Gårding inequalities and their impact on regularity and uniqueness

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Basic problem

Minimization of a Hookian type energy

$$\mathcal{H}(u) = \int_{\Omega}\Big(H(\nabla u(x)) + f(x) \cdot \nabla u(x) \Big)\, \mathrm{d}x$$

over Dirichlet class $u \in \mathrm{W}^{1,2}_{g} = \mathrm{W}^{1,2}_{g}(\Omega, \mathbb{R}^2)$, where

- $H(z) = |z|^2 + h(\mathrm{det} z), \quad z \in \mathbb{R}^{2\times 2}$
- $|z| = \sqrt{z \cdot z}$ and $\mathrm{det} z$ is the determinant of z
- $h \colon \mathbb{R} \to \mathbb{R}$ smooth, convex, $h(t) \sim t^2$ near 0 and $h(t) \sim t$ near ∞.
- Ω smooth bounded domain in $\mathbb{R}^2$
- data $f \in \mathrm{L}^2(\Omega, \mathbb{R}^{2\times 2})$, $g \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^2)$ given.

Minimizers exist.

Are they unique? Are they regular?

The set-up

Minimize

$$\mathcal{F}(u) = \int_\Omega \big(F(\nabla u) + f \cdot \nabla u\big)\,\mathrm{d}x,$$

over Dirichlet class $u \in \mathrm{W}^{1,2}_g(\Omega, \mathbb{R}^N)$, where

- Ω bounded smooth domain in $\mathbb{R}^n$
- $f \in \mathrm{L}^2(\Omega, \mathbb{M})$, $g \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^N)$
- $F\colon \mathbb{M} \to \mathbb{R}$ a C^2 integrand
- $|F(z)| \leq L\big(|z|^2 + 1\big)$ for all $z \in \mathbb{M}$
- $z \mapsto F(z) - \ell|z|^2$ is quasiconvex

where ℓ, L are positive constants.

A minimizer $\bar{u} \in \mathrm{W}^{1,2}_g(\Omega, \mathbb{R}^N)$ exists.

Basic questions

- Regularity of minimizers $\bar{u}$?
- Uniqueness of minimizers $\bar{u}$?
- Dependence on the data f, g?
- What about extremals/stationary extremals?

The Euler-Lagrange system:

$$-\mathrm{div} F'(\nabla u) = \mathrm{div} f \quad \text{in } \Omega$$

Stationarity condition when $f = 0$:

$$\int_{\Omega} \big(F(\nabla u) \mathrm{div}\Phi - F'(\nabla u)[\nabla u \nabla \Phi] \big) \, \mathrm{d}x = 0 \quad \forall\, \Phi \in \mathrm{W}_0^{1,\infty}(\Omega, \mathbb{R}^n).$$

Answer to all these questions is no in general!

Variational property forcing strict convexity

Let $F \colon \mathbb{M} \to \mathbb{R}$ be quasiconvex and $|F(z)| \leq L\big(|z|^2+1\big)^{\frac{p}{2}}$ for all $z \in \mathbb{M}$, where $L > 0$, $1 < p < \infty$.

If for each $f \in \mathrm{C}_c^\infty(\Omega,\mathbb{R}^N)$ the functional

$$\mathfrak{F}(u) = \int_\Omega \big(F(\nabla u) + \nabla f \cdot \nabla u\big)\,\mathrm{d}x$$

has a unique minimizer $u = u_f \in \mathrm{W}_0^{1,p}(\Omega,\mathbb{R}^N)$, then $\mathfrak{F}$ is strictly convex on $\mathrm{W}_0^{1,p}(\Omega,\mathbb{R}^N)$.

Consequence of general results.
[Solov'ev, 1997], [Saint Raymond, 2013],
[Hiriart-Urruty, Volle & Zălinescu, 2013],
[Correa, Hantoute & Pérez-Aros, 2016]

Definition of functional convexity

Let $F\colon \mathbb{M} \to \mathbb{R}$ be continuous and

$$|F(z)| \leq L\big(|z|^2 + 1\big)^{\frac{p}{2}} \quad \forall\, z \in \mathbb{M}$$

where $L > 0$, $1 \leq p < \infty$.

Then F is (strictly) functionally convex if for all bounded, open $\Omega \subset \mathbb{R}^n$ and all Dirichlet data $g \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^N)$ the functional

$$u \mapsto \int_\Omega F(\nabla u(x))\,\mathrm{d}x$$

is (strictly) convex on the Dirichlet class $\mathrm{W}^{1,p}_g(\Omega, \mathbb{R}^N)$.

Bögelein, Dacorogna, Duzaar, Marcellini & Scheven, 2020: Integral convexity (=functional convexity) plays role for questions of existence for parabolic systems with quasiconvex potential

Three observations on functional convexity

- F is (strictly) functionally convex iff there exist a bounded, open, non-empty $\Omega_0 \subset \mathbb{R}^n$ and a Dirichlet datum $g_0 \in \mathrm{W}^{1,p}(\Omega_0, \mathbb{R}^N)$ such that

$$u \mapsto \int_{\Omega_0} F(\nabla u(x))\,\mathrm{d}x$$

  is (strictly) convex on the Dirichlet class $\mathrm{W}^{1,p}_{g_0}(\Omega_0, \mathbb{R}^N)$.
- F functionally convex $\Rightarrow$ F quasiconvex
- When F is C^1: F functionally convex $\Rightarrow$ F' quasi-monotone

The Hookian again: Non-uniqueness

Minimize

$$\mathcal{H}(u) = \int_\Omega \Big(H(\nabla u(x)) + \nabla f(x) \cdot \nabla u(x) \Big) \, \mathrm{d}x$$

over Dirichlet class $u \in \mathrm{W}^{1,2}_g = \mathrm{W}^{1,2}_g(\Omega, \mathbb{R}^2)$, where

$$H(z) = |z|^2 + h\big(\mathrm{det} z\big), \; z \in \mathbb{R}^{2\times 2},$$

where $h \colon \mathbb{R} \to \mathbb{R}$ is smooth, convex and $h(t) \sim t^2$ near 0 and $h(t) \sim t$ near ∞.

- H is strongly polyconvex
- H need not be functionally convex.
  In fact, H' need not be quasi-monotone [Hamburger, 1995]

Further examples and constructions follow from:
[Müller & Šverak, 2003], [Kirchheim, Müller & Šverak, 2003]
[Szekelyhidi, 2004], [Lawson & Osserman, 1977], [Spadaro, 2009].

Examples of functionally convex integands

- Rank-one convex quadratic forms are functionally convex. [van Hove]
- Quasiconvex envelope of

$$z \mapsto \mathrm{dist}(z, \{\xi, \zeta\})^2, \quad z \in \mathbb{R}^{2\times 2}$$

  is functionally convex for any choice of ξ, $\zeta \in \mathbb{R}^{2\times 2}$. [Kohn]
- Quasiconvex envelope of

$$z \mapsto \mathrm{dist}(z, C)^2, \quad z \in \mathbb{R}^{2\times 2}$$

  need not be functionally convex when $\mathrm{card}\, C \geq 3$.
  [Cordero, Kirchheim, Kolář & K, 2019]

Characterization of functional convexity: subquadratic case

[Cordero, Kirchheim, Kolář & K, 2019]

Assume $F\colon \mathbb{M} \to \mathbb{R}$ satisfies

$$|F(z)| \leq L\big(|z|^2+1\big)^{\frac{p}{2}} \quad \forall\, z \in \mathbb{M}$$

for some exponent $p \in [1,2)$ and $L \geq 0$.

Then F is functionally convex if and only if F is convex.

Non-uniqueness of minimizers for nonconvex quasiconvex variational integrals of subquadratic growth!

Can we find a condition that ensures minimizers are unique? What about regularity?

We do not want to assume convexity and we should work under natural hypotheses on the integrand.

Natural conditions on the integrand $F\colon \mathbb{M} \to \mathbb{R}$:

(H0) $F\colon \mathbb{M} \to \mathbb{R}$ a C^2 integrand

(H1) $|F(z)| \leq L\big(|z|^2 + 1\big)$ for all $z \in \mathbb{M}$

(H2) $z \mapsto F(z) - \ell|z|^2$ is quasiconvex

where ℓ, L are positive constants.

Coercivity in Dirichlet classes

(H2) $z \mapsto F(z) - \ell|z|^2$ is quasiconvex:

$$\int_\Omega \bigl(F(z+\nabla\varphi(x)) - F(z)\bigr)\,\mathrm{d}x \geq \ell \int_\Omega |\nabla\varphi(x)|^2\,\mathrm{d}x$$

holds for all $z \in \mathbb{M}$ and $\varphi \in \mathrm{W}_0^{1,2}(\Omega,\mathbb{R}^N)$. [Morrey, 1952], [Evans, 1986]

(H1), (H2) $\Rightarrow \forall\, g \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^N)\ \exists\, c_1 > 0,\ c_2 \in \mathbb{R}$ so

$$\int_\Omega F(\nabla u(x))\,\mathrm{d}x \geq \int_\Omega \bigl(c_1|\nabla u(x)|^2 + c_2\bigr)\,\mathrm{d}x \quad \forall\, u \in \mathrm{W}_g^{1,2}$$

In fact, the strong quasiconvexity condition (H2) at *some* $z \in \mathbb{M}$ is equivalent to $\mathrm{W}^{1,2}$ coercivity in general. [Chen & K, 2017]

Gårding's inequality for quasiconvex integrands: [Cordero & K, 2016]

Assume

- $F\colon \mathbb{M} \to \mathbb{R}$ satisfies (H0), (H1), (H2)
- $u\colon \overline{\Omega} \to \mathbb{R}^N$ is C^1

Then for each $A_1 \in (0,\ell)$ there exists $A_0 \geq 0$ such that

$$\int_\Omega \left(A_1 |\nabla \varphi|^2 - A_0 |\varphi|^2 \right) \mathrm{d}x \leq \int_\Omega F_{\nabla u}(\nabla \varphi) \, \mathrm{d}x \quad \forall\, \varphi \in \mathrm{W}^{1,2}_0(\Omega, \mathbb{R}^N).$$

Notation: $F_{\nabla u}(z) = F(\nabla u + z) - F(\nabla u) - F'(\nabla u)[z]$

- $A_0 = A_0(A_1, L, \nabla u, F'')$.

Dispensing with growth conditions

Quasiconvex surgery: [Cordero & K, 2020]

Let $F\colon \mathbb{M} \to \mathbb{R}$ be quasiconvex and C^2. Let $r > 0$. Then there exists a quasiconvex C^2 integrand $G\colon \mathbb{M} \to \mathbb{R}$ satisfying

- $G(z) = F(z)$ for all $|z| \leq r$
- $G(z) = G^c(z) = k\big(|z|^2 + 1\big)$ for $|z| \geq R$, where k, R are large.

In general one cannot find G as above so $G \leq F$ everywhere (even when $F(z) \geq |z|^2$ on $\mathbb{M}$ and we only require G to be rank-one convex).

Gårding limits nonconvexity

From validity of Gårding's inequality for u, F and elementary spectral properties of Dirichlet Laplacian on Ω:

$$(A_1\lambda_s(\Omega) - A_0) \int_\Omega |\varphi|^2 \, \mathrm{d}x \leq \int_\Omega F_{\nabla u}(\nabla\varphi) \, \mathrm{d}x$$

holds for all $\varphi \in \mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N) \cap \bigoplus_{j=1}^{s-1} X_j^\perp$ and $s \in \mathbb{N}$.

Hereby

$$\mathcal{F}(v) = \int_\Omega F(\nabla v) \, \mathrm{d}x$$

is convex at u on a subspace of $\mathrm{W}_g^{1,2}(\Omega, \mathbb{R}^N)$ of finite codimension

A conditional uniqueness result: [Cordero & K, 2016]

Assume (H0), (H1), (H2). Then $\exists\, \varepsilon = \varepsilon(\frac{\ell}{L}, F'', \Omega) > 0$ with the following property.

If

(I) $\displaystyle \frac{1}{\mathscr{L}^n(\Omega)} \int_\Omega |\nabla g - (\nabla g)_\Omega|^2 \, \mathrm{d}x < \varepsilon$

(II) u is a minimizer of $\mathcal{F}$ with $f = 0$ over $\mathrm{W}^{1,2}_g(\Omega, \mathbb{R}^N)$

(III) Gårding's inequality holds for u, F ($\Leftarrow$ u is C^1 on $\overline{\Omega}$)

then u is the unique minimizer over $\mathrm{W}^{1,2}_g(\Omega, \mathbb{R}^N)$. In fact, u is a strong minimizer.

Remarks:

- Under the assumption of a Gårding's inequality for u, F the quasiconvexity hypothesis (H2) is not needed.
- When Ω is *starshaped* the result persists for stationary extremals.

Further smallness by interpolation

Smallness in Campanato seminorms by interpolation:

$$[\nabla g]_{\mathcal{L}^{2,\alpha}(\Omega)} \leq [\nabla g]^{1-\alpha}_{\mathrm{BMO},\Omega} [\nabla g]^{\alpha}_{\mathcal{L}^{2,1}(\Omega)} \qquad 0 < \alpha < 1$$

Smallness in Schauder seminorms by interpolation: When Ω is convex, then $\forall\ \alpha \in (0,1)$, $p \geq 1$ we have with $c = 4\big(\mathrm{diam}(\Omega)\big)^{\frac{n(1-\alpha)}{n+p}}$ that

$$[\nabla g]_{\mathrm{C}^{0,\alpha}} \leq c \left(\frac{1}{\mathscr{L}(\Omega)} \int_{\Omega} |\nabla g - (\nabla g)_{\Omega}|^{p} \, \mathrm{d}x \right)^{\frac{1-\alpha}{n+p}} \|\nabla^2 g\|_{\mathrm{L}^{\infty}}^{\frac{n+p\alpha}{n+p}}$$

Hörmander, 1976

Further assumptions on the domain and the boundary datum

(H3) $\Omega \subset \mathbb{R}^n$ bounded $C^{1,1}$ domain:

$\exists\, \Phi \in C^{1,1}(\mathbb{R}^n)$ such that $\Omega = \{x \in \mathbb{R}^n : \Phi(x) < 0\}$,
$\mathbb{R}^n \setminus \overline{\Omega} = \{x \in \mathbb{R}^n : \Phi(x) > 0\}$ and $\nabla \Phi(x) \neq 0 \quad \forall x \in \partial\Omega$.

(H4) $g \in C^{1,1}(\overline{\Omega}, \mathbb{R}^N)$:

g is C^1 in a neighbourhood of $\overline{\Omega}$ and

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|\nabla g(x) - \nabla g(y)|}{|x-y|} < \infty$$

Uniqueness from IFT

Theorem: [Kewei Zhang, 1991...sample of special cases]
Assume (H0)–(H3), and

F is $C^{3,1}_{\mathrm{loc}}$.

Let $a\colon \mathbb{R}^n \to \mathbb{R}^N$ be an affine map and $\alpha \in (0,1)$. Then $\exists\, \varepsilon = \varepsilon(\alpha) > 0$ such that for $g\colon \Omega \to \mathbb{R}^N$, $f\colon \overline{\Omega} \to \mathbb{R}^N$ with $f|_{\partial\Omega} = 0$ satisfying

$$\|g - a\|_{C^{1,\alpha}} + \|f\|_{C^{1,\alpha}} < \varepsilon$$

the Euler-Lagrange system

$$\begin{cases} -\mathrm{div} F'(\nabla u) = \Delta f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

admits a unique solution in a $C^{1,\alpha}$ neighbourhood of a. Furthermore, this is the unique minimizer of $\mathcal{F}$ over $W^{1,2}_g(\Omega, \mathbb{R}^N)$.

Related references on uniqueness include

- F. John, 1972
- R.J. Knops & C.A. Stuart, 1984
- F. Stoppelli, 1954
- T. Valent, 1988
- P.G. Ciarlet, 1983
- J.E. Marsden & T.J.R. Hughes, 1983
- P.G. Ciarlet & C. Mardare, 2010
- A. Taheri, 2003
- P. Marcellini, 1983
- H. Parks, 1986
- T. Kilpeläinen, 2001
- L. Lussardi & E. Mascolo, 2017
- D. Spector & S. Spector, 2019

Minimizers inherit smallness from boundary datum

From (H0), (H1), (H2) and minimality of $u \in \mathrm{W}^{1,2}_{g}(\Omega, \mathbb{R}^N)$:

$$\int_\Omega F(\nabla u)\,\mathrm{d}x \leq \int_\Omega F(\nabla v)\,\mathrm{d}x$$

for all $v \in \mathrm{W}^{1,2}_{g}(\Omega, \mathbb{R}^N)$.

- $\exists\ c = c(\frac{\ell}{L})$ such that

$$\int_\Omega |\nabla u - (\nabla u)_\Omega|^2\,\mathrm{d}x \leq c \int_\Omega |\nabla g - (\nabla g)_\Omega|^2\,\mathrm{d}x.$$

Remark: When Ω is starshaped this is also true for stationary extremals.

Review of ε-regularity result

From (H0), (H1), (H2) and minimality of $u \in \mathrm{W}^{1,2}_g(\Omega, \mathbb{R}^N)$:

$$\int_\Omega F(\nabla u)\,\mathrm{d}x \le \int_\Omega F(\nabla v)\,\mathrm{d}x$$

for all $v \in \mathrm{W}^{1,2}_g(\Omega, \mathbb{R}^N)$.

• $\forall\ m > 0\ \exists\ \varepsilon_m = \varepsilon_m(F) > 0$ with property: for any ball $B = B_r(x_0)$ so $B_{2r}(x_0) \subset \Omega$, and

$$|(\nabla u)_B| < m \ \text{ and } \fint_B |\nabla u - (\nabla u)_B|^2\,\mathrm{d}x < \varepsilon_m$$

we have u is $\mathrm{C}^{1,\alpha}$ on $B_{r/2}(x_0)$ for all $\alpha < 1$.
[Evans, 1986], [Acerbi & Fusco, 1986]

A global ε-regularity result: [Cordero & K, 2017]

Assume (H0)–(H4). Then $\forall m > 0 \ \exists \varepsilon_m = \varepsilon_m(F, \Omega) > 0$ such that if

$$\sup_{\Omega} |\nabla g| \leq m$$

and

$$\frac{1}{\mathscr{L}^n(\Omega)} \int_{\Omega} \Big(\exp\big(|\nabla g - (\nabla g)_{\Omega}| \big) - 1 \Big) \, \mathrm{d}y \leq \varepsilon_m$$

then the minimizer u over $\mathrm{W}^{1,2}_{g}(\Omega, \mathbb{R}^N)$ is $\mathrm{C}^{1,\alpha}$ on $\overline{\Omega}$ for each $\alpha < 1$.

Extensions and variants

Global ε-regularity remains valid for ω-minimizers and for L^2 local minimizers

Regularity and uniqueness results in same spirit also hold for variational problems for

$$\int_\Omega \big(F(\nabla u) + h(x,u) \big) \, \mathrm{d}x$$

with suitable h.

Singularity in general

Minimizers only partially regular in general:

De Giorgi, 1968, Maz'ya, 1968, Maz'ya, Nazarow & Plamenewskiĭ, 1982, Frehse, 2009

Nečas, 1975, Hao, Leonardi & Nečas, 1996

Šverák & Yan, 2000, 2003

Mooney & Savin, 2016

Extremals not partially regular in general:

Müller & Šverák, 2003

Szekelyhidi, 2004

Stationary extremals?

De Lellis, De Philippis, Kirchheim & Tione, 2019

Hirsch & Tione, 2020

Higher differentiability: [Cordero & K, 2019]

Assume (H0), (H1), (H2). If $u \in \mathrm{W}^{1,2}_{g}(\Omega, \mathbb{R}^N)$ and

- u is a stationary extremal,
- Gårding's inequality holds for u, F,

then $u \in \mathrm{W}^{2,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for some $p > 2$.

Furthermore, for balls $B_{2R}(x_0) \Subset \Omega$ we have

$$\fint_{B_R(x_0)} |\nabla^2 u|^p \,\mathrm{d}x \leq c \left(\fint_{B_{2R}(x_0)} \frac{|\nabla u - (\nabla u)_{x_0,2R}|^2}{R^2} \,\mathrm{d}x \right)^{\frac{p}{2}} .$$

Remark: Inspired by calculation due to Bitew & Grabovsky, 2008.

Partial regularity: [Cordero & K, 2019]

Assume (H0), (H1), (H2). If $u \in \mathrm{W}^{1,2}_g(\Omega, \mathbb{R}^N)$ and

- u is a stationary extremal,
- Gårding's inequality holds for u, F,

then

- u is $\mathrm{C}^{1,\alpha}$ locally on Ω when dimension $n = 2$, and
- u is $\mathrm{C}^{1,\alpha}$ locally off relatively closed set of $n-2$ dimensional Hausdorff measure 0 when dimension $n \geq 3$.

K & Mingione, 2010: Singular set is uniformly porous when minimizer is assumed Lipschitz.

Direct proof of a result of Grabovsky & Mengesha

Assume

- $F\colon \mathbb{M} \to \mathbb{R}$ satisfies (H0), (H1), (H2)
- $u\colon \overline{\Omega} \to \mathbb{R}^N$ is C^1 and an extremal
- $\exists\, \delta > 0$ such that

$$\int_\Omega F''(\nabla u)[\nabla\varphi, \nabla\varphi]\,\mathrm{d}x \geq 2\delta \int_\Omega |\varphi|^2\,\mathrm{d}x \quad \forall \varphi \in \mathrm{W}^{1,2}_0(\Omega, \mathbb{R}^N)$$

Then $\exists\, \varepsilon > 0$ such that

$$\int_\Omega F(\nabla u + \nabla\varphi)\,\mathrm{d}x \geq \int_\Omega \bigl(F(\nabla u) + \delta|\varphi|^2\bigr)\,\mathrm{d}x$$

for all $\varphi \in \mathrm{W}^{1,2}_0(\Omega, \mathbb{R}^N)$ with $\|\varphi\|_{\mathrm{L}^2} < \varepsilon$.

False if u is only assumed Lipschitz.
[K & Taheri, 2003] based on [Müller & Šverák, 2002].
Strongly polyconvex case treated in [Szekelyhidi, 2004]

Corollary of the Faber-Krahn bound and quantification of result by Kewei Zhang

Assume

- $F\colon \mathbb{M} \to \mathbb{R}$ satisfies (H0), (H1), (H2)
- $u\colon \overline{\Omega} \to \mathbb{R}^N$ is C^1 and an extremal

Then for a $\delta \geq 0$:

$$\int_\Omega F(\nabla u + \nabla\varphi)\,\mathrm{d}x \geq \int_\Omega \bigl(F(\nabla u) + \delta|\varphi|^2\bigr)\,\mathrm{d}x$$

holds for all $\varphi \in \mathrm{W}^{1,2}_0(\Omega,\mathbb{R}^N)$ provided

$$A_0 \leq A_1 \left(\frac{c(n)}{\mathcal{L}^n(\Omega)}\right)^{\frac{2}{n}} - \delta,$$

where A_0, A_1 are the constants from Gårding's inequality.

Comments on proofs: Focus on conditional uniqueness result

Assume (H0), (H1), (H2) and that $u \in \mathrm{W}^{1,2}_g(\Omega, \mathbb{R}^N)$ is a minimizer. Then $\exists\ \varepsilon = \varepsilon(\frac{\ell}{L}, F'', \Omega) > 0$ with the following property.

If

(I) $\displaystyle \frac{1}{\mathscr{L}^n(\Omega)} \int_\Omega |\nabla g - (\nabla g)_\Omega|^2 \, \mathrm{d}x < \varepsilon$

(II) Gårding's inequality holds for u, F ($\Leftarrow u$ is C^1 on $\overline{\Omega}$)

then u is the unique minimizer over $\mathrm{W}^{1,2}_g(\Omega, \mathbb{R}^N)$.

The intuition behind why Gårding's inequality allows us to use the smallness condition:

- nonconvexity is the issue
- Gårding reduces the nonconvexity to a finite dimensional subspace

Comments on proofs: Gårding's inequality

If it holds, then it expresses a Fredholm property of F and, given $A_1 \in (0, \ell)$, the optimal constant A_0 such that

$$\int_\Omega \big(A_1 |\nabla \varphi|^2 - A_0 |\varphi|^2 \big) \, \mathrm{d}x \leq \int_\Omega F_{\nabla u}(\nabla \varphi) \, \mathrm{d}x \quad \forall \, \varphi \in \mathrm{W}_0^{1,2}(\Omega, \mathbb{R}^N).$$

is related to a nonlinear eigenvalue problem.

Notation: $F_{\nabla u}(z) = F(\nabla u + z) - F(\nabla u) - F'(\nabla u)[z]$

Comments on proofs: Bound on Rayleigh type quotient

$$R[\varphi] = \frac{\int_\Omega F_{\nabla u}(\nabla\varphi)\,\mathrm{d}x}{\int_\Omega |\varphi|^2\,\mathrm{d}x} \quad \varphi \in \mathrm{W}_0^{1,2}(\Omega,\mathbb{R}^N)\setminus\{0\}$$

For $\varphi \in \mathrm{W}_0^{1,2}(\Omega,\mathbb{R}^N)$ with $\|\varphi\|_{\mathrm{L}^2} = 1$ and $t > 0$:

$$\begin{aligned} R[t\varphi] &= \int_\Omega t^{-2} F_{\nabla u}(t\nabla\varphi)\,\mathrm{d}x \\ &\geq \int_\Omega \big(A_1|\nabla\varphi|^2 - A_0\big)\,\mathrm{d}x \end{aligned}$$

Equi-coercivity in $t > 0$: Direct method yields minimizer φ_t

Comments on proofs: Uniform higher integrability

There exist $p > 2$, $c > 0$ such that

$$\left(\int_\Omega |\nabla \varphi_t|^p \, \mathrm{d}x \right)^{\frac{2}{p}} \leq c \int_\Omega |\nabla \varphi_t|^2 \, \mathrm{d}x.$$

holds for all $t > 0$.

Proof is variant of standard approach to higher integrability–only issue is constraint $\|\varphi\|_{\mathrm{L}^2} = 1$.

Thank you for the attention and happy New Year