Non-Newtonian fluids: from ketchup to convex integration

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Physical laws and simplifications

- mass balance

$$\dot{\varrho} + \varrho \mathrm{div}\, v = 0 \qquad \partial_t \varrho + \mathrm{div}\,(\varrho v) = 0$$

- linear momentum balance

$$\rho \dot{v} = \mathrm{div}\, T + b \qquad \rho(\partial_t v + v \cdot \nabla v) = \mathrm{div}\, T + b$$

- angular momentum balance supplement:

$$T \text{ symmetric}$$

$T = \mathring{T} - pI$ is Cauchy stress (for contact forces), b are body forces (e.g. fields)

Working assumptions

- incompressible $\mathrm{div}\, v = 0 \implies \dot{\varrho} = 0$
- homogenous $\varrho \equiv const$

Specific models

$$\partial_t v + v \cdot \nabla v + \nabla p = \operatorname{div} \mathring{T} + b, \qquad \operatorname{div} v = 0$$

Aim: choose $\mathring{T}$

- $\mathring{T}(\nabla v)$, so by frame indifference $\mathring{T}(Dv)$ and $\mathring{T}$ isotropic
- representation of isotropic functions

$$\mathring{T}(Dv) = \alpha I + \beta Dv + \gamma (Dv)^2$$

α, β, γ scalar functions depending on invariants of Dv:
$tr, \frac{1}{2}((tr)^2 - tr(^2)), det$

Example of choices

- $\alpha = \beta = \gamma = 0$: Euler's equation
- $\alpha = \gamma = 0$, $\beta = \nu_0$: Navier-Stokes equation
- $\alpha = \gamma = 0$, $\beta = (\nu_0 + |Dv|)^{q-2}$: non-Newtonian fluid

Newtonian vs non-Newtonian fluids

$$\partial_t v + v \cdot \nabla v + \nabla p = \mathrm{div}\, \nu_0 D v + b, \qquad \mathrm{div}\, v = 0$$

ν_0: constant viscosity (Newtonian)

(clip)

but viscosity may change under applied forces (non-Newtonian), e.g.

$$\partial_t v + v \cdot \nabla v + \nabla p = \mathrm{div}\, \Big((\nu_0 + |Dv|)^{q-2} D v \Big) + b, \qquad \mathrm{div}\, v = 0$$

power-law model: 1929 Norton for molten steel, Ostwald for polymers

- $q < 2$ forces decrease viscosity (paints, ketchup, ice)
- $q > 2$ forces increase viscosity (corn starch+water, silicone solutions)

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job of an (applied) mathematician

take a reasonable model and check its basic analytical properties

(i) existence of solutions

(ii) uniqueness of solutions (for reasonable initial data)

(iii) stability on data

(iv) dynamical/regularity properties

for Euler and Navier-Stokes: only (i) satisfactorily answered

Ladyzhenskaya ICM 1966 suggestion: think of power-law fluids

Rules of thumb

$$\partial_t v + v \cdot \nabla v - \operatorname{div}\left((\nu_0 + |Dv|)^{q-2} Dv\right) = \nabla p$$
$$\operatorname{div} v = 0$$

Energy

$$\int |v|^2(t) + 2\int_0^t \int (\nu_0 + |Dv|)^{q-2}|Dv|^2$$

Scaling (case $\nu_0 = 0$)

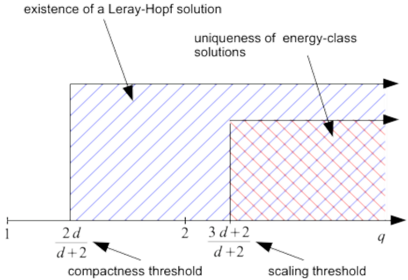
$$v_\lambda := \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t) \quad \text{with} \quad \alpha = \frac{q-1}{3-q}$$

Suggest:

- $W^{1,q} \subset\subset L^2$ for $q > \frac{2d}{d+2} \implies$ existence of a solution
- energy of v_λ as $\lambda \to \infty$ blows up iff $q > \frac{3d+2}{d+2} \implies v \cdot \nabla v$ plays no role for $q > \frac{3d+2}{d+2}$, i.e. uniqueness

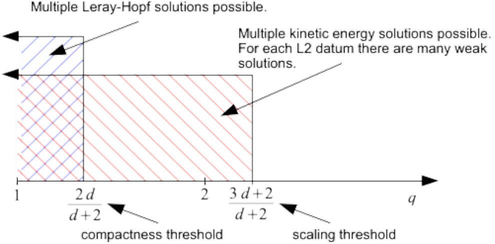
results overview

rigorous proofs of $q > \frac{2d}{d+2}$ and $q > \frac{3d+2}{d+2}$ thresholds 1969–2020: Ladyzhenskaya, Nečas, Malek, Diening, Buliček

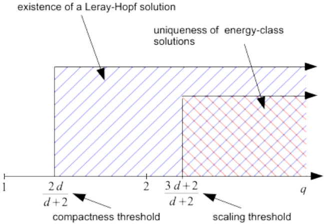


our contribution (B, Modena, Székelyhidi)

dual picture in q



recall:



h-principle in fluid dynamics

- relax PDE to PDRelation with error R
- correct solution to PDR with fast-oscillating function to reduce R
- if problem is flexible enough, we can produce a solution to PDE, which is ‘close’ to PDR
- many PDRs $\implies$ many solutions to PDEs

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$$\partial_t v + \mathrm{div}\,(v \otimes v) - \mathrm{div}\, A(Dv) = \nabla p$$

average

$$\partial_t \bar{v} + \mathrm{div}\,(\bar{v} \otimes \bar{v}) - \mathrm{div}\, A(D\bar{v}) - \nabla \bar{p} = \mathrm{div}\left(\bar{v} \otimes \bar{v} - \overline{v \otimes v}\right) - \mathrm{div}\,(A(D\bar{v}) - \overline{A(Dv)}) =: \mathrm{div}\, R$$

$\bar{v}$ - ‘laminar flow’, R - Reynolds stress (measure of turbulence)

Reducing the error

take (u_0, q_0, R_0) solving Non-Newtonian-Reynolds

$$\partial_t u_0 + \mathrm{div}\,(u_0 \otimes u_0) - \mathrm{div}\,((\nu_0 + |Du_0|)^{q-2} Du_0) + \nabla p = -\mathrm{div}\, R_0$$

aim: produce (u_1, q_1, R_1) via

$$u_1 := u_0 + u_p$$

so that $u_p \otimes u_p - R_0$ small.

$$M = \sum_{k \in K} \Gamma_k^2(M) k \otimes k$$

$$u_p = \sum_k \sqrt{|R_0|}\Gamma_k(\frac{R_0}{|R_0|}) W^k, \qquad \int_{\mathbb{T}^d} W^k \otimes W^k = k \otimes k$$

$$u_p \otimes u_p - R_0 = \sum_k |R_0| \Gamma_k^2(\frac{R_0}{|R_0|}) P_{\neq 0} W^k \otimes W^k$$

$$\mathrm{div}\,(W^k \otimes W^k) = 0 \implies \mathrm{div}\,(u_p \otimes u_p - R_0) = \sum_k \nabla \left(|R_0| \Gamma_k^2(\frac{R_0}{|R_0|}) \right) P_{\neq 0} W^k \otimes W^k$$

$$R_1 = \mathrm{div}^{-1} \left(\sum_k \nabla \left(|R_0| \Gamma_k^2(\frac{R_0}{|R_0|}) \right) P_{\neq 0} W^k \otimes W^k \right)$$

Full derivative by concentrated Mikado flows

$$u_p = \sum_k \sqrt{|R_0|}\Gamma_k\left(\frac{R_0}{|R_0|}\right)W^k$$

$$\mathrm{div}\, W^k_{\mu,\lambda} = 0, \quad \mathrm{div}\,(W^k_{\mu,\lambda} \otimes W^k_{\mu,\lambda}) = 0,$$

$$\fint_{\mathbb{T}^d} W^k_{\mu,\lambda} = 0, \quad \fint_{\mathbb{T}^d} W^k_{\mu,\lambda} \otimes W^k_{\mu,\lambda} = k \otimes k$$

$W^k_{\mu,\lambda}$ and $W^r_{\mu,\lambda}$ have disjoint supports for $k \neq r$ and

$$|\nabla^s W^k_{\mu,\lambda}|_{L^r(\mathbb{T}^d)} \leq C(s,|K|)\lambda^s \mu^{s+\frac{d-1}{2}-\frac{d-1}{r}}.$$

If $d-1 \to d$ above, then

$$|\nabla W^k_{\mu,\lambda}|_{L^{\frac{2d}{d+2}-}(\mathbb{T}^d)} \leq \lambda\mu^{-\epsilon}$$

$d = 3$ by concentrated, localized, traveling Mikado flows

Aim: $d \to d-1$. Localisation destroys Euler-like properties.

$$\operatorname{div}(W^k \otimes W^k) = 0 \implies \operatorname{div}(u_p \otimes u_p - R_0) = \sum_k \nabla(a_k^2) P_{\neq 0} W^k \otimes W^k$$

$$\operatorname{div}(u_p \otimes u_p - R_0) \sim \sum_{k \in K} (P_{\neq 0} W^k \otimes W^k) \nabla(a_k^2) + \sum_{k \in K} \Big(\fint W^k \otimes W^k - k \otimes k \Big) \nabla(a_k^2) + \sum_{k \in K} a_k^2 \operatorname{div}(W^k \otimes W^k).$$

$$Y^k \sim -\frac{1}{\omega} \operatorname{div}(W^k \otimes W^k)$$

Iteration Step

Fix any $e \in C^\infty([0,1];[\frac{1}{2},1])$. (u_0, π_0, R_0) solves

$$\partial_t u_0 + \operatorname{div}(u_0 \otimes u_0) - \operatorname{div} A(Du_0) + \nabla \pi_0 = -\operatorname{div} R_0,$$
$$\operatorname{div} u = 0.$$

Take any $\delta, \eta \in (0,1]$. Assume

$$\frac{3}{4}\delta e(t) \le e(t) - \left(\int_{\mathbb{T}^d} |u_0|^2(t) + 2\int_0^t \int_{\mathbb{T}^d} A(Du_0)Du_0 \right) \le \frac{5}{4}\delta e(t)$$

and

$$|R_0(t)|_{L_1} \le \frac{\delta}{2^7 d}.$$

Then, $\exists$ solution (u_1, π_1, R_1)

$$|(u_1 - u_0)(t)|_{L^2} \le M\delta^{\frac{1}{2}}$$

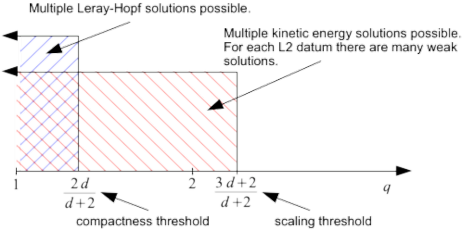
$$|(u_1 - u_0)(t)|_{W^{1,q}} \le \eta$$

$$|R_1(t)|_{L_1} \le \eta.$$

$$\frac{3}{8}\delta e(t) \le e(t) - \left(\int_{\mathbb{T}^d} |u_1|^2(t) + 2\int_0^t \int_{\mathbb{T}^d} A(Du_1)Du_1 \right) \le \frac{5}{8}\delta e(t).$$

Highlights

- non-uniqueness picture, sharp in powers



- improves regularity of NSE non-unique weak solutions by Buckmaster&Vicol
- avoids Fourier side
- avoids meticulous control of decays
- introduces concentration mechanism into fluid-dynamics convex integration
- provides improved antidivergence operators