Boundedness results for degenerate elliptic integrals

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Outline

- Quasi-minimizers and local minimizers
- A brief review of local boundedness results under (p, q)-growth
- Local boundedness for degenerate (better: non-uniformly) elliptic integrals
- DeGiorgi's technique vs Moser's iteration method
- De Giorgi's method in the vectorial case: local boundedness results

Introduction

Consider an integral functional of the Calculus of Variations:

$$F(u) := \int_\Omega f(x, u, Du)\, dx$$

Definition

$u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a *quasi-minimizer* of F if

- $x \mapsto f(x, u(x), Du(x)) \in L^1_{\mathrm{loc}}(\Omega)$
- $\exists Q \geq 1$:

$$\int_{\mathrm{supp}\,\varphi} f(x, u, Du)\, dx \leq Q \int_{\mathrm{supp}\,\varphi} f(x, u + \varphi, D(u + \varphi))\, dx,$$

for every $\varphi \in W^{1,1}(\Omega)$ with $\mathrm{supp}\,\varphi \Subset \Omega$.

If $Q = 1$, we say that u is a *local minimizer* of F.

Local boundedness: a negative result

Scalar case $u : B_1(0) \to \mathbb{R}$, $B_1(0) \subseteq \mathbb{R}^n$

Example (Giaquinta (1987), Marcellini (1987))

$$\int_{B_1(0)} \Big(\sum_{i=1}^{n-1} |u_{x_i}|^2 + c|u_{x_n}|^q\Big)\, dx, \qquad q > 2$$

has an unbounded minimizer if $\dfrac{1}{q} < \dfrac{1}{2} - \dfrac{1}{n-1} \Leftrightarrow q > \overline{p}^*$.

$\overline{p}$ = harmonic mean of $(\underbrace{2, \dots, 2}_{n-1}, q)$: $\quad \dfrac{1}{\overline{\mathbf{p}}} = \dfrac{1}{n}\left(\underbrace{\dfrac{1}{2} + \cdots + \dfrac{1}{2}}_{n-1} + \dfrac{1}{q}\right)$

The integrand f satisfies a (p_i, p_i)-growth:

$$\sum_{i=1}^{n} |u_{x_i}|^{p_i} \le f(Du) \le c\Big(\sum_{i=1}^{n} |u_{x_i}|^{p_i} + 1\Big) \qquad p_i \ge 1.$$

with $(p_1, \cdots, p_n) = (\underbrace{2, \dots, 2}_{n-1}, q)$.

(p_i, p_i)-growth: local boundedness in the scalar case

Theorem (Boccardo-Marcellini-Sbordone (1990), Fusco-Sbordone (1990/3))

$$F(u) = \int_\Omega f(x, u, Du)\, dx$$

$$\sum_{i=1}^n |u_{x_i}|^{p_i} \le f(x, u, Du) \le c(\sum_{i=1}^n |u_{x_i}|^{p_i} + 1) \qquad p_i \ge 1.$$

If $\max\{p_i\} \le (\overline{p})^*$ *then local minimizers are locally bounded.*

$\overline{p}$ = harmonic mean of $(p_1, \dots, p_n)$

See also Stroffolini (1991) for the bound $\overline{p} < (\overline{p})^*$.

Remark:
The energy density has also (p, q)-growth:

$$\boxed{|z|^p \le f(x, u, z) \le c(|z|^q + 1)}, \qquad p := \min\{p_i\},\ q := \max\{p_i\}.$$

The bound $q \le (\overline{p})^*$ is better than $q \le p^*$.
The more hypotheses on the structure there are, the better the condition on the exponents.

(p_i, q)-growth: local boundedness

$$\sum_{i=1}^{n} |u_{x_i}|^{p_i} \leq f(x,u,Du) \leq c\left(\sum_{i=1}^{n} |u_{x_i}|^{q} + a(x)\right) \qquad 1 \leq p_i \leq q$$

Scalar case C.-Marcellini-Mascolo 2016

Let f be convex in $(u,z) \in \mathbb{R} \times \mathbb{R}^n$

• $1 \leq p_i \leq q \leq \overline{p}^*$

• $a \in L^s_{\mathrm{loc}}(\Omega)$, with $s > \frac{n}{\overline{p}}$.

Then the **quasi-minimizers** of F are **locally bounded** in Ω.

Vectorial case $(u = (u^1, \cdots, u^m))$ C.-Marcellini-Mascolo 2013

Additional assumptions are needed:

- $f(x,Du) = g(x, |u_{x_1}|, \cdots, |u_{x_n}|)$
- Δ_2-condition: $\exists \gamma > 1 \; : \; f(x,tz) \leq t^{\gamma} f(x,z) \qquad \forall t > 1$
- $z \mapsto f(x,z)$ of class C^1.

Then the **local minimizers** of F are **locally bounded** in Ω.

A recent improvement in the scalar case

In the **scalar case**, with **pure** (p, q)**-growth**

$$|z|^p \leq f(x, u, z) \leq L \left\{ |z|^q + a(x) \right\},$$

it is proved in Hirsch-Schäffner (2020) that the local boundedness of minimizers holds if

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n-1}$$

(also a Δ_2-type condition is assumed).

This condition is weaker than

$$\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n} \Leftrightarrow q \leq p^*,$$

and it is **sharp**, because of the counterexamples by Giaquinta and Marcellini.

Local boundedness of local/quasi-minimizers of **degenerate** elliptic integral functionals

VECTORIAL CASE:
C.-Marcellini-Mascolo *Nonlinear Anal.* 2018

SCALAR CASE:
Biagi-C.-Mascolo *J. Math. Anal. Appl.* 2020

A motivation

A motivation to study the regularity of degenerate elliptic functionals is given by the so called **double phase functional**.

The double phase functional is

$$\int_\Omega f(x, Du)\, dx = \int_\Omega \{|Du(x)|^p + b(x)|Du(x)|^q\}\, dx \qquad 0 \leq b(x) \leq L,\ 1 < p < q.$$

The functional is usually studied as a functional satisfying the (p, q)-growth condition:

$$\boxed{|z|^p \leq f(x, z) \leq L(|z|^q + 1)}$$

If $b \in C^{0,\alpha}$, $0 < \alpha \leq 1$, and $1 < p < \boxed{q < p + \frac{p\alpha}{n}}$ then local minimizers are in $C^{1,\beta}_{\text{loc}}$ [Colombo-Mingione (first paper) 2015]

The **a priori local boundedness of minimizers** allows to relax the conditions on the exponents.
If $b \in C^{0,\alpha}$ and $1 < p < \boxed{q \leq p + \alpha}$ then locally bounded minimizers are in $C^{1,\beta}_{\text{loc}}$ [Colombo-Mingione (second paper) 2015]

$$\int_\Omega f(x, Du)\,dx = \int_\Omega \left\{|Du(x)|^p + b(x)|Du(x)|^q\right\} dx, \qquad 0 \le b(x) \le L,\ 1 < p < q.$$

$$\boxed{|z|^p \le f(x,z) \le L(|z|^q + 1)}$$

Another growth condition can be considered:

$$\boxed{b(x)|z|^q \le f(x,z) \le L(|z|^q + 1)}$$

Roughly, the double phase functional has a “double growth condition”.

An example

$$\int_{B_1(0)} f(x, Du)\,dx = \int_{B_1(0)} \left\{|Du(x)|^p + |x|^\alpha |Du(x)|^q\right\} dx,$$

where $1 < p \leq q,\ \alpha > 0,\ B_1(0) \subset \mathbb{R}^n,\ n \geq 2.$

Two growth conditions:

$$\boxed{|z|^p \leq f(x,z) \leq c\left(|z|^q + 1\right)}$$

$$\boxed{|x|^\alpha |z|^q \leq f(x,z) \leq |z|^q}$$

This second point of view gives different conditions on the data than the other more classical approach:

$$\alpha < \min\{q, n(q-1)\} \quad \text{vs} \quad q < p^*.$$

To study the regularity for functionals whose energy densities satisfy the growth condition

$$|z|^p \leq f(x,z) \leq L(|z|^q + 1), \qquad 1 < p \leq q$$

or

$$b(x)|z|^q \leq f(x,z) \leq L(|z|^q + 1), \qquad 0 \leq b(x) \leq L, \quad 1 < p \leq q$$

with a unified approach, it is possible to consider functionals satisfying the growth

$$b(x)|z|^p \leq f(x,z) \leq L(|z|^q + 1), \qquad 0 \leq b(x) \leq L, \quad 1 < p \leq q,$$

or, more general,

$$\boxed{\lambda(x)|z|^p \leq f(x,z) \leq \mu(x)(|z|^q + 1)} \qquad 0 \leq \lambda(x) \leq \mu(x), \quad 1 < p \leq q$$

A pioneering paper on this subject is [Trudinger 1971] about the elliptic equation in divergence form:

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = 0,$$

where $a : \Omega \to \mathbb{R}^{n \times n}$ is a measurable symmetric matrix field on $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$.

Main assumption:

$$\lambda(x)|z|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) z_i z_j \leq \mu(x)|z|^2, \qquad 0 \leq \lambda \leq \mu$$

with $\lambda, \mu \geq 0$ measurable functions.

$$\lambda(x)|z|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\, z_i z_j \leq \mu(x)|z|^2, \qquad 0 \leq \lambda \leq \mu.$$

If $0 < c \leq \lambda(x) \leq \mu(x) \leq L$, i.e.

$$\frac{1}{\lambda}, \mu \in L^\infty$$

then a is **uniformly elliptic**.

In this case, by DeGiorgi and Nash, the weak solutions are Hölder continuous.
Moser showed that the Harnack inequality holds and that this implies the Hölder continuity.

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = 0.$$

$$\lambda(x)|z|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) z_i z_j \leq \mu(x)|z|^2 \qquad \lambda, \mu \geq 0.$$

Theorem [Trudinger 1971]

Assume

$$\frac{1}{\lambda} \in L^r \qquad \text{and} \qquad \mu \in L^s$$

with

$$1 < r, s \leq +\infty \qquad \frac{1}{r} + \frac{1}{s} < \frac{2}{n}.$$

Then the weak solutions are **locally bounded**.

The functional of the Calculus of Variations related to

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = 0$$

is

$$F(u) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i} u_{x_j}\, dx.$$

If

$$\lambda(x)|z|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\, z_i z_j \leq \mu(x)|z|^2, \qquad 0 \leq \lambda \leq \mu$$

then the growth condition of the energy density is

$$\lambda(x)|z|^2 \leq f(x,z) \leq \mu(x)|z|^2.$$

In Biagi-C.-Mascolo 2020 we consider

$$\boxed{\lambda(x)\, |z|^p \leq f(x,u,z) \leq \mu(x)\, (|z|^p + |u|^\tau) + a(x)} \qquad 0 \leq \lambda \leq \mu$$

$$\int_\Omega f(x,u,Du)\,dx, \qquad u \in W^{1,1}(\Omega,\mathbb{R})$$

At first, we give our result in **simplified form**:

$$\boxed{\lambda(x)\,|z|^p \le f(x,u,z) \le \mu(x)\,(|z|^p + |u|^p) + 1}, \qquad \lambda \ge 0.$$

- f Carathéodory
- $f(x,\cdot,\cdot)$ is convex in (u,ξ) for a.e. $x \in \Omega$.

Theorem [Biagi-C.-Mascolo 2020]

Let $\frac{1}{\lambda} \in L^r_{\rm loc}(\Omega)$, $\quad \mu \in L^s_{\rm loc}(\Omega)$.

Assume:

$$\max\{1, \frac{1}{p-1}\} \le r \le +\infty, \qquad 1 < s \le +\infty$$

$$\boxed{\frac{1}{r} + \frac{1}{s} < \frac{p}{n}.}$$

Then the **scalar quasi-minimizers** are locally bounded.

Remark

- The relationship between the exponents is

$$\frac{1}{r} + \frac{1}{s} < \frac{p}{n}.$$

Therefore, if $p = 2$, we get

$$\frac{1}{r} + \frac{1}{s} < \frac{2}{n}$$

that is the same assumption given by Trudinger.

Example

$$F(u) = \int_{B_1(0)} |x|^\alpha \left(|\nabla u|^p + |u|^p\right) dx, \qquad p > 1, \quad \alpha > 0$$

If

$$\alpha < \min\{p, n(p-1)\}$$

then the quasi-minimizers of F are locally bounded.

Example

$$F(u) = \int_{B_1(0)} \frac{1}{|x|^\alpha} \left(|\nabla u|^p + |u|^p\right) dx, \qquad p > 1, \quad \alpha > 0.$$

If

$$\alpha < p$$

then the quasi-minimizers of F are locally bounded.

General version

$$\boxed{\lambda(x)\,|z|^p \le f(x,u,z) \le \mu(x)\,(|z|^p + |u|^\tau) + a(x)} \qquad 0 \le \lambda \le \mu$$

with $\tau \ge p > 1$.

Theorem [Biagi-C.-Mascolo, *J. Math. Anal. Appl.* 2020]

Let $\frac{1}{\lambda} \in L^r_{\mathrm{loc}}(\Omega)$, $\quad \mu \in L^s_{\mathrm{loc}}(\Omega)$, $\quad a \in L^\sigma_{\mathrm{loc}}(\Omega)$.

Assume:

$$\max\{1, \frac{1}{p-1}\} \le r \le +\infty, \qquad 1 < s, \sigma \le +\infty$$

$$\boxed{\frac{1}{r} + \max\left\{\frac{1}{s}, \frac{1}{\sigma}\right\} < \frac{p}{n}}$$

$$\tau \frac{s}{s-1} < \left(\frac{pr}{r+1}\right)^* .$$

Then the **scalar quasi-minimizers** are locally bounded.

Remark

- If $\lambda^{-1}, \mu \in L^\infty$, then

$$\boxed{c_1 |z|^p \leq f(x,u,z) \leq c_2 \left(|z|^p + |u|^\tau + a(x)\right)}$$

and our result is:
If $1 < p$ and

$$\tau < p^*$$

and

$$a(x) \in L^\sigma, \qquad \text{with } \sigma > \frac{n}{p}$$

then the quasi-minimizers are locally bounded.

Example

$$F(u) = \int_\Omega h(x)\Big(|Du(x)|^p + |u(x)|^\tau\Big)\,dx \qquad 1 < p \le \tau,\ h \ge 0.$$

If

$$\frac{1}{h} \in L^r_{\rm loc}(\Omega), \qquad h \in L^s_{\rm loc}(\Omega) \qquad r \ge \max\left\{1, \frac{1}{p-1}\right\},\ s > 1$$

and

$$\frac{1}{r} + \frac{1}{s} < \frac{p}{n}$$

$$\tau\frac{s}{s-1} < \left(\frac{pr}{r+1}\right)^*,$$

then the quasi-minimizers of F are locally bounded.

Vectorial case

C.-Marcellini-Mascolo *Nonlinear Analysis* (2018), dedicated to C. Sbordone

We consider vectorial local minimizers of

$$F(v;\Omega) = \int_\Omega f(x, Dv)\, dx$$

where

$$\lambda(x)|z|^p \leq f(x,z) \leq \mu(x)\left(|z|^q + 1\right) \qquad 0 \leq \lambda \leq \mu,\ 1 < p \leq q$$

with

$$\boxed{f(x,z) = g(x,|z|)}$$

with $g : \Omega \times [0,\infty) \to [0,\infty)$,

$t \mapsto g(x,t)$ convex and C^1, of class Δ_2.

Theorem [C.-Marcellini-Mascolo, *Nonlinear Analysis* 2018]

Suppose u is a **vectorial** local minimizer of

$$F(v;\Omega) = \int_\Omega f(x, Dv)\,dx$$

where f is as described above, in particular

$$\begin{cases} \lambda(x)|z|^p \le f(x,z) \le \mu(x)\,(|z|^q + 1) \\ \frac{1}{\lambda} \in L^r_{\mathrm{loc}}(\Omega), \qquad \mu \in L^s_{\mathrm{loc}}(\Omega) \end{cases}$$

If $1 \le r \le +\infty$, $1 < s \le +\infty$ (if $p \le 2$ then we also require $r > \frac{1}{p-1}$) and

$$\boxed{\frac{1}{pr} + \frac{1}{qs} + \frac{1}{p} - \frac{1}{q} < \frac{1}{n}}$$

then u is locally bounded.

- If $r = s = +\infty$, then

$$c_1 |z|^p \leq f(x,z) \leq c_2 \left(|z|^q + 1\right)$$

The condition on the exponents

$$\frac{1}{pr} + \frac{1}{qs} + \frac{1}{p} - \frac{1}{q} < \frac{1}{n} \, ;$$

becomes

$$\boxed{q < p^*}.$$

- Assume $p = q$

The condition on the exponents becomes

$$\boxed{\frac{1}{r} + \frac{1}{s} < \frac{p}{n}},$$

that coincides with the condition assumed in the paper Biagi-C.-Mascolo related to the scalar case.

Example

$$F(u) = \int_{B_1(0)} \{|Du(x)|^p + |x|^\alpha |Du(x)|^q\}\, dx, \qquad 1 < p < q, \;\; \alpha > 0$$

where $1 < p < n$.
If

$$\alpha < \min\{q, n(q-1)\}$$

then the local minimizers are locally bounded.

Example

$$F(u) = \int_{B_1(0)} f(x, Du)\, dx$$

$$|x|^\alpha |Du(x)|^p \leq f(x, Du) \leq |Du(x)|^q, \qquad 1 < p < q, \;\; \alpha > 0$$

with $1 < p < n$.

If

$$q\left(1 + \frac{\alpha}{n-p}\right) < p^*$$

then the local minimizers are locally bounded.

Trudinger's result has been recently extended.

$$\lambda(x)|z|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) z_i z_j \leq \mu(x)|z|^2.$$

Theorem [Bella-Schäffner 2021]

If

$$\frac{1}{\lambda} \in L^r \qquad \text{and} \qquad \mu \in L^s$$

with

$$\frac{1}{r} + \frac{1}{s} < \frac{2}{n-1}$$

then the weak solutions are **locally bounded**.

By a result in **Franchi-Serapioni-Serra Cassano (1998)**, this result is essentially optimal.

Existence

$$\boxed{\lambda(x)\,|z|^{p} \leq f(x,z)} \quad \lambda \geq 0.$$

Assume $\frac{1}{\lambda} \in L^r(\Omega)$ with $r > 1$.

With simple computations:

$$\|\lambda^{-1}\|_{L^r(\Omega)}^{-1}\,\|Du\|^p_{L^{\frac{pr}{r+1}}(\Omega,\mathbb{R}^n)} \leq \int_\Omega f(x,Du)\,dx.$$

Therefore

$$W^{1,F}(\Omega) \subseteq W^{1,\frac{pr}{r+1}}(\Omega).$$

If $\frac{pr}{r+1} > 1$ and if we impose convex assumptions on f and appropriate boundary conditions, then from standard direct methods of the Calculus of Variations we get the existence of minimizers in $W^{1,\frac{pr}{r+1}}(\Omega)$.

In the proofs it is useful the following lemma.

Lemma.

Consider a bounded open set $\Omega \subset \mathbb{R}^n$ and let p and r be such that $p > 1$ and $r \in [1, \infty]$ (if $p \leq 2$ then we also require $r \geq \frac{1}{p-1}$).

Let

- $v \in W_0^{1,\frac{pr}{r+1}}(\Omega; \mathbb{R}^m)$ ($v \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ if $r = \infty$)
- $\lambda : \Omega \to [0, \infty)$ measurable function such that $\lambda^{-1} \in L^r(\Omega)$.

Then

$$\left\{ \int_\Omega |v|^{\sigma^*} \, dx \right\}^{\frac{p}{\sigma^*}} \leq c \|\lambda^{-1}\|_{L^r(\Omega)} \int_\Omega \lambda \, |Dv|^p \, dx,$$

where $\sigma := \frac{pr}{r+1}$ ($\sigma := p$ if $r = \infty$).

Strategy of the Proofs

In the scalar case we use of the **DeGiorgi**'s method (DeGiorgi 1957), since we can consider the super/sub-level sets of scalar functions.

As a consequence, we can skip the Euler's equation, so

- we do not need $z \mapsto f(x, z)$ to be smooth,
- we can consider quasi-minimizers and not only local minimizers.

In the vectorial case we use the Moser's iteration method, since we cannot consider the super/sub-level sets of vectorial functions.

As a consequence,

- we use the first variation of the functional, so we need $z \mapsto f(x, z)$ smooth,
- we can only consider local minimizers.

Moreover, we need to assume the dependence of f on the modulus of Du, since it is known that in the vectorial case, without this assumption, the local boundedness of minimizers may fail (example by DeGiorgi 1968).

A question

Is it possible to use the De Giorgi's method to study the regularity of vectorial minimizers?

Yes, in very special cases, as in:

- C.-Leonetti-Mascolo, *Arch. Rat. Mech. Anal.* (2017)
- Carozza-Gao-Giova-Leonetti, *J. Optim. Theory Appl.* (2018)

  Local boundedness of vectorial minimizers of polyconvex problems

- C.-Focardi-Leonetti-Mascolo, *Adv. Nonlinear Anal.*, (2020)

  Local Hölder continuity of vectorial minimizers of polyconvex and rank one convex problems

The De Giorgi's method is applied to each component of the minimizer u.

C.-Leonetti-Mascolo, (2017)

$$\mathcal{F}(u) = \int_\Omega \left(\sum_{\alpha=1}^{3} F_\alpha(x, Du^\alpha) + \sum_{\alpha=1}^{3} G_\alpha(x, (\mathrm{adj}_2\, Du)^\alpha) + H(x, \det Du) \right) dx,$$

$F_\alpha,\ G_\alpha : \Omega \times \mathbb{R}^3 \to [0, +\infty)$, with $\alpha \in \{1, 2, 3\}$, $H : \Omega \times \mathbb{R} \to [0, +\infty)$

F_α, G_α, H convex in the last variable

The energy density is a polyconvex function.

The proof of the local boundedness result relies on the De Giorgi's method (1957).

$$\mathcal{F}(u) = \int_\Omega \left(\sum_{\alpha=1}^{3} F_\alpha(x, Du^\alpha) + \sum_{\alpha=1}^{3} G_\alpha(x, (\mathrm{adj}_2\, Du)^\alpha) + H(x, \det Du) \right) dx,$$

$F_\alpha,\ G_\alpha : \Omega \times \mathbb{R}^3 \to [0, +\infty)$, with $\alpha \in \{1, 2, 3\}$, $H : \Omega \times \mathbb{R} \to [0, +\infty)$

F_α, G_α, H convex in the last variable

$$|\lambda|^p \lesssim F_\alpha(x, \lambda) \lesssim |\lambda|^p + a(x) \qquad \forall \lambda \in \mathbb{R}^3$$

$$|\lambda|^q \lesssim G_\alpha(x, \lambda) \lesssim |\lambda|^q + b(x) \qquad \forall \lambda \in \mathbb{R}^3$$

$$0 \le H(x, t) \lesssim |t|^{\mathbf{r}} + c(x) \qquad \forall t \in \mathbb{R}$$

where $1 < p \le 3$, $1 < q \le 3$, $1 \le \mathbf{r} \le 3$ and $a, b, c \in L^s(\Omega)$, $s > 1$.

Theorem [C.-Leonetti-Mascolo, *Arch. Rat. Mech. Anal.* (2017)]

Assume

$$\frac{p}{p^*} < \min\left\{ 1 - \frac{qp^*}{p(p^* - q)}, 1 - \frac{rp^*}{q(p^* - r)}, 1 - \frac{1}{s} \right\},$$

Then the **local minimizers** $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^3)$ of $\mathcal{F}$ are **locally bounded**.

Notice that

$$\frac{p}{p^*} < 1 - \frac{1}{s} \iff s > \frac{3}{p} \quad \text{(as in the scalar case).}$$

Example

Consider

$$\min\left\{\mathcal{F}(u)\ :\ u\in\bar{u}+W_0^{1,14/5}(\Omega;\mathbb{R}^3)\right\},\qquad \bar{u}\in W^{1,14/5}(\Omega;\mathbb{R}^3),$$

where

$$\mathcal{F}(u)=\int_\Omega\left(\sum_{\alpha=1}^{3}|Du^\alpha|^{14/5}+|\operatorname{adj}_2 Du|^2+|\det Du|^{3/2}\right)dx.$$

Then there **exists** a solution to this minimization problem.

Moreover, the solutions are **locally bounded**.

Sketch of the proof

The idea is to prove, **separately**, that u^1, u^2 and u^3 are locally bounded

To prove that u^1 is locally bounded we use the **De Giorgi method**.

Step 1. Caccioppoli inequality for u^1

Step 2. Decay of the *excess* on super(sub)-level sets for u^1

Step 3. Iteration argument

$$\Rightarrow \; u^1 \text{ is locally bounded.}$$

C.-Focardi-Leonetti-Mascolo, (2020)

Hölder continuity of minimizers of polyconvex and rank one convex problems

• the **proof follows the C.-Leonetti-Mascolo proof** and it relies on the De Giorgi's method (1957).

• we get the Hölder continuity by proving that each component of the minimizers are in the suitable **De Giorgi class**

• in the **polyconvex** case we have more structure than in the **rank one convex** case: so better conditions on the exponents.

The polyconvex case (n=m=3)

$$\mathcal{F}(u) = \int_\Omega \left(\sum_{\alpha=1}^{3} F_\alpha(x, Du^\alpha) + \sum_{\alpha=1}^{3} G_\alpha(x, (\mathrm{adj}_2\, Du)^\alpha) \right) dx,$$

$F_\alpha, G_\alpha : \Omega \times \mathbb{R}^3 \to [0, +\infty)$, $F_\alpha(x, \cdot)$ and $G_\alpha(x, \cdot)$ convex

$$|\lambda|^p \lesssim F_\alpha(x, \lambda) \lesssim |\lambda|^p + a(x) \qquad \forall \lambda \in \mathbb{R}^3$$
$$0 \leq G_\alpha(x, \lambda) \lesssim |\lambda|^q + b(x) \qquad \forall \lambda \in \mathbb{R}^3$$

where $1 \leq q < p \leq 3$ and $a, b \in L^s(\Omega)$, $s > 1$.

Theorem [C.-Focardi-Leonetti-Mascolo (2020)]

Assume

$$1 \leq q < \frac{p^2}{p+3}, \qquad s > \frac{3}{p}.$$

Then the **local minimizers** $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^3)$ of $\mathcal{F}$ are **locally Hölder continuous**.

Example (Polyconvex, but not convex)

$n = m = 3$

$$\mathcal{F}(u) = \int_\Omega \left\{ \sum_{\alpha=1}^{3} |Du^\alpha|^p + \left(1 + ((\mathrm{adj}_2\, Du)_{11} - 1)^2\right)^{\frac{q}{2}} \right\} dx$$

If

$$1 \leq q < \frac{p^2}{3+p},$$

then **local minimizers** $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^3)$ of $\mathcal{F}$ are **locally Hölder continuous**.

The rank one convex case

$$\mathcal{F}(u) = \int_\Omega \left(\sum_{\alpha=1}^{m} F_\alpha(x, Du^\alpha) + G(x, Du) \right) dx,$$

$F_\alpha : \Omega \times \mathbb{R}^m \to [0, +\infty)$, $F_\alpha(x, \cdot)$ convex
$G : \Omega \times \mathbb{R}^{m \times n} \to \mathbb{R}$, $G(x, \cdot)$ rank one convex

$$|\lambda|^p \lesssim F_\alpha(x, \lambda) \lesssim |\lambda|^p + a(x)$$

$$|G(x, \xi)| \lesssim |\xi|^q + b(x)$$

where $1 \leq q < p \leq n$ and $a, b \in L^s(\Omega)$, $s > 1$.

Theorem [C.-Focardi-Leonetti-Mascolo (2020)]

Assume

$$1 < p \leq n, \quad 1 \leq q < \frac{p^2}{n} \quad \text{and} \quad s > \frac{n}{p}.$$

Then the **local minimizers** $u \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^m)$ of $\mathcal{F}$ are **locally Hölder continuous**.

Example (Non quasiconvex lower order term)

$n \geq 5$, $m \geq 3$

Consider

$$\mathcal{F}(u) = \int_\Omega \left\{ \sum_{\alpha=1}^{m} (\mu + |Du^\alpha|^2)^{\frac{p}{2}} + G(Du) \right\} dx$$

where G is rank one convex, not quasi convex (Šverák).

The integrand is convex if $\mu \geq \mu_0(G, p) > 0$.

By Theorem 3, if

$$2\sqrt{n} < p \leq n$$

then **local minimizers** $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ of $\mathcal{F}$ are **locally Hölder continuous**.