Gradient estimates for fully nonlinear models with non-homogeneous degeneracy

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Outline

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Literature review

Before presenting our main problems and results, let us to revisit some well-known elliptic PDEs models:

Scenario	Divergence form	Non-Divergence form
Uniformly Elliptic	$\mathrm{div}(\mathbb{A}(x)\nabla u)$	$\mathrm{Tr}(\mathbb{A}(x)D^2u)$
Single Degeneracy Law	$\mathrm{div}(\|\nabla u\|^{p-2}\nabla u) \quad p>2$	$\|Du\|^p\mathrm{Tr}(\mathbb{A}(x)D^2u) \quad p>0$
Double Degeneracy Law	$\mathrm{div}((\|\nabla u\|^{p-2}+\mathfrak{a}(x)\|\nabla u\|^{q-2})\nabla u)$	What should be the model case?

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Double Degeneracy Law	$\mathrm{div}((\|\nabla u\|^{p-2} + \mathfrak{a}(x)\|\nabla u\|^{q-2})\nabla u)$	What should be the model case?

In this point will be natural to consider the following non-homogeneous model case:

$$\mathcal{L}[u] = [|Du|^p + \mathfrak{a}(x)|Du|^q]\,\mathrm{Tr}(\mathbb{A}(x)D^2 u) \quad \text{for} \quad 0 < p \le q < \infty \quad \text{and} \quad 0 \le \mathfrak{a} \in C^0(\Omega),$$

i.e. the counterpart of certain variational problems from Calculus of Variations with double phase structure.

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Literature review

In the last decades have appeared a huge amount of literature on double phase problems [a]

$$(w,f) \mapsto \min \int_\Omega \left(\frac{1}{p}|\nabla w|^p + \frac{\mathfrak{a}(x)}{q}|\nabla w|^q - fw\right) dx \quad \Rightarrow \quad \mathcal{L}[u] = -\mathrm{div}\left(\left(|\nabla u|^{p-2} + \mathfrak{a}(x)|\nabla u|^{q-2}\right)\nabla u\right) = f(x)$$

a

- L. Beck, **Elliptic regularity theory**. A first course. *Lecture Notes of the Unione Matematica Italiana*, 19. Springer, Cham; Unione Matematica Italiana, Bologna, 2016. xii+201 pp. ISBN: 978-3-319-27484-3; 978-3-319-27485-0.
- I. Chlebicka *et al*, **Partial Differential Equations in Anisotropic Musielak-Orlicz Spaces**. Monograph.
- **C. De Filippis' contributions** - https://sites.google.com/view/cristianadefilippis/home
- **P. Marcellini's contributions** - http://web.math.unifi.it/users/marcell/index.html
- **G. Mingione's contributions** - https://sites.google.com/site/giuseppemingionemath/
- V.V. Zhikov, , *Lavrentiev phenomenon and homogenization for some variational problems*. C. R. Acad. Sci. Paris Sér. I Math. 316 (1993), no. 5, 435-439.

Presenting the problem

In this Lecture we are interested in studying quantitative features for fully nonlinear elliptic models of double degenerate type as follows:

$$\mathcal{G}[u] := \mathcal{H}(x, Du)F\left(x, D^2u\right) = f(x,u) \quad \text{in} \quad \Omega \subset \mathbb{R}^n \text{ (bounded domain)}, \tag{1.1}$$

where we will suppose the following **Structural Conditions** (SC):

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where we will suppose the following **Structural Conditions** (SC):

- ✓ $f \in C^0(\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$;
- ✓ $F : \Omega \times \text{Sym}(n) \to \mathbb{R}$ is a $(\lambda, \Lambda)-$elliptic operator with $\omega-$continuous coefficients:

$$\lambda\|X - Y\| \leq F(x, X) - F(x, Y) \leq \Lambda\|X - Y\| \quad \text{and} \quad \Theta_F(x,y) := \sup_{\substack{X \in Sym(n)\\ X \neq 0}} \frac{|F(x,X) - F(y,X)|}{\|X\|} \leq C_F\omega(|x-y|).$$

- ✓ $L_1\,[|\xi|^p + \mathfrak{a}(x)|\xi|^q] \leq \mathcal{H}(x,\xi) \leq L_2\,[|\xi|^p + \mathfrak{a}(x)|\xi|^q]$;
- ✓ $0 < p \leq q < \infty$, $0 < L_1 \leq L_2 < \infty$ and $0 \leq \mathfrak{a} \in C^0(\overline{\Omega})$.

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Some Motivations

A key issue in linear/nonlinear PDEs consists in inferring which is the expected regularity to corresponding solutions.

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By way of motivation, let us visit the uniformly elliptic theory: Let u be a solution to

$$\mathcal{G}[u] := \mathrm{Tr}\left(\mathbb{A}(x)D^2u\right) = f(x) \quad \text{in} \quad B_1. \tag{1.2}$$

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There are two important aspects which we must take into account:

A priori estimate to Hom. PDE with "frozen" coeff. & Integrability of the source term

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As a matter of fact, $v(x) := \frac{u(\rho x)}{\rho^{\kappa}}$, for $\kappa \in (0,2]$ verifies:

$$\mathcal{G}_{\rho}[v] = \mathrm{Tr}\left(\mathbb{A}_{\rho}(x)D^2v\right) = \rho^{2-\kappa}f(\rho x) := f_{\rho}(x) \quad \Rightarrow \quad \|f_{\rho}\|_{L^r(B_1)} \le \rho^{2-\kappa-\frac{n}{r}}\|f\|_{L^r(B_1)}.$$

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Sharp regularity estimates

Better integrability/regularity of f (resp. $\mathbb{A}$) $\Rightarrow$ Better (local) regularity of u

Theorem (**[E.V. Teixeira, Arch. Ration. Mech. Anal., 2014]**)

Let u be a bounded viscosity solution[a] *to* (1.2) *then*

$\mathbf{f \in L^r(B_1)}$	**Sharp Regularity**
$\frac{n}{2} < r < n$	$C^{0,\varsigma}_{loc}(B_1)$
$r = n$	$C^{0,Log\text{-}Lip}_{loc}(B_1)$
$n < r < \infty$	$C^{1,\varsigma}_{loc}(B_1)$
$BMO \supset L^\infty$	$C^{1,Log\text{-}Lip}_{loc}(B_1)$

[a] $u \in C^0(B_1)$ is a viscosity super-solution (resp. sub-solution) to (1.2) if whenever $\varphi \in C^2(B_1)$ and $x_0 \in B_1$ such that $u - \varphi$ has a local minimum (resp. local maximum) at x_0, then

$$\mathrm{Tr}\left(\mathbb{A}(x_0)D^2\varphi(x_0)\right) \leq f(x_0) \qquad \text{resp. } \mathrm{Tr}\left(\mathbb{A}(x_0)D^2\varphi(x_0)\right) \geq f(x_0).$$

Explicit representation of the moduli of continuity

Theorem (**[E.V. Teixeira, Arch. Ration. Mech. Anal., 2014]**)

Let u be a bounded viscosity solution to (1.2) *then*

$\mathbf{f} \in \mathbf{L}^{\mathbf{r}}(\mathbf{B_1})$	**Sharp Regularity**
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$BMO \supset L^{\infty}$	$C^{1,Log\text{-}Lip}_{loc}(B_1)$

$$\varsigma := 2 - \frac{n}{r} \quad \text{and} \quad \zeta := \min\left\{\alpha_{\text{Hom}}^{-}, 1 - \frac{n}{r}\right\}$$

Sharp Lipschitz Logarithmical moduli of continuity

Theorem (**[E.V. Teixeira, Arch. Ration. Mech. Anal., 2014]**)

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$n < r < \infty$	$C^{1,\zeta}_{loc}(B_1)$
$BMO \supset L^\infty$	$C^{1,Log\text{-}Lip}_{loc}(Q_1^-)$

$$\tau(s) := s|\log s| \qquad \textit{and} \qquad \psi(s) := s^2|\log s|.$$

Similar borderline/regularity results were addressed by Daskalopoulos et al[a]

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P. Daskalopoulos, T. Kuusi & G. Mingione, *Borderline estimates for fully nonlinear elliptic equations.* **Comm. Partial Differential Equations** 39 (2014), n.3, 574-590..

From uniformly elliptic to doubly degenerate theory

What should we expect from Doubly Degenerate Scenery?

Recently, by combining geometric methods and analytic techniques[a] the $C^{1,\alpha}_{\text{loc}}$ regularity estimate was addressed by De Filippis for equations as follow

$$[|Du|^p + \mathfrak{a}(x)|Du|^q]\, F(D^2u) = f \in L^{\infty}(B_1) \cap C^0(B_1), \quad \text{for some } \alpha(\textit{universal}) \in (0,1).$$

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a

I. Birindelli, F. Demengel & F. Leoni, $C^{1,\gamma}$ *regularity for singular or degenerate fully nonlinear equations and applications*. **NoDEA Nonlinear Differential Equations Appl**. 26 (2019), no. 5, Paper No. 40, 13 pp.

C. De Filippis, *Regularity for solutions of fully nonlinear elliptic equations with nonhomogeneous degeneracy*. **Proc. Roy. Soc. Edinburgh Sect. A**, 151 (2021), no. 1, 110-132.

C. Imbert & L. Silvestre, $C^{1,\alpha}$ *regularity of solutions of degenerate fully non-linear elliptic equations*. **Adv. Math**. 233 (2013), 196-206.

Open sceneries

Nevertheless, De Filippis' work leaves some open issues regards the general scenario:

$$\mathcal{G}[u] := \mathcal{H}(x, Du)F\left(x, D^2u\right) = f(x,u) \quad \text{in} \quad \Omega \qquad \text{under the Structural Conditions (SC).}$$

We stress that the regularity theory for uniformly elliptic models is available in the Caffarelli and Trudinger's works[a]

[a]

L.A. Caffarelli, *Interior apriori estimates for solutions of fully nonlinear equations*. **Ann. of Math.** (2) 130 (1989), 189-213.

L.A. Caffarelli & X. Cabré, *Fully nonlinear elliptic equations*. American Mathematical Society Colloquium Publications, 43. American Mathematical Society, Providence, RI, 1995. vi+104 pp. ISBN: 0-8218-0437-5.

N.S. Trudinger, *Fully nonlinear, uniformly elliptic equations under natural structure conditions*. **Trans. Amer. Math. Soc**. 278 (1983), no. 2, 751-769.

Our contributions

We will provide an **affirmative answer** in following sceneries:

Assumptions	Sharp Regularity
(SC) in force	$C^{1,\min\left\{\frac{1}{p+1},\alpha^{-}_{\mathrm{Hom}}\right\}}_{\mathrm{loc}}$
(SC) + F a concave/convex operator	$C^{1,\frac{1}{p+1}}_{\mathrm{loc}}$

Another pivotal question:

Are there significant changes between De Filippis' approach and ours?

1. Compactness (Hölder estimate) via Ishii-Lions method;
2. $C^{1,\alpha}$ regime via affine approximation scheme (Deviation by Planes);
3. Degeneracy character of operator: a priori estimates for small/large translations;

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Our Main Focus and its possible implications

Our Main Impetus

Therefore, we will establish sharp (geometric) $C^{1,\alpha}$ estimates for solution to (1.1) by making use a systematic and alternative approach[a], as well as address some improvements.

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[a]

D.J. Araújo, E.V. Teixeira & J.M. Urbano, *Towards the $C^{p'}$ regularity conjecture*, **Int. Math. Res. Not. IMRN** 2018, no. 20, 6481-6495.

E. Lindgren, P. Lindqvist, *Regularity of the p-Poisson equation in the plane.* **J. Anal. Math**. 132 (2017), 217-228.

A. Attouchi, M. Parviainen & E. Ruosteenoja, *$C^{1,\alpha}$ regularity for the normalized p-Poisson problem.* **J. Math. Pures Appl.** (9) 108 (2017), no. 4, 553-591.

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Our Main Theorem

Theorem ([da S. - Ricarte, Calc. Var. PDEs, 59 (2020), n. 5, Paper No. 161, 33 pp.])

Let $K \subset\subset B_1$, u be a bounded solution of (1.1) *in* B_1 *and suppose that (SC) are in force. Then u is $C^{1,\alpha}_{loc}$, i.e., there exists a (universal) constant $\mathrm{M} > 0$ such that*

$$[u]^{*}_{C^{1,\alpha}(K)} \leq \mathrm{M}.\left[\|u\|_{L^{\infty}(B_1)} + \|f\|^{\frac{1}{p+1}}_{L^{\infty}(B_1)} + 1\right] \quad \textit{where,}$$

$$[u]^{*}_{C^{1,\alpha}(K)} := \sup_{0<\rho\leq\rho_0}\left(\inf_{x_0\in K}\frac{\|u-\mathfrak{l}_{x_0}(u)\|_{L^{\infty}(B_\rho(x_0)\cap K)}}{\rho^{1+\alpha}}\right) \text{ and } \mathfrak{l}_{x_0}(u) := u(x_0)+Du(x_0)\cdot(x-x_0).$$

Our regularity estimates generalize, to some extent, earlier ones via a different approach[a]

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a

D.J. Araújo, G.C. Ricarte & E.V. Teixeira, *Geometric gradient estimates for solutions to degenerate elliptic equations.* **Calc. Var. Partial Differential Equations** 53 (2015), 605-625.

Our Main Focus and its possible implications

It is worth highlight that such estimates play an essential role in proving[a]:

1. Blow-up and Liouville type results;
2. Weak geometric properties and Hausdorff measure estimates;
3. Sharp regularity in certain free boundary problems (FBPs for short).

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a

J. Andersson, E. Lindgren & H Shahgholian, *Optimal regularity for the obstacle problem for the $p-$Laplacian.* **J. Differential Equations** 259 (2015), n. 6, 2167-2179.

L. Caffarelli & S. Salsa, *A geometric approach to free boundary problems.* Graduate Studies in Mathematics, 68. American Mathematical Society, Providence, RI, 2005. x+270 pp. ISBN: 0-8218-3784-2.

J.V. da Silva & A. Salort, *Sharp regularity estimates for quasi-linear elliptic dead core problems and applications.* **Calc. Var. Partial Differential Equations** 57 (2018), no. 3, 57: 83.

Chapter 1: An approximation Scheme

A key step in accessing the regularity theory available for "frozen" coefficient, homogeneous operators is the following.

Lemma (**Approximation Lemma)**

If u is a solution of (1.1) *in B_1 with $\|u\|_{L^\infty(B_1)} \le 1$, then $\forall \varepsilon > 0$ there exists $\delta = \delta(p,q,n,\lambda,\Lambda,\varepsilon) > 0$ such that whenever $\max\left\{\Theta_F(x), \|f\|_{L^\infty(B_1\times\mathbb{R})}\right\} \le \delta_\varepsilon$ there exists a $\mathfrak{F}$-harmonic function $\phi : B_{\frac{1}{2}} \to \mathbb{R}$, i.e., $F(D^2\phi) = 0$, such that*

$$\max\left\{\|u-\phi\|_{L^\infty\left(B_{\frac{1}{2}}\right)}, \|D(u-\phi)\|_{L^\infty\left(B_{\frac{1}{2}}\right)}\right\} < \varepsilon \quad \textit{where} \quad \left(\|\phi\|_{C^{1,\alpha_{\text{Hom}}}(\Omega')} \le C(n,\lambda,\Lambda)\cdot\|\phi\|_{L^\infty(\Omega)}\right).$$

Proof "Just waving hands":

The proof is based on a *Reductio ad absurdum* and makes use of compactness, stability, *a priori* estimates and uniqueness of Dirichlet problem. □

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Chapter 1: An approximation Scheme

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Chapter 1: An approximation result

Remark (**Normalization and "flatness regime"**)

Assumptions in the Lemma 3 are not restrictive. Indeed, fixed $\delta_\varepsilon > 0$, there exist $\kappa, \tau > 0$ such that the function

$$v(x) = \frac{u(\tau x + x_0)}{\kappa},$$

fall into in the conditions of Lemma 3, where

$$\kappa := \|u\|_{L^\infty(\Omega)} + 1 + \delta_\varepsilon^{-1} \|f\|_{L^\infty(\Omega\times\mathbb{R})}^{\frac{1}{p+1}} \quad \text{and} \quad \tau = \min\left\{\frac{1}{2}, \left(\frac{\delta_\varepsilon}{\|f\|_{L^\infty(\Omega\times\mathbb{R})}+1}\right)^{\frac{1}{p+2}}, \omega^{-1}\left(\frac{\delta_\varepsilon}{C_F+1}\right)\right\}.$$

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Chapter 2: A first approximating estimate

In the sequel, the purpose will be to make use of an $\mathfrak{F}-$harmonic approximation in a C^1-fashion (Approximation Lemma) to ensure that viscosity solutions are "geometrically close" to their tangent plane in a suitable manner, i.e.

$$C^1 - \text{closeness} \quad \overset{\text{Geometric estimate}}{\Longrightarrow} \quad \sup_{B_\rho(x_0)} \frac{|u(x) - u(x_0) - Du(x_0)\cdot(x-x_0)|}{\rho^{1+\alpha}} \leq 1,$$

thereby getting a geometric estimate.

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thereby getting a geometric estimate.

Lemma (**"Pseudo" first step of induction**)

Let u be a viscosity solution of (1.1) in B_1 with $\|u\|_{L^\infty(B_1)} \leq 1$. There exist $\delta_\varepsilon > 0$ and $\rho \in \left(0, \frac{1}{2}\right)$ such that if $\max\left\{\Theta_F(x), \|f\|_{L^\infty(B_1\times\mathbb{R})}\right\} \leq \delta_\varepsilon$, then

$$\sup_{B_\rho(x_0)} \left|u(x) - \mathfrak{l}_{x_0}(u)(x)\right| \leq \rho^{1+\alpha}.$$

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Chapter 2: A first approximating estimate

(Philosophical) Idea of proof:

$$\begin{aligned}
\left\| u - \mathfrak{l}_{x_0}(u) \right\|_{L^\infty(B_\rho(x_0))} &\le \left\| \phi - \mathfrak{l}_{x_0}(\phi) \right\|_{L^\infty(B_\rho(x_0))} + |(u-\phi)(x_0)| \\
&+ \|u-\phi\|_{L^\infty(B_\rho(x_0))} + |D(u-\phi)(x_0)| \\
&\le C \sup_{B_\rho(x_0)} |x-x_0|^{1+\alpha_{\text{Hom}}} + 3\varepsilon \\
&\le C\rho^{1+\alpha_{\text{Hom}}} + 3\varepsilon \ \text{(What is the expected estimate?)}
\end{aligned}$$

□

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Chapter 2: A first approximating estimate

(Philosophical) Idea of proof:

$$\begin{aligned}\left\|u - \mathfrak{l}_{x_0}(u)\right\|_{L^\infty(B_\rho(x_0))} &\leq \left\|\phi - \mathfrak{l}_{x_0}(\phi)\right\|_{L^\infty(B_\rho(x_0))} + |(u-\phi)(x_0)| \\ &+ \|u-\phi\|_{L^\infty(B_\rho(x_0))} + |D(u-\phi)(x_0)| \\ &\leq C \sup_{B_\rho(x_0)} |x-x_0|^{1+\alpha_{\text{Hom}}} + 3\varepsilon \\ &\leq C\rho^{1+\alpha_{\text{Hom}}} + 3\varepsilon \quad (\text{What is the expected estimate?})\end{aligned}$$

□

Coming back to last estimate we can conclude:

$$\left\|u - \mathfrak{l}_{x_0}(u)\right\|_{L^\infty(B_\rho(x_0))} \leq \rho^{1+\alpha}$$

provided $\rho \in \left(0, \min\left\{\frac{1}{2}, \left(\frac{1}{2C}\right)^{\frac{1}{\alpha_{\text{Hom}}-\alpha}}\right\}\right)$ and $\varepsilon \in \left(0, \frac{1}{6}\rho^{1+\alpha}\right)$.

Chapter 3: The gap in the standard induction process

Proceeding with the iteration process

Different from $C^{1,\alpha}$ regularity estimates from linear setting, we can no longer proceed with an iterative scheme, i.e.

$$\sup_{B_{\rho^k}(x_0)} \frac{|u(x) - \mathfrak{l}_k(x)|}{\rho^{k(1+\alpha)}} \leq 1 \quad \underset{\Longrightarrow}{\text{Dini-Campanato embedding}} \quad u \text{ is } C^{1,\alpha} \text{ at } x_0,$$

because *a priori* we do not know the equation which is satisfied by

$$B_1(0) \ni x \mapsto \frac{(u - \mathfrak{l}_k)(\rho^k x)}{\rho^{k(1+\alpha)}}, \text{ for } \{\mathfrak{l}_k\}_{k \in \mathbb{N}} \text{ affine functions, since } \mathcal{H}(x, Dv)F(x, D^2 v) \text{ is not invariant by affine maps.}$$

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For this reason, an alternative approach must be undertaken: quantitative information on the oscillation of u

$$\sup_{B_\rho(x_0)} \frac{\rho^{-1}|u(x) - u(x_0)|}{\rho^\alpha + |Du(x_0)|} \leq 1 \quad \overset{\text{Iteration}}{\Longrightarrow} \quad \sup_{B_{\rho^k}(x_0)} \frac{\rho^{-k}|u(x) - u(x_0)|}{\rho^{k\alpha} + \frac{|Du(x_0)|(1-\rho^{(k-1)\beta})}{1-\rho^\alpha}} \leq 1,$$

which proves to be the proper estimate for continuing with an iterative process, provided we get a sort of suitable control under the magnitude of the gradient (point-wisely)

Chapter 4: Iteration scheme: A new oscillation mechanism

Corollary (**The (real) First step of induction)**

Suppose that the assumptions of previous Lemma are in force. Then,

$$\sup_{B_{\rho}(x_0)} |u(x) - u(x_0)| \leq \rho^{1+\alpha} + \rho|Du(x_0)|.$$

Chapter 4: Iteration scheme: A new oscillation mechanism

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In order to obtain a precise control on the influence of magnitude of the gradient of u, we iterate solutions (using the previous Corollary) in corrected ρ-adic cylinders.

Lemma (**Iterative process)**

Under the assumptions of previous Corollary one has

$$\sup_{B_{\rho^k}(x_0)} |u(x) - u(x_0)| \leq \rho^{k(1+\alpha)} + |Du(x_0)| \sum_{j=0}^{k-1} \rho^{k+j\alpha}.$$

Proof.

By induction process – Here we make use of assumption $\alpha \leq \frac{1}{p+1}$. □

Chapter 4: Iteration scheme: A new oscillation mechanism

Our next result provides the geometric regularity estimate inside critical zone. We define the critical zone as follows:

$$\mathcal{C}_{\rho}^{\alpha}(B_1) := \{x \in B_1;\ |Du(x)| \leq \rho^{\alpha}\}.$$

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$$\mathcal{C}_{\rho}^{\alpha}(B_1) := \{x \in B_1;\ |Du(x)| \leq \rho^{\alpha}\}.$$

Lemma (Estimate inside critical zone)

Suppose that the assumptions of previous Lemma are in force. Then, there exists $\mathrm{M}(\textit{universal}) > 1$ *such that*

$$\sup_{B_{\rho_0}(x_0)} |u(x) - u(x_0)| \leq \mathrm{M}{\rho_0}^{1+\alpha}\left(1 + |Du(x_0)|{\rho_0}^{-\alpha}\right),\ \forall \rho_0 \in (0, \rho).$$

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Concluding: Proof of Main Theorem

Proof of the main Theorem.

WLOG, we may assume that $K = B_{\frac{1}{2}}$ and $x_0 = 0 \in \mathcal{C}^{\alpha}_{\rho_0}(K)$. Using previous Lemma, we estimate

$$\begin{aligned}
\sup_{B_{\rho_0}} \frac{|u(x) - \mathfrak{l}_0 u(x)|}{\rho_0^{1+\alpha}} &\leq \sup_{B_{\rho_0}} \frac{|u(x) - u(0)|}{\rho_0^{1+\alpha}} + \frac{|Du(0)|\rho_0}{\rho_0^{1+\alpha}} \\
&\leq \mathrm{M}\left(1 + |Du(0)|\rho_0^{-\alpha}\right) + 1 \\
&\leq 3\mathrm{M}.
\end{aligned}$$

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&\leq \mathrm{M}\left(1 + |Du(0)|\rho_0^{-\alpha}\right) + 1 \\
&\leq 3\mathrm{M}.
\end{aligned}$$

On the other hand, if the gradient has a uniform lower bound, i.e. $|Du| \geq L_0 > 0$, then Caffarelli-Trudinger's classical estimates (or Teixeira's work) can be enforced since the operator becomes uniformly elliptic:

$$\mathcal{P}^{-}_{\lambda,\Lambda}(D^2u) \leq C_0\left(L_0^{-1}, p, q, \|\mathfrak{a}\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)}\right) \quad \text{and} \quad \mathcal{P}^{+}_{\lambda,\Lambda}(D^2u) \geq -C_0\left(L_0^{-1}, p, q, \|\mathfrak{a}\|_{L^\infty(B_1)}, \|f\|_{L^\infty(B_1)}\right).$$

□

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Final Chapter: The Journey continues...

Closing remarks:

Coming back to the open issues (yet):

$\mathbf{f \in L^r(B_1) \cap C^0(B_1)}$	**Regularity Estimates**
(SC) and $r = n$	$C^{0,\alpha}_{loc}$
(SC) and $n < r < \infty$	Open Problem
(SC)	$C_{\text{loc}}^{1,\min\left\{\frac{1}{p+1},\, \alpha^{-}_{\text{Hom}}\right\}}$
(SC) + F a concave/convex (or Asympt. convex) operator	$C_{\text{loc}}^{1,\frac{1}{p+1}}$
(SC) + $f(x) = f_0(x)u_+^{\mu}(x)$	Better estimates?

Hölder estimates are a consequence of Harnack inequality proved in the following references:[a]

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[a]

I. Birindelli & F. Demengel, *Eigenfunctions for singular fully nonlinear equations in unbounded domains.* **NoDEA Nonlinear Differential Equations Appl**. 17 (2010), n. 6, 697-714.

J.V. da Silva & H. Vivas, *The obstacle problem for a class of degenerate fully nonlinear operators*, To appear in **Revista Matemática Iberoamericana**, 2021. DOI 10.4171/rmi/1256.

J.V. da Silva, G.C. Rampasso, G.C. Ricarte & H. Vivas, *Free boundary regularity for a class of one-phase problems with non-homogeneous degeneracy*, **Submitted Article**.

Final Chapter: An improved regularity estimate

Let us consider the dead-core problem for fully nonlinear models with non-homogeneous degeneracy, whose source term presents an absorption term:

$$u \geq 0 \quad \text{and} \quad \mathcal{H}(x, Du).F(x, D^2u) = f_0(x) \cdot u^{\mu} \chi_{\{u>0\}} \quad \text{in} \quad \Omega, \tag{3.1}$$

where $0 < \mu < p+1$ is the order of reaction and f_0 is the Thiele modulus, which is bounded away from zero and infinity.

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where $0 < \mu < p+1$ is the order of reaction and f_0 is the Thiele modulus, which is bounded away from zero and infinity.

What are the expect regularity estimates for dead core models like (3.1)?

We shall establish an improved regularity estimate for non-negative solutions of (3.1) along their touching ground boundary $\partial\{u > 0\} \cap \Omega'$:

Final Chapter: An improved regularity estimate

Theorem (**Improved regularity along free boundary)**

Let u *be a nonnegative and bounded viscosity solution to* (3.1) *and consider* $z_0 \in \partial\{u > 0\} \cap \Omega'$ *a free boundary point with* $\Omega' \Subset \Omega$*. Then for* $r \ll \min\left\{1, \frac{\mathrm{dist}(\Omega', \partial\Omega)}{2}\right\}$ *there holds*

$$\sup_{B_r(z_0)} u(x) \leq \mathrm{C} \cdot r^{\frac{p+2}{p+1-\mu}},$$

where $\mathrm{C} > 0$ *depends only on* $n, \lambda, \Lambda, p, q, \mu, \|f_0\|_{L^\infty(\Omega)}$ *and* $\mathrm{dist}(\Omega', \partial\Omega)$.

Notice that $\frac{p+2}{p+1-\mu} > 1 + \frac{1}{p+1}$. Moreover, such regularity estimates are natural extension for ones in [a]

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[a]

J.V. da Silva, R.A. Leitão & G.C. Ricarte, *Geometric regularity estimates for fully nonlinear elliptic equations with free boundaries*. **Mathematische Nachrichten**, Vol. 294(1) 2021 p. 38-55.

E.V. Teixeira, *Regularity for the fully nonlinear dead-core problem*. **Math. Ann.** 364 (2016), no. 3-4, 1121-1134.

Final Chapter: An improved regularity estimate

Next result regards the first step of a sharp geometric decay, which is a powerful device in nonlinear (geometric) regularity theory and plays a pivotal role in our approach.

Lemma (**Flatness improvement regime**)

Suppose that the assumptions (SC) are in force. Given $0 < \eta < 1$, *there exists a* $\delta = \delta(n, \lambda, \Lambda, \eta) > 0$ *such that if* ϕ *satisfies* $0 \le \phi \le 1$, $\phi(0) = 0$ *and*

$$\mathcal{H}(x, D\phi).F(x, D^2\phi) = f_0(x) \cdot (\phi^+)^{\mu}, \qquad (3.2)$$

in the viscosity sense in $B_1(0)$, *with* $\|f_0\|_{L^\infty(B_1(0))} \le \delta$. *Then,*

$$\sup_{B_{1/2}(0)} \phi \le 1 - \eta. \qquad (3.3)$$

Final Chapter: An improved regularity estimate

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Proof.

Consequence of Harnack inequality: $\displaystyle\sup_{B_{1/2}(0)} \phi \leq \mathrm{C} \cdot \left(\inf_{B_{1/2}(0)} \phi + (q+1)^{\frac{1}{q+1}} \max\left\{ \|f_0\phi^{\mu}\|^{\frac{1}{p+1}}_{L^\infty(B_1)}, \|f_0\phi^{\mu}\|^{\frac{1}{q+1}}_{L^\infty(B_1)} \right\} \right).$ □

Finally, by applying Lemma 9 recursively in dyadic balls $B_{\frac{1}{2^k}}(0)$ with $\eta := 1 - \left(\frac{1}{2}\right)^{\frac{p+2}{p+1-\mu}}$, we are able to establish improved regularity estimates along touching ground points.

Final Chapter: An improved regularity estimate

We can to show that the maximum of a solution in a ball of radius $r \lll 1$ growths precisely as $r^{\frac{p+2}{p+1-\mu}}$.

Theorem (Non-degeneracy)

Let u be a nonnegative, bounded viscosity solution to (3.1) in $B_1(0)$ with $f(x) \geq \mathfrak{m} > 0$ and let $x_0 \in \overline{\{u > 0\}} \cap B_{\frac{1}{2}}(0)$ be a point in the closure of the non-coincidence set. Then for any $0 < r < \frac{1}{2}$, there holds

$$\sup_{\partial B_r(x_0)} u(x) \geq \mathrm{C} \cdot r^{\frac{p+2}{p+1-\mu}},$$

where $\mathrm{C} = \mathrm{C}(\mathfrak{m}, \|\mathfrak{a}\|_{L^\infty(\Omega)}, L_1, n, \lambda, \Lambda, p, q, \mu, \Omega) > 0.$

Final Chapter: An improved regularity estimate

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Proof.

The proof is a consequence of Birindelli-Demengel's Comparison Principle applied to following profiles:

$$\Xi(x) := \mathrm{C} \cdot |x|^{\frac{p+2}{p+1-\mu}} \quad \text{and} \quad u_r(x) := \frac{u(x_0 + rx)}{r^{\frac{p+2}{p+1-\mu}}} \quad \text{for} \quad x \in B_1(0)$$

□

Thank you very much for your attention $: -)$!

We are waiting your visit as soon as possible at UNICAMP's Math. Department!

Please, follow me on ResearchGate $; -)$

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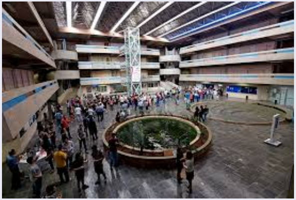

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