega |\mathbb{M}G|^{p-2}\mathbb{M}G \cdot (\mathbb{M}\nabla v)\,dx.$

Natural energy norm $\|\omega\nabla v\|_p$ with $\omega(x) := |\mathbb{M}(x)|.$

History: Weighted p-Laplacian

Weighted p-Laplacian: $-\operatorname{div}\left(|\mathbb{M}(x)\nabla u|^{p-2}\mathbb{M}^2(x)\nabla u\right) = -\operatorname{div}\left(|\mathbb{M}(x)G|^{p-2}\mathbb{M}^2(x)G\right)$.

Goal: Regularity transfer $\|\nabla u\|_{\mathbb{X}} \lesssim \|G\|_{\mathbb{X}}$.

$\mathbb{X} = L^q,\ q \in [p,\infty)$	$\mathbb{M} = \mathrm{Id}$	Iwaniec; Caffarelli, Peral
$\mathbb{X} = L^q,\ q \in [p-\delta,\infty)$	$\mathbb{M} = \mathrm{Id}$	Iwaniec, Lewis; Kinnunen, Zhou
$\mathbb{X} = L^q,\ q \in [p,\infty)$	$\mathbb{M} \in \mathrm{VMO}$ + unif. elliptic	Kinnunen, Zhou 1999
$\mathbb{X} = \mathrm{BMO}$	$\mathbb{M} = \mathrm{Id}$	Di Benedetto, Manfredi 1993 Diening, Kaplický, Schwarzacher 2012

Riesz potential estimates $\mathbb{M} = \mathrm{Id}$:	Kuusi, Mingione; Duzaar, Malý, …
Higher regularity $\mathbb{M} = \mathrm{Id}$, $\mathbb{X} = F^s_{p,\rho}$:	Balci, Diening, Weimar, …
Weights in Sobolev spaces:	Baisó, Clop, Giova, Orobitg, Passarelli di Napoli, …
Muckenhoupt weights, $u \in C^{0,\alpha}$:	Modica; Cruz-Uribe, Moen, Naibo, Rodney, …

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Main Result – Non-Linear Case

Let u solve $\quad -\operatorname{div}\big(|\mathbb{M}(x)\nabla u|^{p-2}\mathbb{M}^2(x)\nabla u\big) = -\operatorname{div}\big(|\mathbb{M}(x)G|^{p-2}\mathbb{M}^2(x)G\big).$

Degenerate elliptic, i.e. $|\mathbb{M}(x)|\,|\mathbb{M}^{-1}(x)| \leq \Lambda,\quad$ and $\omega(x) := |\mathbb{M}(x)|$.

Theorem [Balci, Diening, Giova, Passarelli di Napoli '20]

If $p \leq q < \infty$ and $|\log \mathbb{M}|_{\mathrm{BMO}} \leq \frac{\kappa_\Lambda}{q}$, then $\|\omega\,\nabla u\|_q \lesssim \|\omega\, G\|_q$.

Condition scales well and is sharp.

Linear Case: all $1 < q < \infty$.

Non-Linear Case: only $q \geq p$.

Reason: Duality for linear equation.

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Properties of Logarithmic Weights

John-Nirenberg Type Estimates: (recall $\langle \mathbb{M}\rangle_B^{\log} := \exp(\langle \log \mathbb{M}\rangle_B)$)

If $q\,|\log \mathbb{M}|_{\mathrm{BMO}} \leq \varepsilon_0$, then

$$\left(\fint_B \left(\frac{|\mathbb{M} - \langle \mathbb{M}\rangle_B^{\log}|}{|\langle \mathbb{M}\rangle_B^{\log}|} \right)^q dx \right)^{\frac{1}{q}} \leq c_3\, q\, |\log \mathbb{M}|_{\mathrm{BMO}}.$$

Muckenhoupt Weights:

If $|\log \omega|_{\mathrm{BMO}} \leq \gamma \min\{\frac{1}{p}, \frac{1}{p'}\}$, then $\omega^p \in A_p$ and

$$[\omega^p]_{A_p}^{\frac{1}{p}} = \sup_B \left(\fint_B \omega^p\, dx \right)^{\frac{1}{p}} \left(\fint_B \omega^{-p'}\, dx \right)^{\frac{1}{p'}} \leq 4.$$

PDE Technique in the Proof

- Small $|\log \mathbb{M}|_{\mathrm{BMO}}$ implies Muckenhoupt.
- Muckenhoupt weighted Poincaré inequalities.
- Cacciopoli estimates.
- Reverse Hölder estimates.
- Improved integrability $\nabla u \in L^{p+\delta}_\omega$.
- Compare solution locally with weak solution (recall $\langle \mathbb{M} \rangle_B^{\log} := \exp(\langle \log \mathbb{M} \rangle_B)$)

$$\begin{aligned} -\operatorname{div}\big(\big|\langle \mathbb{M} \rangle_B^{\log} \nabla h\big|^{p-2} (\langle \mathbb{M} \rangle_B^{\log})^2 \nabla h\big) &= 0 && \text{in } B, \\ h &= u\zeta^{p'} && \text{on } \partial B. \end{aligned}$$

- Use decay estimate of h and closeness of h to $u\zeta^{p'}$.

Maximal Function Estimates

Let $z = u\zeta^{p'}$ and $V(\xi) := |\xi|^{\frac{p-2}{2}}\xi$.

$$\mathcal{M}_2 f(x) := \sup_{B(x)} \bigg(\fint_B |f|^2 dy \bigg)^{\frac{1}{2}}, \qquad \mathcal{M}_2^\sharp f(x) := \sup_{B(x)} \bigg(\fint_B |f - \langle f \rangle_B|^2 dy \bigg)^{\frac{1}{2}}.$$

By comparison estimates for small $\delta > 0$

$$\mathcal{M}_2^\sharp\big(V(\mathbb{M}\nabla z)\big) \leq \big(|\log \mathbb{M}|_{\mathrm{BMO}} + \delta\big)\mathcal{M}_2\big(V(\mathbb{M}\nabla z)\big) + c\,\delta^{1-p}\mathcal{M}_{2s}\big(V(\mathbb{M}G)\big) + \dots$$

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For $q \gg 2$: $\|f\|_q \leq c\,\|\mathcal{M}_2 f\|_q \leq c\,q\|\mathcal{M}_2^\sharp f\|_q$ based on $\langle f, g\rangle \leq c\langle \mathcal{M}^\sharp f, \mathcal{M}g\rangle$.

Absorb blue into left-hand side if $q\,|\log \mathbb{M}|_{\mathrm{BMO}}$ is small.

We obtain as desired $\|\omega\,\nabla u\|_q \lesssim \|\omega\, G\|_q$.

Summary

Weighted equations:

$$-\operatorname{div}\big(\mathbb{A}(x)\nabla u\big) = -\operatorname{div}\big(\mathbb{A}(x)G\big),$$

$$-\operatorname{div}\big(|\mathbb{M}(x)\nabla u|^{p-2}\mathbb{M}^2(x)\nabla u\big) = -\operatorname{div}\big(|\mathbb{M}(x)G|^{p-2}\mathbb{M}^2(x)G\big) \qquad 1 < p < \infty.$$

Degenerate elliptic weights: $|\mathbb{M}(x)|\,|\mathbb{M}^{-1}(x)| \leq \Lambda,$

Weighted Laplacian [Balci, Diening, Giova, Passarelli di Napoli '20]

If $1 < q < \infty$ and $|\log \mathbb{A}|_{\mathrm{BMO}} \leq \kappa_\Lambda \min\left\{\frac{1}{q}, 1 - \frac{1}{q}\right\}$, then $\|\omega\,\nabla u\|_q \lesssim \|\omega\, G\|_q$.

Weighted p-Laplacian [Balci, Diening, Giova, Passarelli di Napoli '20]

If $p \leq q < \infty$ and $|\log \mathbb{M}|_{\mathrm{BMO}} \leq \frac{\kappa_\Lambda}{q}$, then $\|\omega\,\nabla u\|_q \lesssim \|\omega\, G\|_q$.

Condition scales well and estimates are sharp. Model example $\mathbb{M}(x) = |x|^{\pm\delta}\mathrm{Id}$.