Regularity issues for orthotropic functionals

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References

The research presented is part of a project started in 2011, mainly in collaboration with Pierre Bousquet (Toulouse)

1. Bousquet - B., **Rev. Mat. Iberoam.** (2019)
2. Bousquet - B. - Leone - Verde, **Calc. Var. PDE** (2018)
3. Bousquet - B., **Anal. PDE** (2018)
4. B. - Leone - Pisante - Verde, **Nonlinear Anal.** (2017)
5. Bousquet - B. - Julin, **Ann. SNS** (2016)
6. B. - Carlier, **Adv. Calc. Var.** (2014)

Outline

1. The functionals
2. Isotropic VS. Orthotropic
3. Lipschitz regularity
4. Beyond standard growth

Orthotropic functionals

We want to consider local minimizers of a functional with **orthotropic structure**

$$\sum_{i=1}^{N} \int f_i(u_{x_i})\,dx \qquad f_i \text{ convex}, \quad u_{x_i} = \frac{\partial u}{\partial x_i}$$

Example

By taking $f_i(t) = t^2/2$, we get

$$\sum_{i=1}^{N} \frac{1}{2} \int |u_{x_i}|^2\,dx = \frac{1}{2} \int |\nabla u|^2\,dx \qquad \text{Dirichlet integral}$$

A well-known functional without orthotropic structure

For $p \neq 2$, the classical

$$\frac{1}{p} \int |\nabla u|^p\,dx \qquad p-\text{Dirichlet integral}$$

does not fall in this class

Leading example

Orthotropic $p-$Dirichlet integral

$$\sum_{i=1}^{N} \frac{1}{p} \int |u_{x_i}|^p \, dx$$

This is a natural generalization of the Dirichlet integral

The orthotropic $p-$Laplacian operator

Local minimizers are weak solutions of the Euler-Lagrange equation

$$\sum_{i=1}^{N} \left(|u_{x_i}|^{p-2} \, u_{x_i} \right)_{x_i} = 0$$

Remark

This equation looks similar to the more familiar one

$$\sum_{i=1}^{N} \left(|\nabla u|^{p-2} \, u_{x_i} \right)_{x_i} = 0$$

but **they are quite different**

So similar, yet so different

Let us set

$$\mathcal{I}(z) = |z|^p \qquad \text{and} \qquad \mathcal{O}(z) = \sum_{i=1}^{N} |z_i|^p$$

Similarities: growth

Both of them are **strictly convex**, with $p-$**growth**, i. e.

$$\mathcal{O}(z) \simeq |z|^p = \mathcal{I}(z)$$

For basic regularity (i.e. L^∞ and $C^{0,\alpha}$ estimates, Harnack inequalities, Gehring-type gradient integrability etc.)

$$\sum_{i=1}^{N} \left(|\nabla u|^{p-2}\, u_{x_i}\right)_{x_i} \qquad \text{and} \qquad \sum_{i=1}^{N} \left(|u_{x_i}|^{p-2}\, u_{x_i}\right)_{x_i}$$

can be **treated in exactly the same manner** and there is nothing new (*see Chapters* 6 & 7 *of Giusti's book*)

Differences: ellipticity ($p \geq 2$)

- *isotropic*

$$\langle D^2\mathcal{I}(z)\,\xi,\xi\rangle \simeq |z|^{p-2}\,|\xi|^2$$

least eigenvalue of $D^2\mathcal{I}(z)$ becomes 0 **only at** $z = 0$

- *orthotropic*

$$\langle D^2\mathcal{O}(z)\,\xi,\xi\rangle \simeq \sum_{i=1}^{N} |z_i|^{p-2}\,|\xi_i|^2$$

least eigenvalue of $D^2\mathcal{O}(z)$ becomes 0 **each time** $z_i = 0$

For higher regularity (i. e. Lipschitz and $C^{1,\alpha}$) these are **completely different**

– I this talk I will be interested in **Lipschitz regularity** –

Some variations on the theme

Our motivation for this orthotropic functional was a problem in **Optimal Transport**, but once we opened the hell's gates....

1. General norms

$$\int \|\nabla u\|^p \, dx$$

where $\| \cdot \|$ is any norm

The relevant $p-$Laplacian behaves like the isotropic one only when **the norm $\| \cdot \|$ is uniformly convex**, otherwise it is a completely different story

2. Orthotropic & non-standard growth

$$\sum_{i=1}^{N} \int |u_{x_i}|^{p_i} \, dx, \qquad 1 < p_1 \leq p_2 \leq \ldots p_N < +\infty$$

For **gradient regularity**, this is one of the nastiest functionals (introduced by the Soviet school already in the 70s and independently by Marcellini in Western Countries)

A handful of (old) references

1. Orthotropic $p-$Laplacian has been considered for example in
   - Visik, **Mat. Sbornik** (1962)
   - Lions' book "*Quelques méthodes de résolution etc.*" (1969)
   - Zeidler's book "*Nonlinear functional analysis and its applications*" (1990)

   They tackle the **existence issue** for its parabolic version

$$\sum_{i=1}^{N}\left(|u_{x_i}|^{p-2}\,u_{x_i}\right)_{x_i}=u_t$$

2. For **higher regularity** (i.e. Lipschitz & $C^{1,\alpha}$), this equation has been overlooked or neglected, apart for
   - Uralt'seva - Urdaletova, **Vest. Leningr. Univ. Math.** (1984)

   They proved Lipschitz regularity for $p \geq 4$, without using energy methods, but Bernstein's one

Disclaimer

- From now on, I will manipulate solutions as if they were C^2
- I will focus on formally obtaining **a priori estimates**
- everything can then be rigorously justified by approximations

One step back: isotropic case

Consider a local weak solution of the **standard $p-$Laplacian**

$$-\sum_{i=1}^{N}(|\nabla u|^{p-2}\,u_{x_i})_{x_i}=0$$

How to prove that $\nabla u \in L^\infty_{\rm loc}$?

Equation for the gradient

We still use the notation $\mathcal{I}(z)=|z|^p$, then the equation rewrites

$$-\mathrm{div}\nabla\mathcal{I}(\nabla u)=0$$

Differentiate the equation in direction x_k, we get that u_{x_k} solves

$$-\mathrm{div}(D^2\mathcal{I}(\nabla u)\,\nabla u_{x_k})=0$$

We can think of this as *degenerate linear equation*, with coefficients matrix $D^2\mathcal{I}(\nabla u)$

One step back: isotropic case II

Subsolutions

For every $f:\mathbb{R}\to\mathbb{R}$ **convex**

$$\int \langle D^2\mathcal{I}(\nabla u)\,\nabla f(u_{x_k}),\nabla\varphi\rangle \le 0 \qquad \text{for every } \varphi\ge 0$$

that is, $f(u_{x_k})$ is a **subsolution** of the **linearized equation**

$$-\mathrm{div}(D^2 I(\nabla u)\,\nabla\psi)=0$$

Caccioppoli for the gradient

Take the test function $\varphi=\eta^2\,f(u_{x_k})$, then we get

Caccioppoli inequality for convex functions of u_{x_k}

$$\int \langle D^2\mathcal{I}(\nabla u)\,\nabla f(u_{x_k}),\nabla f(u_{x_k})\rangle\,\eta^2 \lesssim \int \langle D^2\mathcal{I}(\nabla u)\,\nabla\eta,\nabla\eta\rangle\,f(u_{x_k})^2$$

One step back: isotropic case III

We are in troubles, since

$$D^2\mathcal{I}(\nabla u) \simeq |\nabla u|^{p-2}$$

thus Caccioppoli for the gradient is **apparently useless** when $\nabla u = 0$

Absorption trick

We can bypass this problem by **absorbing** $D^2\mathcal{I}(\nabla u)$ **into the subsolution**. More precisely, find suitable **convex functions** f and F such that

$$\begin{aligned}\langle D^2\mathcal{I}(\nabla u)\,\nabla f(u_{x_k}), \nabla f(u_{x_k})\rangle &\simeq |\nabla u|^{p-2}\,|\nabla f(u_{x_k})|^2\\ &\geq |u_{x_k}|^{p-2}\,|\nabla f(u_{x_k})|^2\\ &= |\nabla F(u_{x_k})|^2\end{aligned}$$

Ok....but which kind of f and F work?

One step back: isotropic case IV

Power functions! Take $f(u_{x_k}) = |u_{x_k}|^\beta$, then F is still a power

By using this **trick**, from “Caccioppoli for the gradient” we get

$$\int_{B_\varrho} \left|\nabla |u_{x_k}|^{\beta+\frac{p-2}{2}}\right|^2 \lesssim \int_{B_R} |\nabla u|^{2\,\beta+p-2}$$

and combining with Sobolev inequality

$$\left(\int_{B_\varrho} |u_{x_k}|^{(2\,\beta+p-2)\frac{2^*}{2}}\right)^{\frac{2}{2^*}} \lesssim \int_{B_R} |\nabla u|^{2\,\beta+p-2}$$

iterative scheme of reverse Hölder inequalities $(2^*/2 > 1)$

Moser's iteration

Start with $\beta = 1$ and iterate infinitely many times

Now move forward

For the orthotropic case, we try to mimick the same strategy

Equation for the gradient

We have a look at the equation solved by u_{x_k}

By differentiating the equation with respect to x_k, we get

$$\sum_{i=1}^{N} \int |u_{x_i}|^{p-2} \, (u_{x_k})_{x_i} \, \varphi_{x_i} = 0$$

a **linear degenerate** elliptic equation with **diagonal** coefficient matrix

$$D^2\mathcal{O}(\nabla u) = \begin{bmatrix} |u_{x_1}|^{p-2} & & \\ & \ddots & \\ & & |u_{x_N}|^{p-2} \end{bmatrix}$$

The **least eigenvalue is** 0 each time a component of ∇u vanishes

Subsolutions

For every $f : \mathbb{R} \to \mathbb{R}$ **convex**

$$\sum_{i=1}^{N} \int |u_{x_i}|^{p-2} \Big(f(u_{x_k}) \Big)_{x_i} \varphi_{x_i} \leq 0 \qquad \text{for every } \varphi \geq 0$$

that is $f(u_{x_k})$ is a **subsolution** of the **linearized equation**

$$\mathrm{div}(D^2\mathcal{O}(\nabla u)\, \nabla \psi) = \sum_{i=1}^{N} \Big(|u_{x_i}|^{p-2}\, \psi_{x_i} \Big)_{x_i} = 0$$

Caccioppoli inequality for the gradient

Take the test function $\varphi = \eta^2\, f(u_{x_k})$, then we get

Caccioppoli inequality for convex functions of u_{x_k}

$$\sum_{i=1}^{N} \int |u_{x_i}|^{p-2} \left| \Big(f(u_{x_k}) \Big)_{x_i} \right|^2 \eta^2 \lesssim \sum_{i=1}^{N} \int |u_{x_i}|^{p-2}\, f(u_{x_k})^2\, |\eta_{x_i}|^2$$

A major obstruction

In the **isotropic case** Caccioppoli for the gradient gave a control on

$$|\nabla u|^{p-2}\,|\nabla f(u_{x_k})|^2$$

but **now it is much worse**!

We only control

$$\sum_{i=1}^{N} |u_{x_i}|^{p-2}\left|\Big(f(u_{x_k})\Big)_{x_i}\right|^2$$

i.e. a weighted gradient of $f(u_{x_k})$...too much degeneracy

No way that the "absorption trick" works as before

Lipschitz regularity for $p \geq 2$

Theorem (Bousquet-B.-Leone-Verde)

Let $p \geq 2$. If u is a local minimizer of

$$\sum_{i=1}^{N} \frac{1}{p} \int |u_{x_i}|^p \, dx$$

then $\nabla u \in L^\infty_{\mathrm{loc}}$ and

$$\|\nabla u\|_{L^\infty(B_{R/2})} \lesssim \left(\fint_{B_R} |\nabla u|^p \, dx \right)^{\frac{1}{p}}$$

Remark

We want to perform a Moser's iteration, but we need <u>new ideas</u> in order to exploit **Caccioppoli for the gradient** and circumvent the degeneracy of the weights $|u_{x_i}|^{p-2}$

A technical innovation

We cook up **new Caccioppoli inequalities for the gradient**

The method

- as before, take the equation differentiated with respect to x_k

$$\sum_{i=1}^{N} \int |u_{x_i}|^{p-2} \, (u_{x_k})_{x_i} \, \varphi_{x_i} = 0$$

- now insert the **weird test function** $(\alpha \leq \beta)$

$$\varphi = |u_{x_k}|^{2\,\alpha-1} \, |u_{x_j}|^{2\,\beta} \, \eta^2$$

- combine the Caccioppoli inequality so obtained (we call it **weird Caccioppoli**)...
- ...with the Caccioppoli for the gradient (I mean, the one we obtained previously)...
- ... plus a finite iteration on indexes α and β with $\alpha + \beta$ fixed (this is the magic & scaring part that nobody wants to see in a talk)

“The dish is ready”

For every $q = 2^m$ we get for every j, k

$$\sum_{i=1}^{N} \int |u_{x_i}|^{p-2}\, u_{x_i\, x_k}^2\, |u_{x_j}|^{2\,q} \lesssim \int |\nabla u|^{p+2\,q}$$

Why two indices j, k? What we do now?

We can now perform the usual **absorption trick** on the left-hand side!! In the sum, **keep only the term** $i = j$

$$\int |u_{x_j}|^{p-2}\, u_{x_j\, x_k}^2\, |u_{x_j}|^{2\,q} \simeq \int \left| \left(|u_{x_j}|^{\frac{p}{2}+q} \right)_{x_k} \right|^2$$

and **sum over** k **to reconstruct the full gradient** of $|u_{x_j}|^{p/2+p}$!

Conclusion

After all these struggles, we get

Caccioppoli for power-functions

$$\int \left|\nabla |u_{x_j}|^{\frac{p}{2}+q}\right|^2 \lesssim \int |\nabla u|^{p+2\,q}$$

We are in the same situation as for the standard $p-$Laplacian:

- use Sobolev inequality
- get an iterative Moser's scheme
- iterate infinitely many times for a diverging sequence q_n

(I am hiding *sous le tapis* other – lower order yet annoying – technical complications)

Some comments

Related results

The same Lipschitz result has been obtained by means of *viscosity techniques* by *Demengel* [Adv. Differ. Equ. (2016)]

Right-hand side

- Our result also covers much more degenerate situations and the non-homogeneous case

$$-\sum_{i=1}^{N}\left(|u_{x_i}|^{p-2}\,u_{x_i}\right)_{x_i} = f$$

  under some **non-sharp** assumptions on f
- The expected sharp assumption on f to get Lipschitz regularity is $f \in L^q$ with $q > N$ (actually the sharpest assumption should be on the Lorentz scale $f \in L^{N,1}$ as in *Beck - Mingione* [CPAM (2019)])
- At present, this is still an open problem

Other regularity results

Higher differentiability *à la* Uhlenbeck

Local minimizers are such that

$$|u_{x_i}|^{\frac{p-2}{2}}\, u_{x_i} \in W^{1,2}_{\mathrm{loc}}$$

Still true with a right-hand side f, under the sharp assumption $f \in W^{s,p'}_{\mathrm{loc}}$, as in *B.-Santambrogio* [Comm. Cont. Math. (2016)]

C^1 regularity

In dimension $N = 2$, local minimizers are such that (Bousquet - B.)

$$\nabla u \in C^0$$

The proof works with a right-hand side f, as well....but the paper was already quite complicated with $f = 0$

Still on C^1 regularity

- *Lindqvist - Ricciotti* [Nonlinear Anal. (2018)] improved the result to

$$\nabla u \in C^{0,\omega}$$

for some logarithmic modulus of continuity ω

- this is for the homogeneous equation only
- for a right-hand side f, one could try to transfer the **excess-decay estimate**

$$\fint_{B_r} |\nabla u - \overline{\nabla u}_{B_r}|^p \, dx \lesssim \omega(r)$$

from solutions of the homogeneous equation...

- ...but the modulus ω is too weak for this strategy to work (in other words, Campanato's Theorem fails for $C^{0,\omega}$, see *Spanne* [Ann. SNS (1965)])

1. The functionals

2. Isotropic VS. Orthotropic

3. Lipschitz regularity

4. Beyond standard growth

Next challenge

What about the **orthotropic & non-standard growth**?

$$\sum_{i=1}^{N} \frac{1}{p_i} \int |u_{x_i}|^{p_i}\, dx, \qquad 1 < p_1 \le p_2 \le \dots p_N < +\infty$$

Well-known fact

We **can not** expect regularity for local minimizers, when

$$1 \ll \frac{p_N}{p_1}$$

(Giaquinta-Marcellini's counterexamples)
In this case, local minimizers may be **unbounded**

Question

What if we impose *a priori* that a local minimizer is bounded?

Orthotropic & non-standard growth

Theorem (Bousquet - B.)

Let $2 \le p_1 \le \cdots \le p_N$. *If* u *is a* **bounded** *local minimizer of*

$$\sum_{i=1}^{N} \frac{1}{p_i} \int |u_{x_i}|^{p_i}\, dx$$

then $\nabla u \in L^\infty_{\mathrm{loc}}$

Remarks

- **no upper bound** on p_N/p_1 is assumed. Under such a generality, the result is claimed in *Lieberman* [Adv. Diff. Eq. (2005)]
- case $N = 2$ previously proved in *B. - Leone - Pisante - Verde*, by using a different argument i.e. a two-dimensional trick introduced in *Bousquet - B. - Julin*
- for $p_1 \ge 4$ and $p_N < 2\, p_1$, proved by Uralt'seva & Urdaletova (by Bernstein's method)

A glimpse of the proof

The proof is composed of two main steps:

A. a Moser's iteration similar to the one for $p_1 = \cdots = p_N$, to get

$$\|\nabla u\|_{L^\infty(B_{1/2})} \lesssim \left(\int_{B_1} |\nabla u|^\gamma\right)^{\frac{1+\Theta}{\gamma}}$$

γ could be very big (here, we **do not** need $u \in L^\infty_{\mathrm{loc}}$)

B. a self-improving scheme for the gradient *à la Bildhauer-Fuchs-Zhong* [Ann. SNS (2007)]

$$\int_{B_{\sigma R}} |u_{x_k}|^{p_k+2+\alpha}\, dx \le C + C \sum_{i \neq k} \int_{B_R} |u_{x_i}|^{\frac{p_i-2}{p_k}(p_k+2+\alpha)}\, dx$$

The constant C **depends on** $\|u\|_{L^\infty_{\mathrm{loc}}}$

Final gain of **B.**: $\nabla u \in L^q_{\mathrm{loc}}$ for every q

Some comments

L^∞ assumption

- Sharp assumptions in order to get $u \in L^\infty_{\mathrm{loc}}$ are in

  Fusco - Sbordone [Manuscripta Math. (1990)]

- for example, in dimension $N = 2$ local minimizers are always locally bounded

- for more general functionals with nonstandard growth, many authors contributed to local boundedness. Among others, we mention

  Cupini - Marcellini - Mascolo [Nonlinear Anal. (2019)]

  Hirsch - Schäffner [Comm. Contemp. Math. (2020)]

Right-hand side

- Our result does not cover the non-homogeneous case

$$-\sum_{i=1}^{N}\left(|u_{x_i}|^{p_i-2}\,u_{x_i}\right)_{x_i}=f$$

- the proof is very likely to be adapted (with some *sweat & tears*) to include the right-hand side f, without sharp assumptions
- The expected sharp assumption on f to get Lipschitz regularity is...? In view of *Beck* - *Mingione* it is reasonable to expect $f\in L^{N,1}$

Higher differentiability *à la* Uhlenbeck

$L^\infty_{\rm loc}$ local minimizers are such that (Bousquet - B.)

$$|u_{x_i}|^{\frac{p_i-2}{2}}\,u_{x_i}\in W^{1,2}_{\rm loc}$$

C^1 regularity

- in dimension $N = 2$ *Lindqvist - Ricciotti* [Nonlinear Anal. (2018)] proved also
$$\nabla u \in C^{0,\omega}$$
for some logarithmic modulus of continuity ω, even for $2 \leq p_1 \leq p_2$
- again, this is for the homogeneous equation only

Many thanks for your kind attention

"I knew it would take some time to get to that point.
And I worked hard to get there"
C. Schuldiner