Continuity and Harnack's inequality for bounded solutions of elliptic and parabolic equations with non-standard growth under the non-logarithmic Zhikov's condition

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under the non-logarithmic Zhikov's condition

$$(g_{\mu}) \; : \; g(x, v/r) \leqslant c(K)\mu(r)g(y, v/r), x, y \in B_r(x_0), br < v \leqslant K, \exists b \geqslant 0$$

$$\int\limits_0 \mu^{-\gamma}(r)\frac{dr}{r} = +\infty \; , \qquad \mu(r) \sim \ln^L \frac{1}{r}, L \leqslant 1/\gamma$$

We also consider parabolic equations with (p, q) growth

$$u_t - div\left(g(x,t,\mid \nabla u \mid)\frac{\nabla u}{\mid \nabla u \mid}\right) = 0, (x,t) \in \Omega_T$$

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under the non-logarithmic condition

(g_μ) $g(x, t, v/r) \leqslant c(K)\mu(r)g(y, \tau, v/r), (x, t), (y, \tau) \in Q_{r,r}(x_0, t_0), br < v \leqslant K$

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We are interesting in such basic properties of bounded solutions as interior continuity,continuity up to the boundary and Harnack's inequality

• boundedness, anisotropic elliptic and parabolic equations
Kolodiy$(1970, 1971,$MGU Vestnik$)$

$p_{\max} \leqslant \frac{np}{n-p}, \ \frac{1}{p} = \frac{1}{n} \sum_{i=0}^{n} \frac{1}{p_i}, \ p < n$

• homogenization problems, Lavrentiev phenomenon Zhikov$(1983, 1995)$

• C^1 regularity Marcellini(1986)

• Hölder continuity, C^1 regularity, Harnack's inequality
$(g$ independent of $x)$ Lieberman(1991)

log ZHIKOV's CONDITION

Few of the results concerning the regularity of solutions to $p(x)$-Laplace Eq. under the *log* Zhikov's condition

$$g(x, v/r) \leqslant c(K) g(y, v/r), x, y \in B_r(x_0), r < v \leqslant K$$

$$\triangle_{p(x)} u = 0$$

$$\operatorname{osc}_{B_r(x_0)} p(x) \leqslant \frac{L}{\ln \frac{1}{r}}, 0 < L < +\infty \Rightarrow u \in C^{0,\beta}, \beta > 0$$

regularity,Harnack inequality; boundary continuity - Acerbi , Fusco ,Alkhutov , Chiado Piat , Coscia , Mingione , Krasheninnikova , Diening, Kristensen, Harjulehto , Hästo ,Toivanen ,Karppinen,Ok,....

Few of the results concerning the regularity of solutions to double-phase and triple-phase Eqs. under the *log* Zhikov's condition

$$g(x, v/r) \leqslant c(K)g(y, v/r), x, y \in B_r(x_0), 0 < v \leqslant K$$

$$div(| \nabla u |^{p-2} \nabla u + a(x) | \nabla u |^{q-2} \nabla u) = 0$$

$$0 \leqslant a(x) \in C^{0,\alpha}, 1 < p \leqslant q \leqslant p + \alpha \Rightarrow u \in C^{0,\beta}, \beta > 0$$

regularity,Harnack inequality - Esposito , Mingione, Colombo ,Baroni, Harjulehto , Hästo , Ok,Cupini,Marcellini,Mascolo,...

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$$div(\mid \nabla u \mid^{p-2} \nabla u + a_1(x) \mid \nabla u \mid^{q_1-2} \nabla u + a_2(x)|\nabla u|^{q_2-2}\nabla u) = 0$$

regularity - De Filippis,Oh,Fang,Zhang,Zhang ,...

Few of the results concerning the regularity of solutions to degenerate double-phase Eq. under the *log* Zhikov's condition

$g(x, v/r) \leqslant c(K) g(y, v/r), x, y \in B_r(x_0), 0 < v \leqslant K$

$div(| \nabla u |^{p-2} \nabla u (1 + a(x) \ln(1 + | \nabla u |)) = 0$

$0 \leqslant a(x), \operatorname{osc}_{B_r(x_0)} a(x) \leqslant \frac{B}{\ln \frac{1}{r}} \Rightarrow u \in C^{0,\beta}, \beta > 0$

regularity,Harnack inequality - Mingione , Siepe ,Gianetti , Passarelli di Napoli , Baroni,Colombo, Harjulehto , Hästo,Byun, Oh ,Coscia , Balci ,...

Non log ZHIKOV's CONDITION

Zhikov introduced the following condition

$$\operatorname{osc}_{B_r(x_0)} p(x) \leqslant \frac{\ln \mu(r)}{\ln \frac{1}{r}}, \ \lim_{r \to 0} \frac{\ln \mu(r)}{\ln \frac{1}{r}} = 0 \quad , \int_0 \mu^{-n/p}(r) \frac{dr}{r} = +\infty$$

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- density of C^∞ in $W^{1,p(x)}$- Zhikov(2004)

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- Gehring's lemma - Pastuchova and Zhikov (2008)

  $\int |\nabla u|^{p(x)} \ln^{\delta_0}(1 + |\nabla u|) dx < +\infty \ , \ \exists \ \delta_0 \in (0,1),$

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- in general case non *log* Zhikov's condition transforms into

  $g(x, v/r) \leqslant c(K)\mu(r)g(y, v/r), x, y \in B_r(x_0), br < v \leqslant K, \exists b \geqslant 0$

  $\int\limits_0 \mu^{-\gamma}(r) \frac{dr}{r} = +\infty$

Non log ZHIKOV's CONDITION

The first attempt in the direction of non-log condition for the $p(x)$-Laplace was made by Alkhutov and Krasheninnikova (2008), namely,they proved the C_{loc} result under the exp condition

$\triangle_{p(x)} u = 0$

$$\operatorname{osc}_{B_r(x_0)} p(x) \leqslant \frac{\ln \mu(r)}{\ln \frac{1}{r}}, \quad \int\limits_0 e^{-\gamma \mu^\beta(r)} \frac{dr}{r} = +\infty$$

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general case : interior continuity,Harnack's inequality- Skr. and Voitovych (2020),Shan, Skr. and Voitovych (2021)

$$g(x, v/r) \leqslant c(K)\mu(r)g(y, v/r), x, y \in B_r(x_0), br < v \leqslant K$$

DIFFICULTIES

- De Giorgi's method,expansion of the positivity

$u > 0, \exists\ N > 0, \exists\ \alpha \in (0, 1)$, having the information on the measure

$| \{B_r(x_0) : u \geqslant N\} | \geqslant \alpha\mu^{-\gamma}(r) \mid B_r(x_0) \mid$

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we will inevitably arrive at the lower bound of the solution

$$\min_{B_{r/2}(x_0)} u \geqslant \gamma N e^{-\gamma\mu^{\beta}(r)} \ and \ \int_0 e^{-\gamma\mu^{\beta}(r)} dr/r = +\infty$$

DIFFICULTIES

- Moser's method, the appearance of exp condition one can easily seen from the inequality

$$\max_{B_{r/2}(x_0)} \ln \frac{\omega_r}{v_\pm(x)+r} \leqslant \gamma\mu^\gamma(r),\ \ v_+(x) = M_r - u(x),\ \ v_-(x) = u(x) - m_r$$

$$\Rightarrow \omega_{r/2} \leqslant (1 - e^{-\gamma\mu^\gamma(r)})\omega_r + r \Rightarrow \int\limits_0 e^{-\gamma\mu^\gamma(r)} dr/r = +\infty$$

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- So, the direct application of De Giorgi's or Moser's method leads us to the exponential condition, therefore we need to use some workaround

u- solution $\Rightarrow$ Cacciopoli type inequality $+$ Young's inequality $\forall\ a, b, \varepsilon > 0$: $g(x,a)b \leqslant \frac{1}{\varepsilon} g(x,a)a + \max(\varepsilon^{p-1}, \varepsilon^{q-1}) g(x,b)b$

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$\Rightarrow u \in B_{1,g}(\Omega) \ \ : \quad \forall \varepsilon \in (0,1]$

$$\int\limits_{B_r(x_0)\ \cap\{k<u<l\}} g(x, \tfrac{M_-(l,r)}{r}) \mid \nabla u \mid \zeta^q dx \leqslant$$

$$\leqslant \frac{K_1}{\varepsilon} \frac{M_-(l,r)}{r} \int\limits_{B_r(x_0)\ \cap\{k<u<l\}} g(x, \tfrac{M_-(l,r)}{r}) dx +$$

$$+ K_1 \varepsilon^{p-1} \int\limits_{B_r(x_0)} g(x, M_-(l,r) \mid \nabla \zeta \mid)\ (u-l)_- \mid \nabla \zeta \mid \ dx$$

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$u \in W^{1,1}(\Omega) \cap L^\infty(\Omega),\ \forall B_r(x_0) \subset \Omega,\ \forall\ k < l \in \mathbb{R},$
$M_+(k,r) := sup_{B_r(x_0)} (u-k)_+,\ (u-k)_+ := max(u-k, 0)$
$M_-(l,r) := sup_{B_r(x_0)} (u-l)_-,\ (u-l)_- := max(l-u, 0),$

$\forall \zeta \in C_0^1(B_r(x_0))\ ,\ 0 \leqslant \zeta \leqslant 1, |\nabla \zeta| \sim \frac{1}{r}$

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The advantage of such $B_{1,g}(\Omega)$ classes is that we move away from Orlicz spaces,and we do not need any special properties of these spaces.Moreover, in the log case, Hölder continuity and Harnack's inequality is a simple consequence of the definition of $B_{1,g}(\Omega)$ classes (Ladyzgenskaya and Ural'tseva,$B_p(\Omega)$ classes, 1964)

We propose to replace the non *log* Zhikov's condition by

$(g_\lambda) \;\; : \;\; g(x, v/r) \leqslant c(K) g(y, v/r), \;\; x, y \in B_r(x_0) \subset B_R(x_0), br < v \leqslant K\lambda(r)$

$$\exists b \geqslant 0, \;\; \exists \; 0 < \lambda(r) \leqslant 1 : \lim_{r \to 0} \lambda(r) = 0, \lim_{r \to 0} \frac{r^{1-\overline{a}}}{\lambda(r)} = 0, \;\; \exists \; \overline{a} \in (0,1)$$

$\lambda(r) = (\ln \frac{1}{r})^{-\gamma} \; , \gamma > 0$

MAIN RESULT

$$g(x,v/r)\leqslant c(K)g(y,v/r),\ \ x,y\in B_r(x_0), br<v\leqslant K\lambda(r),$$

$$u\in B_{1,g}(\Omega)\Rightarrow \omega_r\leqslant\omega_\rho\exp(-\gamma\int\limits_{2r}^{\rho}\lambda(s)\,\frac{ds}{s})+\gamma b\rho^{\overline{a}},\ \ 0<r\leqslant\rho,$$

if additionally $\int\limits_0 \lambda(s)ds/s=+\infty\ \ \Rightarrow\ \ u\in C_{loc}(\Omega)$

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continuity result is a consequence of elliptic $B_{1,g}$ classes

MAIN RESULT

$g(x, v/r) \leqslant c(K) g(y, v/r),\ \ x, y \in B_r(x_0), br < v \leqslant K\lambda(r),$

$(\mathbf{A}(x,\xi) - \mathbf{A}(x,\eta))(\xi - \eta) > 0, \quad \xi, \eta \in \mathbb{R}^n, \quad \xi \neq \eta$

$$\int_0 g_{x_0}^{-1} \left(\frac{C\big(B_r(x_0) \setminus \Omega, B_{8r}(x_0); \lambda(r)\big)}{r^{n-1}} \right) dr = +\infty,$$

$x_0 \in \partial\Omega, \ \Rightarrow x_0$ is a regular boundary point

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$x_0 \in \partial\Omega, \; \Rightarrow x_0$ is a regular boundary point

$F \subset B_r(x_0) \subset B_\rho(x_0), \mathfrak{M}(F) := \{\varphi \in W_0(B_{8\rho}(x_0)), \varphi \geqslant 1 \; on \; F\}$

$C(F, B_{8\rho}(x_0); m) := \inf\limits_{\varphi \in \mathfrak{M}(F)} \int\limits_{B_{8\rho}(x_0)} g(x, m|\nabla\varphi|)\, |\nabla\varphi|\, dx, m > 0$

MAIN RESULT

$$g(x,v/r)\leqslant c(K)g(y,v/r),\ \ x,y\in B_r(x_0), br<v\leqslant K\lambda(r),$$

$$(\mathbf{A}(x,\xi)-\mathbf{A}(x,\eta))(\xi-\eta)>0,\quad \xi,\eta\in\mathbb{R}^n,\quad \xi\neq\eta$$

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$p(x)$-Laplace,$\lambda(r)=1,\mu(r)=1$: Alkhutov and Krasheninnikova(2004),
$\lambda(r)=1,\mu(r)=(\ln\ln\frac{1}{r})^L$: Alkhutov and Surnachev(2019)

exp condition $\int\limits_0 e^{-\gamma_1\mu^{\gamma_2}(r)}\left(\frac{C_{p(\cdot)}(B_r(x_0)\setminus\Omega)}{r^{n-p(x_0)}}\right)^{\frac{1}{p(x_0)-1}}\frac{dr}{r}=+\infty$

- $F = B_r(x_0),\ br < m \leqslant \lambda(8r) \Rightarrow$

  $C(B_r(x_0), B_{8r}(x_0); m) \asymp r^{n-1} g(x_0, m/r) \Rightarrow$

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  $C(B_r(x_0) \setminus \Omega, B_{8r}(x_0); \lambda(r)) \geqslant \gamma^{-1} C(B_r(x_0), B_{8r}(x_0); \lambda(r)) \Rightarrow$

  $$\int\limits_0 g_{x_0}^{-1}\left(\frac{C\big(B_r(x_0)\setminus\Omega, B_{8r}(x_0);\lambda(r)\big)}{r^{n-1}}\right) dr \geqslant \gamma^{-1} \int\limits_0 \lambda(r)\,\frac{dr}{r}$$

  $\lambda(r) = 1$: Harjulehto and Hästo (2017)

We use old Landis's ideas and his so-called "growth-type" lemma.We construct the auxiliary solutions of potential type :

$$\mathrm{div}\mathbf{A}(x,\nabla v)=0, \quad x\in\mathcal{D}:=B_{8\rho}(x_0)\setminus F,\ F\subset B_r(x_0),\ r<\rho$$

$$v-m\psi\in W_0(\mathcal{D}), b\rho<m\leqslant\lambda(8\rho), \psi\in\mathfrak{M}(F), \psi=1 \ on \ F$$

LOWER and UPPER BOUNDS of AUXILIARY SOLUTIONS

$$g(x, v/r) \leqslant c(K) g(y, v/r),\ \ x, y \in B_r(x_0), br < v \leqslant K\lambda(r),$$

either

$$C(F, B_{8\rho}(x_0); m) \leqslant c(1+b)\rho^{n-1},$$

or

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$$\gamma^{-1} g_{x_0}^{-1}\left(\frac{C(F, B_{8\rho}(x_0); m)}{\rho^{n-1}}\right)\rho \leqslant v(x) \leqslant \gamma\, g_{x_0}^{-1}\left(\frac{C(F, B_{8\rho}(x_0); m)}{\rho^{n-1}}\right)\rho$$

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LOWER and UPPER BOUNDS of AUXILIARY SOLUTIONS

$$g(x, v/r) \leqslant c(K) g(y, v/r),\ \ x, y \in B_r(x_0), br < v \leqslant K\lambda(r),$$

either

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$$p - Laplace\ : v(x) \asymp m\left(\frac{C_p(F, B_{8\rho}(x_0))}{\rho^{n-p}}\right)^{\frac{1}{p-1}}, x \in B_{4\rho}(x_0) \setminus B_{2\rho}(x_0)$$

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$$(\mathbf{A}(x,\xi) - \mathbf{A}(x,\eta))(\xi - \eta) > 0, \quad \xi, \eta \in \mathbb{R}^n, \quad \xi \neq \eta$$

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this condition is worse than the previous one,but at the same time it is much better,than the exp type condition

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GENERALIZED Non log ZHIKOV'S CONDITION

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  $\lambda(r) \sim r \exp(\frac{1}{\ln\ln \frac{1}{r}})$ but the condition fails: $\int\limits_0 \lambda(r)\frac{dr}{r} < +\infty$

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- The difficulties completely similar to that of C^1 regularity (Acerbi,Mingione $(2001, 2005)$,Bögelein,Habermann (2010), Ok $(2016, 2017)$),

  by using the auxiliary solutions,we need to estimate the integrals

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- different behavior with respect to non-logarithmic condition

  $v^{p-1}(1 + a(\cdot)\ln(1+v)) \;\sim\; p(\cdot)$-Laplace (only (g_μ) condition)

  $v^{p-1}(1 + \ln(1 + a(\cdot)v)) \;\sim\;$ double-phase (both (g_μ) and (g_λ) conditions)

$l_1, l_2 > 0$,following Krylov and Safonov consider the equation

$\max\limits_{B_{\rho\tau}(x_0)} u = \frac{u(x_0)}{2(1-\tau)^{l_1}} \left(\frac{\mu\left((1-\tau)\rho\right)}{\mu(\rho)}\right)^{l_2}, \tau \in (0,1)$; let τ_0 be the max.root

and $\overline{x}$: $u(\overline{x}) = \max\limits_{B_{\rho\tau_0}(x_0)} u = \frac{u(x_0)}{2(1-\tau_0)^{l_1}} \left(\frac{\mu\left((1-\tau_0)\rho\right)}{\mu(\rho)}\right)^{l_2} \qquad , r := \frac{(1-\tau_0)\rho}{2}$

Claim. $\exists\, \nu \in (0,1) : \left|\left\{x \in B_r(\overline{x}) : u(x) \geqslant \frac{u(\overline{x})}{2}\right\}\right| \geqslant \nu\, \mu^{-2n}(r)|B_r(\overline{x})|,$

SKETCH OF THE PROOF

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We stop here because a direct application of this inequality leads us to

condition with exp,instead,we use auxiliary solutions
v , $v = \frac{u(\overline{x})\lambda(\rho)}{2}$ on F,

$F := \{x \in B_r(\overline{x}) : u(x) \geqslant \lambda(\rho) \frac{u(\overline{x})}{2}\}$, $|F| \geqslant \nu\mu^{-2n}(r)|B_r(\overline{x})|$,

using lower bound of auxiliary solution and only after we use the Claim, we arrive at the required Harnack's inequality

- $\log$ Zhikov's condition

  C^1-regularity and Hölder continuity

  Antontsev,Zhikov,Acerbi,Mingione,Xu,Cheng,Alkhutov,Bögelein, Duzaar,Marcellini,Yao,Diening,Scharle,Schwarzacher, Hästo,Ok

  Wang,Bahja-$p(x,t) - Laplace$ (Harnack)

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  Hwang,Lieberman Hölder continuity :

  $u_t - div\Big(g(|\nabla u|)\nabla u/|\nabla u|\Big) = 0$,$(p,q)$-growth

  $p \geqslant 2$ or $q \leqslant 2$,

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- non-$\log$ Zhikov's condition of "$\exp$" type- Skr.,Voitovych

u-solution $\Rightarrow$ (Cacciopoli type inequality + Young's inequality),from the elliptic term of Eq. $\Rightarrow$ $\forall \varepsilon \in (0,1)$

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$$\leqslant \ \tfrac{\gamma}{\varepsilon}\tfrac{M_+(k)}{r} \iint\limits_{Q_{r,\theta}(x_0,t_0)\cap(k<u<l)} g(x,t,\tfrac{M_+(k)}{r})dxdt+$$
$$+\gamma\varepsilon^{p-1}\Big\{ \int\limits_{B_r(x_0)} (u-k)_+^2\zeta^q(x)\varphi(t_0-\theta)dx+$$
$$+ \iint\limits_{Q_{r,\theta}(x_0,t_0)} (u-k)_+^2\zeta^q(x) \mid \varphi_t \mid dxdt+$$
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similar inequality for $(u-l)_-$,

$M_+(k) := sup_{Q_{r,\theta}(x_0,t_0)}(u-k)_+$, $Q_{r,\theta}(x_0,t_0) := B_r(x_0)\times(t_0-\theta,t_0)$

From the parabolic term of Eq. ,by Young's inequality

$$\int\limits_{B_r(x_0)\times\{t\}} (u-k)_\pm^2 \zeta^q(x)\varphi(t)dx$$
$$\leqslant \int\limits_{B_r(x_0)\times\{t_0-\theta\}} (u-k)_\pm^2 \zeta^q(x)\varphi(t_0-\theta)dx+$$

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The same as in the elliptic case,the advantage of such $B_{1,g}(\Omega_T)$ classes is that we move away from Orlicz spaces,and we do not need any special properties of these spaces

$$\int\limits_{B_r(x_0)\times\{t\}} \ln^2 \frac{M_\pm(k)}{M_\pm(k)-(u-k)_\pm+a}\zeta^q(x)dx \leqslant$$

$$\leqslant \int\limits_{B_r(x_0)\times\{t_0-\theta\}} \ln^2 \frac{M_\pm(k)}{M_\pm(k)-(u-k)_\pm+a}\zeta^q(x)dx+$$

$$+\frac{\gamma}{r^2}\ln\frac{M_\pm(k)}{a}\iint\limits_{A_k^\pm}\psi\Big(x,t,(M_\pm(k)-(u-k)_\pm+a)|\nabla\zeta|\Big)dxdt$$

$$\psi(x,t,v) := \frac{g(x,t,v)}{v}$$

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$p > 2 \;\Rightarrow$ by (p,q)-growth condition $\psi \nearrow$

$q < 2 \;\Rightarrow$ by (p,q)-growth condition $\psi \searrow$

$p < 2 < q$??? $\Rightarrow$ additional assumptions

Fix (x_0, t_0) ,additionally assume : $\exists R = R(x_0, t_0) > 0$,$b_0, a_0 \geqslant 0$

PARABOLIC $B_{1,g}(\Omega_T)$ CLASSES

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if $p \geqslant 2 \Rightarrow \quad \forall\ (x, t) \in \Omega_T$ degenerate case $(b_0, a_0 = 0)$

if $q \leqslant 2 \Rightarrow \quad \forall\ (x, t) \in \Omega_T$ singular case $(b_0, a_0 = 0) \quad \Leftarrow$

$$\Leftarrow \left(\frac{w}{v}\right)^{p-1} \leqslant \frac{g(x,t,w)}{g(x,t,v)} \leqslant \left(\frac{w}{v}\right)^{q-1}, (x,t) \in \Omega_T, w \geqslant v > 0$$

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these additional assumptions arise naturally(see examples below)

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these additional assumptions arise naturally(see examples below)

- in the proof we use DiBenedetto's approach and his innovative scaling method

- $\psi(x,t,v) = v^{p-2} + a(x,t)v^{q-2}$, $\operatorname{osc}_{Q_{r,r}(x_0,t_0)} a(x,t) \leqslant \; A r^{q-p} \ln^{\beta} \frac{1}{r}$

CASE $p < 2 < q$

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  $a(x_0,t_0) > 0, \psi'(x_0,t_0,v) = v^{p-3}(p-2+(q-2)a(x_0,t_0)v^{q-p}) \geqslant 0,$
  if $v \geqslant (\frac{2-p}{(q-2)a(x_0,t_0)})^{\frac{1}{q-p}}$. $R : AR^{q-p} \ln^{\beta} \frac{1}{R} = 1/2a(x_0,t_0) \Rightarrow$
  degenerate if $v \geqslant b_0(A,p,q)R^{-1}$ $(a_0 = 1)$,

  $a(x_0,t_0) \asymp a(x,t)$ in $Q_{R,R}(x_0,t_0)$

CASE $p < 2 < q = p + 1$

- $\psi(x,t,v) = v^{p-2}(1 + \ln(1 + a(x,t)v)),\ \operatorname{osc}_{Q_{r,r}(x_0,t_0)} a(x,t) \leqslant\ Br \ln^{\beta} \frac{1}{r}$

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  All another cases can be considered completely similar...

MAIN RESULT

$g(x,t,v/r) \leqslant c(K)g(y,\tau,v/r),\ (x,t),(y,\tau) \in Q_{r,r}(x_0,t_0), br < v \leqslant K\lambda(r),$

$u \in B_{1,g}(\Omega_T)$,fix $(x_0,t_0) \in \Omega_T \Rightarrow$ if additionally $\int\limits_0 \lambda(s)ds/s = +\infty \ \Rightarrow$

u is continuous at (x_0,t_0)

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the conditions on the coefficients in the double phase,$(l+1)$-phase,degenerate double phase,.... are completely similar to that of elliptic case

HARNACK's INEQUALITY,SINGULAR CASE

- singular case : $2n/(n+1) < p < \min(2, q)$

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  case $p < 2 < q,\ a(x_0,t_0) = 0$-double-phase ??? still open

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non log case $a(x_0,t_0) > 0$ is also included

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p-Laplace evolution equation, $p < 2$- DiBenedetto,Gianazza and Vespri (2008)

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$$g(x,t,\tfrac{v}{r}) \leqslant c(K)\mu(r) g(y,\tau,\tfrac{v}{r}),\ (x,t),(y,\tau) \in Q_{r,r}(x_0,t_0), br < v \leqslant K$$

$$(\mathbf{A}(x,t,\xi) - \mathbf{A}(x,t,\eta))(\xi-\eta) > 0, \quad \xi,\eta \in \mathbb{R}^n, \quad \xi \neq \eta$$

$$\int\limits_0 \lambda_1(r) dr/r = +\infty, \lambda_1(r) = \lambda(\rho)\ \mu^{-\beta}(\rho) \quad \Rightarrow$$

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$$u(x_0,t_0) \leqslant \gamma\lambda_1(\rho)\{\min_{B_\rho(x_0)} u(\cdot,t) + (b+b_0)\rho^{1-\frac{a_0}{a_1}}\}, \quad a_1 > a_0/\overline{a}$$

$$\forall \ \ t: \quad t_0 - \frac{\rho^2}{\psi(x_0,t_0,\frac{\delta\lambda_1(\rho)u(x_0,t_0)}{\rho})} \leqslant t \leqslant t_0 + \frac{\rho^2}{\psi(x_0,t_0,\frac{\delta\lambda_1(\rho)u(x_0,t_0)}{\rho})}.$$

$$\beta = \tfrac{1}{2-p} + \tfrac{n(1+2(2-p))}{p+n(p-2)},\ p+n(p-2) > 0$$

DIFFICULTIES

The difficulties arising in the parabolic case completely similar to that of elliptic case,namely having information on the measure of the "positivity set" of u over the ball $B_r(y)$ for some time level τ:

$$|\{B_r(y) : u(\cdot, \tau) \geqslant N\}| \geqslant \alpha(r)|B_r(y)|,$$

with some $r > 0, N > 0$ and $\alpha(r) \in (0,1), \alpha(r) \to 0,$ as $r \to 0$, and using the standard DiBenedetto's arguments, we inevitably arrive to the estimate

$$u(x,t) \geqslant N\gamma_1^{-1} e^{-\gamma_2 \alpha^{-\beta}(r)}\ ,\ x \in B_{2r}(y),$$

for some time level $t > \tau$ and with some $\gamma_1, \gamma_2, \beta > 1$. This estimate leads us to exp type condition

$$\int\limits_0 e^{-\gamma_2 \alpha^{-\beta}(r)\mu^{\gamma}(r)} \frac{dr}{r} = +\infty$$

An important step in the proof of Harnack's inequality is the so-called $L^1 - L^\infty$ Harnack's inequality, DiBenedetto (p-Laplace evolution equation, 1993) :

$$\sup_{Q^+_{r,t-s}(y,s)} u \leqslant \gamma\left(\frac{r^p}{t-s}\right)^{\frac{n}{\varkappa}} \left(r^{-n} \inf_{2s-t<\tau<t} \int_{B_{2r}(y)} u(x,\tau)dx\right)^{\frac{p}{\varkappa}} +$$

$$+\gamma\left(\frac{t-s}{r^p}\right)^{\frac{1}{2-p}}, \ \varkappa = p + n(p-2) > 0$$

$$\varphi_Q(v) := \frac{v^{n+1}}{[G_Q^{-1}(v^2)]^n}, \quad G_Q(v) := \inf_{(x,t)\in Q} G(x,t,v), \ \varphi_Q(v) \nearrow$$

$$\psi_Q(v) := \sup_{(x,t)\in Q} \psi(x,t,v), \ G(x,t,v) := \int_0^v g(x,t,z)dz$$

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$$\sup_{Q^+_{r,t-s}(y,s)} u \leqslant$$

$$\gamma(t-s)^{\frac{1}{2}} \varphi^{-1}_{Q_{r,t-s}(y,s)} \left((t-s)^{-\frac{n+1}{2}} \inf_{2s-t<\tau<t} \int_{B_{2r}(y)} u(x,\tau)dx \right)$$

$$+\gamma(t-s)^{\frac{1}{2}} \varphi^{-1}_{Q_{r,t-s}(y,s)} \left(\left(\frac{r^2}{t-s}\right)^{\frac{n+1}{2}} \psi^{-1}_{Q_{r,t-s}(y,s)}\left(\frac{r^2}{t-s}\right) \right) +$$

$$+\gamma r \psi^{-1}_{Q_{r,t-s}(y,s)}\left(\frac{r^2}{t-s}\right),$$

provided $\psi^{-1}_{Q_{r,t-s}(y,s)}\left(\frac{r^2}{t-s}\right) \geqslant (b+b_0) r^{-\frac{a_0}{a}}$

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$\varphi_Q(v) \to (p\text{-Laplace})v^{\frac{p+n(p-2)}{p}}, r\psi^{-1}_{Q_{r,t-s}(y,s)}\left(\frac{r^2}{t-s}\right) \to \left(\frac{t-s}{r^p}\right)^{\frac{1}{2-p}}$

L^1 form of the HARNACK INEQUALITY

- if $\psi^{-1}_{Q_{r,t-s}(y,s)}(\frac{r^2}{t-s}) \geqslant (b+b_0)r^{-\frac{a_0}{a}}, \quad \Rightarrow$

$$\sup_{s<\tau<t} \int\limits_{B_r(y)} u(x,\tau)dx \leqslant \gamma \inf_{s<\tau<t} \int\limits_{B_{2r}(y)} u(x,\tau)dx$$

$$+\gamma r \psi^{-1}_{Q_{r,t-s}(y,s)}\left(\frac{r^2}{t-s}\right),$$

p-Laplace evolution Eq. :

$$r\psi^{-1}_{Q_{r,t-s}(y,s)}\left(\frac{r^2}{t-s}\right) = \left(\frac{t-s}{r^p}\right)^{\frac{1}{2-p}}$$

The next important step in the proof of Harnack's inequality is the so-called theorem on the expansion of positivity

$g(x_1, t_1, \frac{v}{r}) \leqslant c(K) g(x_2, t_2, \frac{v}{r}),\ (x_1, t_1), (x_2, t_2) \in Q_{r,r}(y, s), br < v \leqslant K\lambda(r)$

$\psi(y, s, v)$ is non increasing $for\ v \geqslant b_0 R^{-a_0}$
and let for some $0 < \rho \leqslant R^{a_1}, a_1 > a_0/\overline{a},\ 0 < N \leqslant \lambda(\rho), \alpha \in (0, 1)$

$$|\{B_\rho(y) : u(\cdot, t) \geqslant N\}| \geqslant \alpha |B_\rho(y)|, \forall\ t \in (s - \frac{\rho^2}{\psi(y, s, \frac{\delta N}{\rho})}, s)$$

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$$|\{B_\rho(y):u(\cdot,t)\geqslant N\}|\geqslant\alpha|B_\rho(y)|,\forall\ t\in(s-\frac{\rho^2}{\psi(y,s,\frac{\delta N}{\rho})},s)$$

$\Rightarrow$ either $N\leqslant(b+b_0)\rho^{1-a_0/a_1}$, or $\exists\ \sigma\in(0,1)$

$$u(x,t)\geqslant\sigma N,x\in B_{2\rho}(y),\quad\forall t\in(s-\frac{1}{2}\frac{\rho^2}{\psi(y,s,\frac{\delta N}{\rho})},s)$$

In the proof of this theorem for the function ψ, we used a condition similar to the continuity condition

SKETCH OF THE PROOF, NON log CONDITION

following to DiBenedetto,Gianazza and Vespri for $\tau \in (0,1)$ consider the equation

$$M_\tau := sup_{B_{\rho\tau(x_0)}} u(\cdot, t_0) = N_\tau$$

$$N_\tau := \frac{1}{2} u(x_0, t_0) \frac{\lambda(\rho)}{\lambda((1-\tau)\rho)} \left(\frac{\mu((1-\tau)\rho)}{\mu(\rho)} \right)^\beta (1-\tau)^{-m}$$

τ_0-the maximal root, $\tau_1 = 1 - 4^{-\frac{1}{m}}(1-\tau_0), 2r = (1 - 4^{-\frac{1}{m}}(1-\tau_0))\rho,$

$Claim.$ $\exists\ \overline{x} : |\{B_r(\overline{x}) : u(\cdot, s) \geqslant \varepsilon N_{\tau_0} \overline{\lambda}(2r)\}| \geqslant \alpha(r)|B_r(\overline{x})|$

$\forall s : |s - t_0| \leqslant \overline{\theta}_0,\ \overline{\theta}_0 = \frac{r^2}{\psi(\overline{x}, t_0, \frac{\delta N_{\tau 0}}{r} \overline{\lambda}(2r))},$
$\overline{\lambda}(2r) = \mu^{-m_1}(2r)\lambda(2r), \alpha(r) = \alpha\mu^{-m_2}(2r),\ \exists m_1, m_2 > 0$

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In the case $\lambda(r) = \mu(r) = 1$ (log case) this inequality is applicable similarly to DiBenedetto,Gianazza and Vespri(Acta Math.,2008).

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Direct application leads to condition with exp,instead,we use old Landis's ideas and his so-called growth lemma.We use $L^1 - L^\infty$, L^1 forms of Harnack's inequality and the result on the expansion of positivity,we also use auxiliary solutions $v\;\; \forall s : |s - t_0| \leqslant \overline{\theta}_0$,as initial time , we use the Claim, after iteration we arrive at the required Harnack's inequality

$$g(x,t,\frac{v}{r}) \leqslant c(K) g(t,\tau,\frac{v}{r}),\ (x,t),(y,\tau)\in Q_{r,r}(x_0,t_0), 0<v\leqslant K$$

$\Rightarrow b=0$, $p(x,t)$-Laplace $\{b=1\}$???? $\qquad q<2 \Rightarrow b_0=0$

$Q := Q_{|x-x_0|,(t-t_0)}(x_0,t_0), h(v) := \frac{g^{-1}(v)}{v} \nearrow,\ g(v) := \inf_Q g(\cdot,\cdot,v)$

$B_{8|x-x_0|}(x_0) \subset B_{R_0}(x_0),\ 0 < t-t_0 \leqslant \epsilon R_0 < \epsilon t_0$

on the right-hand side the term $(b+b_0)\rho^{1-\frac{a_0}{a_1}}$ is discarded

$$\frac{u(x,t)}{u(x_0,t_0)} \geqslant \gamma \left\{1+\gamma_1 \frac{u(x_0,t_0)}{h^{-1}\left(\frac{t-t_0}{|x-x_0|^2}\right)} \frac{|x-x_0|}{t-t_0}\right\}^{-1}.$$

this estimate gives the same decay in the space variables as the Barenblatt-type solution

SUB POTENTIAL LOWER BOUND, log CASE

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p-Laplace evolution eq.(DiBenedetto,Gianazza,Vespri,2010)

$$\sim \frac{u(x,t)}{u(x_0,t_0)} \geqslant \gamma \left\{ 1 + \gamma_1 u(x_0,t_0) \left(\frac{|x-x_0|^p}{t-t_0} \right)^{\frac{1}{2-p}} \right\}^{-1}$$

$$\sim \gamma \left\{ 1 + \gamma_1 u(x_0,t_0)^{\frac{2-p}{p-1}} \left(\frac{|x-x_0|^p}{t-t_0} \right)^{\frac{1}{p-1}} \right\}^{-\frac{p-1}{2-p}}$$

THANKS

FOR

ATTENTION