Fourth order *pq*-Laplacian

Jan Lang, The Ohio State University

4/26/2021, Nonstandard Seminar, (Warsaw University)

Fourth order *pq*-Laplacian

Jan Lang, The Ohio State University

4/26/2021, Nonstandard Seminar, (Warsaw University)

(To generalize something means to think. G.W.F.Hegel)

Abstract

Motivated by study of the higher order Sobolev embeddings on interval and their approximation we introduce and study a non-linear pq-biharmonic eigenvalue problem on the unit segment subject to Navier boundary condition. We will disuse existence of periodic symmetric solutions. In the case p, p' we show that all eigenvalues and eigenfunctions can be expressed in terms of generalized trigonometric functions.
(*Theses results were obtained with Lyonell Boulton.*)

Introduction

The Main Problem: pq-bi-Lapalcian (with Navier boundary condition)

$$\begin{aligned} ([u'']^{p-1})'' &= \lambda [u]^{q-1} \qquad 0 \le t \le t_0 \\ u(0) = u(t_0) = [u''(0)]^{p-1} &= [u''(t_0)]^{p-1} = 0, \end{aligned} \tag{1}$$

where $[u(t)]^{p-1} = |u(t)|^{p-1}\operatorname{sgn}(u(t))$, $1 < p, q < \infty$ and $\lambda \in \mathbb{R}$.

pq-Laplacian problem (with Dirichlet boundary condition)

$$\begin{aligned} ([u']^{p-1})' &= \lambda [u]^{q-1} \quad 0 \le t \le t_0 \\ u(0) &= u(t_0) = 0. \end{aligned} \tag{2}$$

Introduction

The Main Problem: pq-bi-Lapalcian (with Navier boundary condition)

$$
\begin{aligned}
([u'']^{p-1})'' &= \lambda [u]^{q-1} \qquad 0 \le t \le t_0 \\
u(0) = u(t_0) = [u''(0)]^{p-1} &= [u''(t_0)]^{p-1} = 0,
\end{aligned}
\tag{1}
$$

where $[u(t)]^{p-1} = |u(t)|^{p-1} \operatorname{sgn}(u(t))$, $1 < p, q < \infty$ and $\lambda \in \mathbb{R}$.

pq-Laplacian problem (with Dirichlet boundary condition)

$$
\begin{aligned}
([u']^{p-1})' &= \lambda [u]^{q-1} \qquad 0 \le t \le t_0 \\
u(0) = u(t_0) &= 0.
\end{aligned}
\tag{2}
$$

Consider Sobolev embeddings

$$E_1 : W_0^{1,p}(\mathcal{I}) \to L^q(\mathcal{I}).$$

Where $\mathcal{I} = [0, t_0]$, $1 < p, q < \infty$ and by $W_0^{1,p}(\mathcal{I})$ we denote the closure of $C_0^\infty(\mathrm{Int}\,\mathcal{I})$ in the Sobolev space $W^{1,p}(\mathcal{I})$ with respect to the usual norm $\|u\|_{W^{1,p}} := \|u\|_{p,\mathcal{I}} + \|u'\|_{p,\mathcal{I}}$. And $W_0^{1,p}$ is equipped with the norm $\|u\|_{W_0^{1,p}} := \|u'\|_{p,\mathcal{I}}$ due to the vanishing of its functions at 0 and t_0.

From compactness of E_1 and reflexivity of the underlying spaces, it follows that there exists an optimal element $u_0 \in W_0^{1,p}(\mathcal{I})$ such that

$$\sup_{u \in W_0^{1,p}(\mathcal{I})} \frac{\|u\|_{q,\mathcal{I}}}{\|u'\|_{p,\mathcal{I}}} = \frac{\|u_0\|_{q,\mathcal{I}}}{\|u_0'\|_{p,\mathcal{I}}}.$$

Consider Sobolev embeddings

$$E_1 : W_0^{1,p}(\mathcal{I}) \to L^q(\mathcal{I}).$$

Where $\mathcal{I} = [0, t_0]$, $1 < p, q < \infty$ and by $W_0^{1,p}(\mathcal{I})$ we denote the closure of $C_0^\infty(\operatorname{Int}\mathcal{I})$ in the Sobolev space $W^{1,p}(\mathcal{I})$ with respect to the usual norm $\|u\|_{W^{1,p}} := \|u\|_{p,\mathcal{I}} + \|u'\|_{p,\mathcal{I}}$. And $W_0^{1,p}$ is equipped with the norm $\|u\|_{W_0^{1,p}} := \|u'\|_{p,\mathcal{I}}$ due to the vanishing of its functions at 0 and t_0.

From compactness of E_1 and reflexivity of the underlying spaces, it follows that there exists an optimal element $u_0 \in W_0^{1,p}(\mathcal{I})$ such that

$$\sup_{u \in W_0^{1,p}(\mathcal{I})} \frac{\|u\|_{q,\mathcal{I}}}{\|u'\|_{p,\mathcal{I}}} = \frac{\|u_0\|_{q,\mathcal{I}}}{\|u_0'\|_{p,\mathcal{I}}}.$$

Introduction

Consider Sobolev embeddings

$$E_1 : W_0^{1,p}(\mathcal{I}) \to L^q(\mathcal{I}).$$

Where $\mathcal{I} = [0, t_0]$, $1 < p, q < \infty$ and by $W_0^{1,p}(\mathcal{I})$ we denote the closure of $C_0^\infty(\operatorname{Int}\mathcal{I})$ in the Sobolev space $W^{1,p}(\mathcal{I})$ with respect to the usual norm $\|u\|_{W^{1,p}} := \|u\|_{p,\mathcal{I}} + \|u'\|_{p,\mathcal{I}}$. And $W_0^{1,p}$ is equipped with the norm $\|u\|_{W_0^{1,p}} := \|u'\|_{p,\mathcal{I}}$ due to the vanishing of its functions at 0 and t_0.

From compactness of E_1 and reflexivity of the underlying spaces, it follows that there exists an optimal element $u_0 \in W_0^{1,p}(\mathcal{I})$ such that

$$\sup_{u \in W_0^{1,p}(\mathcal{I})} \frac{\|u\|_{q,\mathcal{I}}}{\|u'\|_{p,\mathcal{I}}} = \frac{\|u_0\|_{q,\mathcal{I}}}{\|u_0'\|_{p,\mathcal{I}}}.$$

In order to characterize $u_0(t)$, write the quotient of modular as

$$S(u) := \frac{\|u\|_{q,\mathcal{I}}^q}{\|u'\|_{p,\mathcal{I}}^p}, \text{ for } 0 \neq u \in W_0^{1,p}(\mathcal{I}).$$

Let $u \in L^p(\mathcal{I})$. Taking the Gâteux derivative of the L^p norm,

$$\operatorname{grad} \|u\|_{p,\mathcal{I}}(v) = \|u\|_{p,\mathcal{I}}^{1-p} \int_{\mathcal{I}} [u(t)]^{p-1} v(t)\, \mathrm{d}t,$$

yields

$$\operatorname{grad}(\|u\|_{p,\mathcal{I}}^p)(v) = p \int_{\mathcal{I}} [u(t)]^{p-1} v(t)\, \mathrm{d}t,$$

where $[u(t)]^{p-1} = |u(t)|^{p-1} \operatorname{sgn}(u(t))$. Then,

$$\operatorname{grad} S(u) = 0 \iff ([u']^{p-1})' = -\lambda [u]^{q-1} \tag{3}$$

for an appropriate multiplier/eigenvalue $\lambda > 0$.

In other words, the eigenfunctions of a suitable pq-Laplacian eigenvalue problem, are exactly the extremal functions of the Sobolev embedding E_1.

Introduction

In order to characterize $u_0(t)$, write the quotient of modular as

$$S(u) := \frac{\|u\|_{q,\mathcal{I}}^q}{\|u'\|_{p,\mathcal{I}}^p}, \text{ for } 0 \neq u \in W_0^{1,p}(\mathcal{I}).$$

Let $u \in L^p(\mathcal{I})$. Taking the Gâteux derivative of the L^p norm,

$$\operatorname{grad} \|u\|_{p,\mathcal{I}}(v) = \|u\|_{p,\mathcal{I}}^{1-p} \int_{\mathcal{I}} [u(t)]^{p-1} v(t)\, \mathrm{d}t,$$

yields

$$\operatorname{grad}(\|u\|_{p,\mathcal{I}}^p)(v) = p \int_{\mathcal{I}} [u(t)]^{p-1} v(t)\, \mathrm{d}t,$$

where $[u(t)]^{p-1} = |u(t)|^{p-1} \operatorname{sgn}(u(t))$. Then,

$$\operatorname{grad} S(u) = 0 \iff ([u']^{p-1})' = -\lambda [u]^{q-1} \tag{3}$$

for an appropriate multiplier/eigenvalue $\lambda > 0$.

In other words, the eigenfunctions of a suitable pq-Laplacian eigenvalue problem, are exactly the extremal functions of the Sobolev embedding E_1.

Introduction

In order to characterize $u_0(t)$, write the quotient of modular as

$$S(u) := \frac{\|u\|_{q,\mathcal{I}}^q}{\|u'\|_{p,\mathcal{I}}^p}, \text{ for } 0 \neq u \in W_0^{1,p}(\mathcal{I}).$$

Let $u \in L^p(\mathcal{I})$. Taking the Gâteux derivative of the L^p norm,

$$\mathrm{grad}\, \|u\|_{p,\mathcal{I}}(v) = \|u\|_{p,\mathcal{I}}^{1-p} \int_{\mathcal{I}} [u(t)]^{p-1} v(t)\, \mathrm{d}t,$$

yields

$$\mathrm{grad}(\|u\|_{p,\mathcal{I}}^p)(v) = p \int_{\mathcal{I}} [u(t)]^{p-1} v(t)\, \mathrm{d}t,$$

where $[u(t)]^{p-1} = |u(t)|^{p-1} \operatorname{sgn}(u(t))$. Then,

$$\mathrm{grad}\, S(u) = 0 \iff ([u']^{p-1})' = -\lambda [u]^{q-1} \tag{3}$$

for an appropriate multiplier/eigenvalue $\lambda > 0$.

In other words, the eigenfunctions of a suitable pq-Laplacian eigenvalue problem, are exactly the extremal functions of the Sobolev embedding E_1.

Small detour: $\sin_{p,q}$, $\cos_{p,q}$ (specjalne wspaniałe funkcje)

Let $1 < p, q < \infty$ and define a (differentiable) function $F_{p,q} : [0,1] \to \mathbf{R}$ by

$$F_{p,q}(x) = \int_0^x \frac{1}{\sqrt[p]{1-t^q}} dt, \qquad 0 \le x \le 1. \tag{4}$$

Since $F_{p,q}$ is strictly increasing it is a one-to-one function on $[0,1]$ with range $[0, \pi_{p,q}/2]$, where

$$\pi_{p,q} = 2 \int_0^1 \frac{1}{\sqrt[p]{1-t^q}} dt, \qquad 0 \le x \le 1. \tag{5}$$

The inverse of $F_{p,q}$ on $[0, \pi_{p,q}/2]$ we denote by $\sin_{p,q}$ and extend as in the case of sin (p=q=2) to $[0, \pi_{p,q}]$ by defining

$$\sin_{p,q}(x) = \sin_{p,q}(\pi_{p,q} - x) \qquad \text{for } x \in [\pi_{p,q}/2, \pi_{p,q}];$$

further extension is achieved by oddness and $2\pi_{p,q}$-periodicity on the whole of $\mathbf{R}$. By this means we obtain a differentiable function on $\mathbf{R}$ which coincides with sin when $p = q = 2$.

Review: Some special functions: $\sin_{p,q}$, $\cos_{p,q}$

Corresponding to this we define a function $\cos_{p,q}$ by the prescription

$$\cos_{p,q}(x) = \frac{d}{dx}\sin_{p,q}(x), \ x \in \mathbf{R}. \tag{6}$$

Clearly $\cos_{p,q}$ is even, $2\pi_{p,q}$-periodic and odd about $\pi_{p,q}/2$; and $\cos_{2,2} = \cos$. If $x \in [0, \pi_{p,q}/2]$, then from the definition it follows that

$$\cos_{p,q}(x) = (1 - (\sin_{p,q}(x))^q)^{1/p}. \tag{7}$$

Moreover, the asymmetry and periodicity show that

$$|\sin_{p,q}(x)|^q + |\cos_{p,q}(x)|^p = 1, \quad x \in \mathbf{R}. \tag{8}$$

We will use:

$\pi_p := \pi_{p,p}$, $\sin_p := \sin_{p,p}$ and $\cos_p := \cos_{p,p}$, and then we have

$$|\sin_p(x)|^p + |\cos_p(x)|^p = 1, \quad x \in \mathbf{R}. \tag{9}$$

Review: Some special functions: $\sin_{p,q}$, $\cos_{p,q}$

Corresponding to this we define a function $\cos_{p,q}$ by the prescription

$$\cos_{p,q}(x) = \frac{d}{dx}\sin_{p,q}(x), \ \ x \in \mathbf{R}. \tag{6}$$

Clearly $\cos_{p,q}$ is even, $2\pi_{p,q}$-periodic and odd about $\pi_{p,q}/2$; and $\cos_{2,2} = \cos$. If $x \in [0, \pi_{p,q}/2]$, then from the definition it follows that

$$\cos_{p,q}(x) = (1 - (\sin_{p,q}(x))^q)^{1/p}. \tag{7}$$

Moreover, the asymmetry and periodicity show that

$$|\sin_{p,q}(x)|^q + |\cos_{p,q}(x)|^p = 1, \quad x \in \mathbf{R}. \tag{8}$$

We will use:
$\pi_p := \pi_{p,p}$, $\sin_p := \sin_{p,p}$ and $\cos_p := \cos_{p,p}$, and then we have

$$|\sin_p(x)|^p + |\cos_p(x)|^p = 1, \quad x \in \mathbf{R}. \tag{9}$$

Review: Some special functions: $\sin_{p,q}$, $\cos_{p,q}$

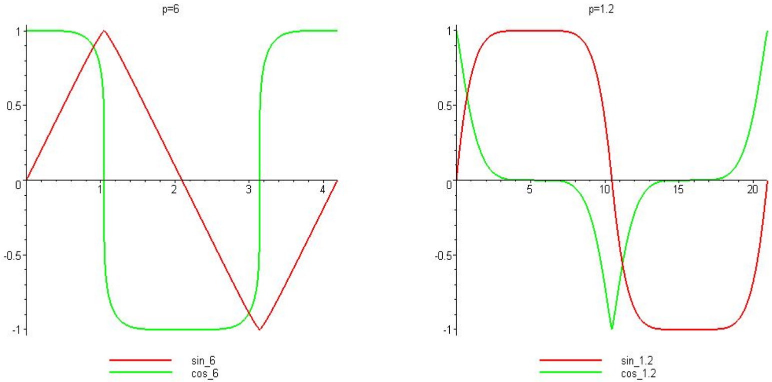


Figure: $\sin_6, \cos_6$ and $\sin_{1.2}, \cos_{1.2}$

History: Erik Lundberg (1879),S. Günther (1881), V.I.Levin (1938), E.Schmidt (1940), R. Grammel (1948), D. Shelupsky (1959), Tichomirov, Makovoz, Buslaev & etc. (1960-90), J.Peetre (1972), A.Elbert (1979), M.Ôtani (1984), P.Lindqvist (1995), P.Drábek (1999)

Review: Some special functions: $\sin_{p,q}$, $\cos_{p,q}$

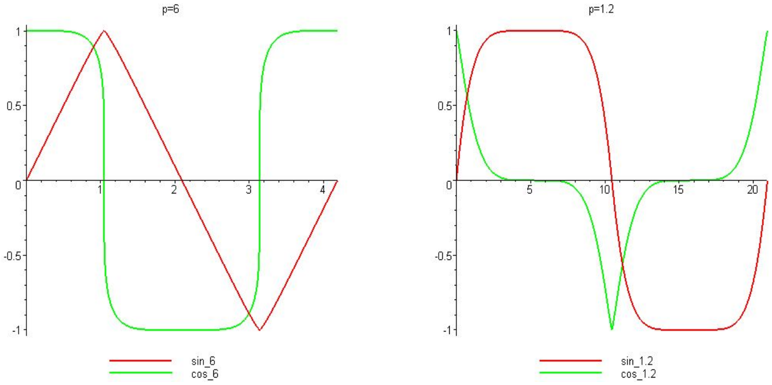


Figure: $\sin_6, \cos_6$ and $\sin_{1.2}, \cos_{1.2}$

History: Erik Lundberg (1879),S. Günther (1881), V.I.Levin (1938), E.Schmidt (1940), R. Grammel (1948), D. Shelupsky (1959), Tichomirov, Makovoz, Buslaev & etc. (1960-90), J.Peetre (1972), A.Elbert (1979), M.Ôtani (1984), P.Lindqvist (1995), P.Drábek (1999)

Theorem (Edmunds, Gurka, L.)

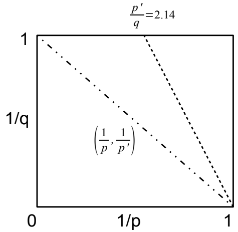
Let $p, q \in (1, \infty)$ *and let*

$$\frac{p'}{q} < \frac{4}{\pi^2 - 8} \approx 2.14. \tag{10}$$

Then the sequence $\left(\sin_{p,q}(n\,\pi_{p,q}\,t)\right)_{n \in \mathbf{N}}$ *is a Schauder basis in* $L^r(0,1)$ *for any* $r \in (1, \infty)$.

The functions $f_{n,p}(x) := \sin_p(n\pi_p x)$ form a basis in $L_q(0,1)$ for every $q \in (1, \infty)$ if $p_0 < p < \infty$, where p_0 is defined by the equation

$$\pi_{p_0} = \frac{2\pi^2}{\pi^2 - 8} \qquad \text{(i.e. } p_0 \approx 1.05) \tag{11}$$



Review: Some special functions: $\sin_{p,q}$, $\cos_{p,q}$

Theorem (Edmunds, Gurka, L.)

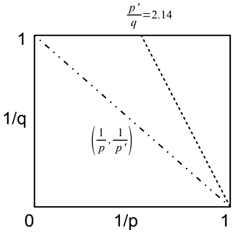
Let $p, q \in (1, \infty)$ *and let*

$$\frac{p'}{q} < \frac{4}{\pi^2 - 8} \approx 2.14. \qquad (10)$$

Then the sequence $\left(\sin_{p,q}(n\,\pi_{p,q}\,t)\right)_{n\in\mathbf{N}}$ *is a Schauder basis in* $L^r(0,1)$ *for any* $r \in (1, \infty)$.

The functions $f_{n,p}(x) := \sin_p(n\pi_p x)$ form a basis in $L_q(0,1)$ for every $q \in (1, \infty)$ if $p_0 < p < \infty$, where p_0 is defined by the equation

$$\pi_{p_0} = \frac{2\pi^2}{\pi^2 - 8} \qquad \text{(i.e. } p_0 \approx 1.05) \qquad (11)$$



We recall the definition of the p-Laplacian which is a natural extension of the Laplacian:

$$\Delta_p u = (|u'|^{p-2}u')' \qquad \text{(evidently } \Delta_2 u = \Delta u\text{).}$$

Then the analogue of (3) is the eigenvalue problem

$$\left.\begin{array}{c} \Delta_p u + \lambda |u|^{p-2}u = 0 \quad \text{on } (0, T), \\ u(0) = 0, u(T) = 0. \end{array}\right\} \tag{12}$$

In [Elbert, Lindqvist, Drabek] it is shown that all eigenvalues of this problem are of the form

$$\lambda_n = \left(\frac{n\pi_p}{T}\right)^p \frac{p}{p'}$$

with corresponding eigenfunctions

$$u_n(t) = \sin_p\left(\frac{n\pi_p}{T}t\right).$$

We recall the definition of the p-Laplacian which is a natural extension of the Laplacian:

$$\Delta_p u = (|u'|^{p-2}u')' \qquad \text{(evidently } \Delta_2 u = \Delta u\text{).}$$

Then the analogue of (3) is the eigenvalue problem

$$\left.\begin{array}{c} \Delta_p u + \lambda|u|^{p-2}u = 0 \quad \text{on } (0,T), \\ u(0) = 0, u(T) = 0. \end{array}\right\} \tag{12}$$

In [Elbert, Lindqvist, Drabek] it is shown that all eigenvalues of this problem are of the form

$$\lambda_n = \left(\frac{n\pi_p}{T}\right)^p \frac{p}{p'}$$

with corresponding eigenfunctions

$$u_n(t) = \sin_p\left(\frac{n\pi_p}{T}t\right).$$

We recall the definition of the p-Laplacian which is a natural extension of the Laplacian:

$$\Delta_p u = (|u'|^{p-2}u')' \qquad \text{(evidently } \Delta_2 u = \Delta u\text{)}.$$

Then the analogue of (3) is the eigenvalue problem

$$\left.\begin{array}{c} \Delta_p u + \lambda |u|^{p-2} u = 0 \quad \text{on } (0, T), \\ u(0) = 0, u(T) = 0. \end{array}\right\} \tag{12}$$

In [Elbert, Lindqvist, Drabek] it is shown that all eigenvalues of this problem are of the form

$$\lambda_n = \left(\frac{n\pi_p}{T}\right)^p \frac{p}{p'}$$

with corresponding eigenfunctions

$$u_n(t) = \sin_p\left(\frac{n\pi_p}{T} t\right).$$

We recall the definition of the p-Laplacian which is a natural extension of the Laplacian:

$$\Delta_p u = (|u'|^{p-2}u')' \qquad \text{(evidently } \Delta_2 u = \Delta u\text{)}.$$

Then the analogue of (3) is the eigenvalue problem

$$\left.\begin{array}{l} \Delta_p u + \lambda |u|^{p-2} u = 0 \quad \text{on } (0,T), \\ u(0) = 0, u(T) = 0. \end{array}\right\} \tag{12}$$

In [Elbert, Lindqvist, Drabek] it is shown that all eigenvalues of this problem are of the form

$$\lambda_n = \left(\frac{n\pi_p}{T}\right)^p \frac{p}{p'}$$

with corresponding eigenfunctions

$$u_n(t) = \sin_p\left(\frac{n\pi_p}{T}t\right).$$

Theorem (Drabek, Manásevic (1999))

Consider the following Dirichlet problem

$$\left.\begin{array}{c} \Delta_p u + \lambda |u|^{q-2} u = 0 \quad \text{on } (0,T), \\ u(0) = 0, u(T) = 0. \end{array}\right\} \tag{13}$$

All eigenvalues are of the form:

$$\lambda_{n,\alpha} = \left(\frac{n\pi_{p,q}}{T}\right)^q \frac{|\alpha|^{p-q} q}{p'}, \ \alpha \in \mathbf{R} \setminus \{0\}, n \in \mathbf{N}$$

with corresponding eigenfunctions

$$u_{n,\alpha}(t) = \frac{\alpha T}{n\pi_{p,q}} \sin_{p,q}\left(\frac{n\pi_{p,q}}{T} t\right)$$

Let $1 < p < \infty$ and $-\infty < a < b < \infty$. Consider the Sobolev embedding on $I = [a, b]$,

$$E_1 : W_0^{1,p}(I) \to L^p(I) \tag{14}$$

Then $u(x) = \sin_p\left(\frac{x-a}{b-a}\pi_p\right)$ is the extremal functions.

Theorem

Let $\widetilde{s_n}$ stands for any strict s-number (i.e. $a_n, d_n, d^n, m_n, b_n, i_n$). Let n be an integer, then

$$s_n(E_1) = \frac{|I|}{n\pi_p} \cdot \left(\frac{p'}{p}\right)^{1/p}$$

and

$$s_n(E_1) = \|(E_1 - R_n)g|L^p(I)\|, \quad \text{where } g(x) = \sin_p\left(\frac{x-a}{b-a}\pi_p n\right).$$

Here

$$R_n f = \sum_{i=1}^{n-1} P_i f, \qquad \text{where } P_i f(x) = \chi_{I_i}(x) f\left(a + i\frac{|I|}{n}\right).$$

Let $1 < p < \infty$ and $-\infty < a < b < \infty$. Consider the Sobolev embedding on $I = [a, b]$,

$$E_1 : W_0^{1,p}(I) \to L^p(I) \tag{14}$$

Then $u(x) = \sin_p \left(\frac{x-a}{b-a}\pi_p\right)$ is the extremal functions.

Theorem

Let $\widetilde{s}_n$ stands for any strict s-number (i.e. $a_n, d_n, d^n, m_n, b_n, i_n$). Let n be an integer, then

$$s_n(E_1) = \frac{|I|}{n\pi_p} \cdot \left(\frac{p'}{p}\right)^{1/p}$$

and

$$s_n(E_1) = \|(E_1 - R_n)g|L^p(I)\|, \quad \text{where } g(x) = \sin_p\left(\frac{x-a}{b-a}\pi_p n\right).$$

Here

$$R_n f = \sum_{i=1}^{n-1} P_i f, \qquad \text{where } P_i f(x) = \chi_{I_i}(x) f\left(a + i\frac{|I|}{n}\right).$$

(n-dim)

Let $1 < p < \infty$ and $-\infty < a_i < b_i < \infty$ and set $R := \prod_{i=1}(a_i, b_i)$ or on $D := \mathbf{R}^k \times \prod_{i=1}^{n-k}(a_i, b_i)$. Consider the Sobolev embedding:

$$E_1 : W_0^{1,p}(\Omega) \to L^p(\Omega), \qquad \text{with } \Omega = R \text{ or } D. \tag{15}$$

and where:

$$\|u\|_{1,p} = \left(\|u\|^p_{L^p(\Omega)} + \||\nabla u|_{l^p}\|^p_{L^p(\Omega)} \right)^{1/p}$$

.

Study of this embedding is related to study of this pseudo-p-Laplacian problem:

$$\tilde{\Delta}_p u = \lambda |u|^{p-2} u, \qquad \text{with } u = 0 \text{ on } \partial\Omega$$

where

$$\tilde{\Delta}_p u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right).$$

Remark

If $\Omega = R$ then these functions are eigenfunctions:

$$\prod_{i=1}^{n} \sin_p \left(\frac{\pi_p k_i (x_i - a_i)}{b_i - a_i} \right), \quad (x_1, x_2, ..., x_n) \in R, \ \textit{for some } k_i \in \mathbf{N}.$$

Question: Are all eigenfunctions of the above problem of the above form?

(n-dim)

Theorem (Edmunds, Mihula, L.)

(i) The case D: There is not an extremal function and

$$\|W_0^{1,p}(D) \to L^p(D)\| = \left(1 + \pi_p^p(p-1)\sum_{i=1}^{n-k}\frac{1}{(a_i - b_i)^p}\right)^{-1/p}.$$

(ii) The case R: The norm of embedding $W_0^{1,p}(R) \to L^p(R)$ *is reached by function*

$$u(x) = \prod_{i=1}^{n}\sin_p\left(\frac{\pi_p(x_i - a_i)}{b_i - a_i}\right)$$

Lemma (Edmunds, Mihula, L.)

The first egenfunction for

$$\tilde{\Delta}_p u = \lambda |u|^{p-2}u, \qquad \textit{with } u = 0 \textit{ on } \partial R$$

is

$$u(x) = \prod_{i=1}^{n}\sin_p\left(\frac{\pi_p(x_i - a_i)}{b_i - a_i}\right)$$

(n-dim)

Theorem (Edmunds, Mihula, L.)

Let $1 < p < \infty$, $k \in \mathbf{N}$, $k \leq n-1$, $D := \mathbf{R}^k \times \prod_{i=1}^{n-k}(a_i, b_i)$. *Then we have*

$$s_n(W_0^{1,p}(D) \to L^p(D)) = \left(1 + \pi_p^p(p-1)\sum_{i=1}^{n-k}\frac{1}{(a_i - b_i)^p}\right)^{-1/p},$$

where s_n *stands for any strict s-number.*

Consider Sobolev embeddings

$$E_2 : W_D^{2,p}(\mathcal{I}) \to L^q(\mathcal{I}),$$

where $\mathcal{I} = [0, t_0]$, $1 < p, q < \infty$ and $W_D^{2,p}(\mathcal{I}) = C_0^1(\operatorname{Int}\mathcal{I}) \cap W^{2,p}(\mathcal{I})$ equipped with the norm $\|u\|_{W_D^{2,p}} := \|u''\|_{p,\mathcal{I}}$.

Due compactness of E_2 and reflexivity of the underlying spaces, it follows that there exists an optimal element $u_0 \in W_D^{2,p}(\mathcal{I})$ such that

$$\sup_{u \in W_D^{1,p}(\mathcal{I})} \frac{\|u\|_{q,\mathcal{I}}}{\|u''\|_{p,\mathcal{I}}} = \frac{\|u_0\|_{q,\mathcal{I}}}{\|u_0''\|_{p,\mathcal{I}}}.$$

Consider Sobolev embeddings

$$E_2 : W_D^{2,p}(\mathcal{I}) \to L^q(\mathcal{I}),$$

where $\mathcal{I} = [0, t_0]$, $1 < p, q < \infty$ and $W_D^{2,p}(\mathcal{I}) = C_0^1(\operatorname{Int}\mathcal{I}) \cap W^{2,p}(\mathcal{I})$ equipped with the norm $\|u\|_{W_D^{2,p}} := \|u''\|_{p,\mathcal{I}}$.

Due compactness of E_2 and reflexivity of the underlying spaces, it follows that there exists an optimal element $u_0 \in W_D^{2,p}(\mathcal{I})$ such that

$$\sup_{u \in W_D^{1,p}(\mathcal{I})} \frac{\|u\|_{q,\mathcal{I}}}{\|u''\|_{p,\mathcal{I}}} = \frac{\|u_0\|_{q,\mathcal{I}}}{\|u_0''\|_{p,\mathcal{I}}}.$$

As before, in order to characterize $u_0(t)$, write the quotient of modular as

$$D(u) := \frac{\|u\|_{q,\mathcal{I}}^q}{\|u''\|_{p,\mathcal{I}}^p}, \text{ for } 0 \neq u \in W_D^{2,p}(\mathcal{I}).$$

Take the Gâteux derivative of $D(u)$. Then

$$\operatorname{grad} D(u) = 0 \iff \|u''\|_p^p \operatorname{grad}(\|u\|_q^q) = \|u\|_q^q \operatorname{grad}(\|u''\|_p^p), \tag{16}$$

which can be re-written as as a differential equation which is reminiscent of the equation form the introduction:

$$\begin{aligned} ([u'']^{p-1})'' &= \lambda [u]^{q-1} \qquad 0 \leq t \leq t_0 \\ u(0) = u(t_0) &= [u''(0)]^{p-1} = [u''(t_0)]^{p-1} = 0 \end{aligned} \tag{17}$$

for $u \neq 0$ and $\lambda \in \mathbb{R}$.

And we can see that the eigenfunctions of a suitable fourth order p, q-Laplacian eigenvalue problem, are exactly the extremal functions of the Sobolev embedding E_2.

As before, in order to characterize $u_0(t)$, write the quotient of modular as

$$D(u) := \frac{\|u\|_{q,\mathcal{I}}^q}{\|u''\|_{p,\mathcal{I}}^p}, \text{ for } 0 \neq u \in W_D^{2,p}(\mathcal{I}).$$

Take the Gâteux derivative of $D(u)$. Then

$$\operatorname{grad} D(u) = 0 \iff \|u''\|_p^p \ \operatorname{grad}(\|u\|_q^q) = \|u\|_q^q \ \operatorname{grad}(\|u''\|_p^p), \tag{16}$$

which can be re-written as as a differential equation which is reminiscent of the equation form the introduction:

$$\begin{aligned} ([u'']^{p-1})'' &= \lambda[u]^{q-1} && 0 \leq t \leq t_0 \\ u(0) = u(t_0) &= [u''(0)]^{p-1} = [u''(t_0)]^{p-1} = 0 \end{aligned} \tag{17}$$

for $u \neq 0$ and $\lambda \in \mathbb{R}$.

And we can see that the eigenfunctions of a suitable fourth order p, q-Laplacian eigenvalue problem, are exactly the extremal functions of the Sobolev embedding E_2.

Idea: Assume that $\lambda > 0$. Writing

$$u_1(t) = u(t), \quad u_2(t) = u'(t),$$
$$w_1(t) = -[u''(t)]^{p-1} \quad \text{and} \quad w_2(t) = -([u''(t)]^{p-1})',$$

we get the system of differential equations

$$\begin{aligned} u_1'(t) &= u_2(t) & u_2'(t) &= -[w_1(t)]^{p'-1} \\ w_1'(t) &= w_2(t) & w_2'(t) &= -\lambda[u_1(t)]^{q-1} \end{aligned}$$

or the system of integral equations

$$\begin{aligned} u_1(t) &= \int_0^t u_2(s)\mathrm{d}s & u_2(t) &= \alpha - \int_0^t [w_1(s)]^{p'-1}\mathrm{d}s \\ w_1(t) &= \int_0^t w_2(s)\mathrm{d}s & w_2(t) &= \beta - \lambda \int_0^t [u_1(s)]^{q-1}\mathrm{d}s, \end{aligned} \tag{18}$$

both subject to the initial and final conditions

$$u_1(0) = u_1(t_0) = w_1(0) = w_1(t_0) = 0. \tag{19}$$

It is useful to fix λ and understand (17) in the context of (18) as a dynamical system seeking for the trajectory

$$\underline{\varphi}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ w_1(t) \\ w_2(t) \end{bmatrix}$$

starting from an initial state at $t = 0$

$$\underline{\varphi}(0) = \underline{\zeta} = \begin{bmatrix} u_1(0) \\ u_2(0) \\ w_1(0) \\ w_2(0) \end{bmatrix}.$$

Example

Let $1 < p < q < \infty$, $\lambda > 0$ and set $r = 2p/(p-q)$. Let $H, K > 0$ be arbitrary (fixed). Then $u(t) = K(H-t)^r$ is a solution of our *pq*-bi-Laplacian with a finite time blow-up at $t_\infty = H > 0$. (Here t_∞ denotes the blow-up time)

It is useful to fix λ and understand (17) in the context of (18) as a dynamical system seeking for the trajectory

$$\underline{\varphi}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ w_1(t) \\ w_2(t) \end{bmatrix}$$

starting from an initial state at $t = 0$

$$\underline{\varphi}(0) = \underline{\zeta} = \begin{bmatrix} u_1(0) \\ u_2(0) \\ w_1(0) \\ w_2(0) \end{bmatrix}.$$

Example

Let $1 < p < q < \infty$, $\lambda > 0$ and set $r = 2p/(p-q)$. Let $H, K > 0$ be arbitrary (fixed). Then $u(t) = K(H-t)^r$ is a solution of our *pq*-bi-Laplacian with a finite time blow-up at $t_\infty = H > 0$.

(Here t_∞ denotes the blow-up time)

It is useful to fix λ and understand (17) in the context of (18) as a dynamical system seeking for the trajectory

$$\underline{\varphi}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ w_1(t) \\ w_2(t) \end{bmatrix}$$

starting from an initial state at $t = 0$

$$\underline{\varphi}(0) = \underline{\zeta} = \begin{bmatrix} u_1(0) \\ u_2(0) \\ w_1(0) \\ w_2(0) \end{bmatrix}.$$

Example

Let $1 < p < q < \infty$, $\lambda > 0$ and set $r = 2p/(p-q)$. Let $H, K > 0$ be arbitrary (fixed). Then $u(t) = K(H-t)^r$ is a solution of our *pq*-bi-Laplacian with a finite time blow-up at $t_\infty = H > 0$. (Here t_∞ denotes the blow-up time)

Using By Picard-Lindelöf or Cauchy-Peano techniques we get:

Lemma

Given λ, p, q, α and β fixed, there exists a unique solution to (18) for all $t \in (0, t_\infty)$.

Then by careful analysis of our system and using observation like:

Lemma

Consider the system of integral equations (18) with $\lambda > 0$ and initial conditions $u_1(0) = w_1(0) = 0$.
(i) If $\alpha > 0$, $\beta < 0$ then u_1 is strictly increasing and w_1 is strictly decreasing for $t > 0$.
(ii) If $\alpha < 0$, $\beta > 0$ then u_1 is strictly decreasing and w_1 is strictly increasing for $t > 0$.
(iii) ...

we obtain that there must exists values $\alpha, \beta > 0$ for which we have a unique positive solution on $(0, t_0)$ with given boundary conditions.
Note about initial value problem:

Using By Picard-Lindelöf or Cauchy-Peano techniques we get:

Lemma

Given λ, p, q, α and β fixed, there exists a unique solution to (18) for all $t \in (0, t_\infty)$.

Then by careful analysis of our system and using observation like:

Lemma

Consider the system of integral equations (18) with $\lambda > 0$ and initial conditions $u_1(0) = w_1(0) = 0$.
(i) If $\alpha > 0$, $\beta < 0$ then u_1 is strictly increasing and w_1 is strictly decreasing for $t > 0$.
(ii) If $\alpha < 0$, $\beta > 0$ then u_1 is strictly decreasing and w_1 is strictly increasing for $t > 0$.
(iii) ...

we obtain that there must exists values $\alpha, \beta > 0$ for which we have a unique positive solution on $(0, t_0)$ with given boundary conditions.
Note about initial value problem:

Lemma

Let λ, p, q be fixed. Consider the evolution systems $(*)$. Let $\alpha_2 \le \alpha_1$ and $\beta_1 \le \beta_2$. Let $t_1 > 0$ be any point such that all the quantities[a] $|u_k^j(t)|$ and $|w_k^j(t)|$ are finite for $0 \le t < t_1$. Then,

$$u_k^2(t) \le u_k^1(t) \quad \text{and} \quad w_k^1(t) \le w_k^2(t) \qquad \forall k = 1,2 \quad t \in (0, t_1). \tag{20}$$

Moreover,

$$\begin{aligned} u_1^1(t) - u_1^2(t) &\ge (\alpha_1 - \alpha_2)t \\ w_1^2(t) - w_1^1(t) &\ge (\beta_2 - \beta_1)t \end{aligned} \qquad \forall t \in (0, t_1). \tag{21}$$

In fact, if one of the inequalities involving α_j or β_j is strict, then $u_1^2(t) < u_1^1(t)$ and $w_1^1(t) < w_1^2(t)$ for $0 < t \le t_1$.

[a] Here and everywhere below, the indices j (on top) refer to corresponding sub-indices of α or β, in context.

Theorem (Existence and Uniqueness)

Let $t_0 > 0$ be fixed. If $p \ne q$, then for all $\lambda > 0$ there exists a unique solution $u(t)$ positive on $(0, t_0)$ satisfying (17). If $p = q$, then there exists a unique $\lambda \equiv \lambda(t_0) > 0$ such that a solution $u(t)$ positive on $(0, t_0)$ satisfying (17) exists. Moreover this solution is unique up to multiplication by a constant.

Lemma

Let λ, p, q be fixed. Consider the evolution systems $(*)$. Let $\alpha_2 \leq \alpha_1$ and $\beta_1 \leq \beta_2$. Let $t_1 > 0$ be any point such that all the quantities[a] $|u_k^j(t)|$ *and* $|w_k^j(t)|$ *are finite for* $0 \leq t < t_1$. *Then,*

$$u_k^2(t) \leq u_k^1(t) \quad \text{and} \quad w_k^1(t) \leq w_k^2(t) \qquad \forall k = 1,2 \quad t \in (0, t_1). \tag{20}$$

Moreover,

$$\begin{aligned} u_1^1(t) - u_1^2(t) &\geq (\alpha_1 - \alpha_2)t \\ w_1^2(t) - w_1^1(t) &\geq (\beta_2 - \beta_1)t \end{aligned} \qquad \forall t \in (0, t_1). \tag{21}$$

In fact, if one of the inequalities involving α_j or β_j is strict, then $u_1^2(t) < u_1^1(t)$ and $w_1^1(t) < w_1^2(t)$ for $0 < t \leq t_1$.

[a] Here and everywhere below, the indices j (on top) refer to corresponding sub-indices of α or β, in context.

Theorem (Existence and Uniqueness)

Let $t_0 > 0$ be fixed. If $p \neq q$, then for all $\lambda > 0$ there exists a unique solution $u(t)$ positive on $(0, t_0)$ satisfying (17). If $p = q$, then there exists a unique $\lambda \equiv \lambda(t_0) > 0$ such that a solution $u(t)$ positive on $(0, t_0)$ satisfying (17) exists. Moreover this solution is unique up to multiplication by a constant.

Theorem (Existence and Uniqueness)

Let $t_0 > 0$ be fixed. If $p \neq q$, then for all $\lambda > 0$ there exists a unique solution $u(t)$ positive on $(0, t_0)$ satisfying (17). If $p = q$, then there exists a unique $\lambda \equiv \lambda(t_0) > 0$ such that a solution $u(t)$ positive on $(0, t_0)$ satisfying (17) exists. Moreover this solution is unique up to multiplication by a constant.

Theorem (Symmetricity)

Let $u(t)$ be a positive solution of (17) on $(0, t_0)$. Then, $u(t) = u(t_0 - t)$ for all $0 < t < \frac{t_0}{2}$. Moreover, $u(t)$ can be extended to a $2t_0$-periodic function $u_* \in C^1(\mathbb{R})$ satisfying

$$([u_*''(t)]^{p-1})'' = \lambda[u_*(t)]^{q-1} \qquad \forall t \in \mathbb{R}.$$

Corollary

Let $u(t)$ be a solution of (17) with exactly n zeros in $(0, t_0)$. Then, these zeros are all simple and located at

$$t_j = \frac{jt_0}{n+1} \qquad j = 1, \ldots, n.$$

Moreover, $u''(t)$ also vanish exactly at the points t_j.

pq-bi-Laplacian

Theorem (Existence and Uniqueness)

Let $t_0 > 0$ be fixed. If $p \neq q$, then for all $\lambda > 0$ there exists a unique solution $u(t)$ positive on $(0, t_0)$ satisfying (17). If $p = q$, then there exists a unique $\lambda \equiv \lambda(t_0) > 0$ such that a solution $u(t)$ positive on $(0, t_0)$ satisfying (17) exists. Moreover this solution is unique up to multiplication by a constant.

Theorem (Symmetricity)

Let $u(t)$ be a positive solution of (17) on $(0, t_0)$. Then, $u(t) = u(t_0 - t)$ for all $0 < t < \frac{t_0}{2}$. Moreover, $u(t)$ can be extended to a $2t_0$-periodic function $u_* \in C^1(\mathbb{R})$ satisfying

$$([u_*''(t)]^{p-1})'' = \lambda [u_*(t)]^{q-1} \qquad \forall t \in \mathbb{R}.$$

Corollary

Let $u(t)$ be a solution of (17) with exactly n zeros in $(0, t_0)$. Then, these zeros are all simple and located at

$$t_j = \frac{j t_0}{n+1} \qquad j = 1, \ldots, n.$$

Moreover, $u''(t)$ also vanish exactly at the points t_j.

The case $q = p'$

Theorem

Let $1 < p < \infty$ and $T > 0$. Then for any given $c \neq 0$ and $n \in \mathbf{N}$, the n-eigenvalue of problem (17) on $[0, T]$ is

$$\lambda_n(c) = \frac{\left(\pi_{2,p'}\pi_{2,p}n^2\right)^p}{T^{2p}}|c|^{p-p'},$$

and the corresponding n-eigenfunction is (i.e. eigenfunction which change $n-1$ times sign on $(0, T)$) is

$$f_{n,c}(x) = c\sin_{2,p'}(\pi_{2,p'}nx/T), \tag{22}$$

and this eigenfunction is the unique n-eigenfunction to the given eigenvalue $\lambda_n(c)$.

Note

$\left\{\sin_{2,p'}(\pi_{2,p'}nx/T)\right\}_n$ is a basis in $L^r(0, T)$ for any $1 < r < \infty$.

The case $q = p'$

Theorem

Let $1 < p < \infty$ and $T > 0$. Then for any given $c \neq 0$ and $n \in \mathbf{N}$, the n-eigenvalue of problem (17) on $[0, T]$ is

$$\lambda_n(c) = \frac{\left(\pi_{2,p'}\pi_{2,p}n^2\right)^p}{T^{2p}}|c|^{p-p'},$$

and the corresponding n-eigenfunction is (i.e. eigenfunction which change $n-1$ times sign on $(0, T)$) is

$$f_{n,c}(x) = c\sin_{2,p'}(\pi_{2,p'}nx/T), \tag{22}$$

and this eigenfunction is the unique n-eigenfunction to the given eigenvalue $\lambda_n(c)$.

Note

$\left\{\sin_{2,p'}(\pi_{2,p'}nx/T)\right\}_n$ is a basis in $L^r(0, T)$ for any $1 < r < \infty$.

Approximation of $W_D^{2,p} \to L^q$

Definition

Given any continuous function f on closed interval I we denote by $Z(f)$ the number of distinct zeros of f on interior of I, and by $P(f)$ the number of sign changes of f on interval I.
By $SP_n(I,p,q)$ we denote the set of all spectral couples (f,λ) with $Z(f)=n$ on interval I, and by $sp_n(I,p,q)$ a set of all corresponding spectral numbers λ.

Lemma

*For each $n \in N$, $SP_n(I,p,q)$ contains only one spectral couple (f,λ).
Let (f_1,λ_1) be a spectral couple from $SP_0([0,1],p,q)$ and let us consider that f_1 is periodically extended on $\mathbf{R}$, then*

$$\tilde{\lambda} := n^{2q}\lambda_1 \text{ and } \tilde{f}(t) := f_1(nt)/n^2$$

*is a spectral couple for $SP_n([0,1],p,q)$.
If $(f,\lambda) \in SP_n([0,1],p,q)$, then $\tilde{\lambda} := d^{q/p-1-2q}\lambda$ and $\tilde{f}(t) := d^{2-1/p}f(t/d)$ is a spectral couple for $SP_n([0,d],p,q)$.
Then we have that*

$$\bar{\lambda} := n^{2q}d^{q/p-1-2q}\lambda_1 \text{ and } \bar{f}(t) := d^{2-1/p}f_1(nt/d)$$

is a spectral couple for $SP_n([0,d],p,q)$.

Approximation of $W_D^{2,p} \to L^q$

Theorem

Let $1 < p, q < \infty$, $I = [a, b]$ *and* $E_2 : W_D^{2,p}(I) \to L^q(I)$. *Then*

$$\|E_2\| := \sup_{u \in W^{2,p}(I)} \frac{\|u\|_{q,\mathcal{I}}}{\|u''\|_{p,\mathcal{I}}} = \frac{\|f\|_{q,\mathcal{I}}}{\|f''\|_{p,\mathcal{I}}} = \lambda^{-1/q} = |I|^{1/q-1/p+2}\lambda_0^{-1/q},$$

where $(f, \lambda) \in SP_0(I, p, q)$ *and* $\lambda_0 \in sp_0([0, 1], p, q)$.

Theorem

Let $I = [a, b]$ *and* $E_2 : W_D^{2,p}(I) \to L^q(I)$.
If $1 < p < q < \infty$ *then*

$$i_n(E_2) = b_n(E_2) = \lambda_n^{-1/q} = \frac{|I|^{1/q+2-1/p}}{n^2 \lambda_0^{1/q}},$$

and if $1 < q \leq p < \infty$ *then*

$$a_n(E_2) = d_n(E_2) = \lambda_n^{-1/q} = \frac{|I|^{1/q+2-1/p}}{n^2 \lambda_0^{1/q}},$$

where $\lambda_n \in sp_n(I, p, q)$ *and* $\lambda_0 \in sp_0([0, 1], p, q)$.