POLYCONVEX INTEGRALS AND REGULARITY OF MINIMIZERS

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March 8th

Best wishes to all women!

$$u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N \qquad n \geq 2$$

$$I(u) = \int_\Omega f(Du(x))\,dx$$

$0 \leq f$ continuous

$$\min\left\{ I(u) : \quad u = u_* \text{ on } \partial\Omega \right\}$$

Direct Method

$$\nu |z|^p - C \leq f(z)$$

minimizing sequences are bounded in $W^{1,p}$

$p > 1$

weak compactness

Weak lower semicontinuity in $W^{1,p}$

$$\Downarrow$$

existence of minimizers

$$u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N$$

$$I(u) = \int_\Omega f(Du)\,dx$$

$\boxed{\text{when } N=1}$ scalar case

I weakly lower semicontinuous on $W^{1,p}$

$$\Updownarrow$$

f convex

Dacorogna's book

$$u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N$$

$$I(u) = \int_\Omega f(Du)\, dx$$

$\boxed{\text{when } N \geq 2}$ vectorial case

I weakly lower semicontinuous in $W^{1,p}$

$$\Downarrow$$

f quasi convex

$$\Updownarrow$$

$$f(A) \leq \frac{1}{|D|} \int_D f(A + D\varphi(x))\, dx$$

$\forall A \in \mathbb{R}^{N \times n}, \quad \forall \varphi \in W_0^{1,\infty}(D), \quad \forall D \subset \mathbb{R}^n$

I weakly lower semicontinuous in $W^{1,p}$

f quasiconvex

$0 \le f(z) \le L|z|^p + c$

$0 \le f$ convex

$$f \text{ convex} \Rightarrow f \text{ quasiconvex}$$

$$\int_D f(A)\,dx \le \int_D f(A + D\varphi(x))\,dx$$

affine functions $Ax+B$ are local minimizers

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non locality of quasiconvexity   Kristiansen 1999

f convex $\Rightarrow$ f polyconvex $\Longrightarrow$ f quasiconvex

$$f(z) = g(z, \mathrm{adj}_2 z, \dots, \mathrm{adj}_{N \wedge m} z)$$

g convex

$$z \in \mathbb{R}^{N \times m}$$

$$\mathrm{adj}_k z = \{\text{all } k \times k \text{ minors of } z\}$$

$N = n = 2 \qquad Z \in \mathbb{R}^{2\times 2}$

$$f(Z) = |Z|^2 + 4\,|\det Z|$$

f is not convex

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \tilde{Z} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f(Z) = 1 = f(\tilde{Z})$$

$$\frac{1}{2} Z + \frac{1}{2}\tilde{Z} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

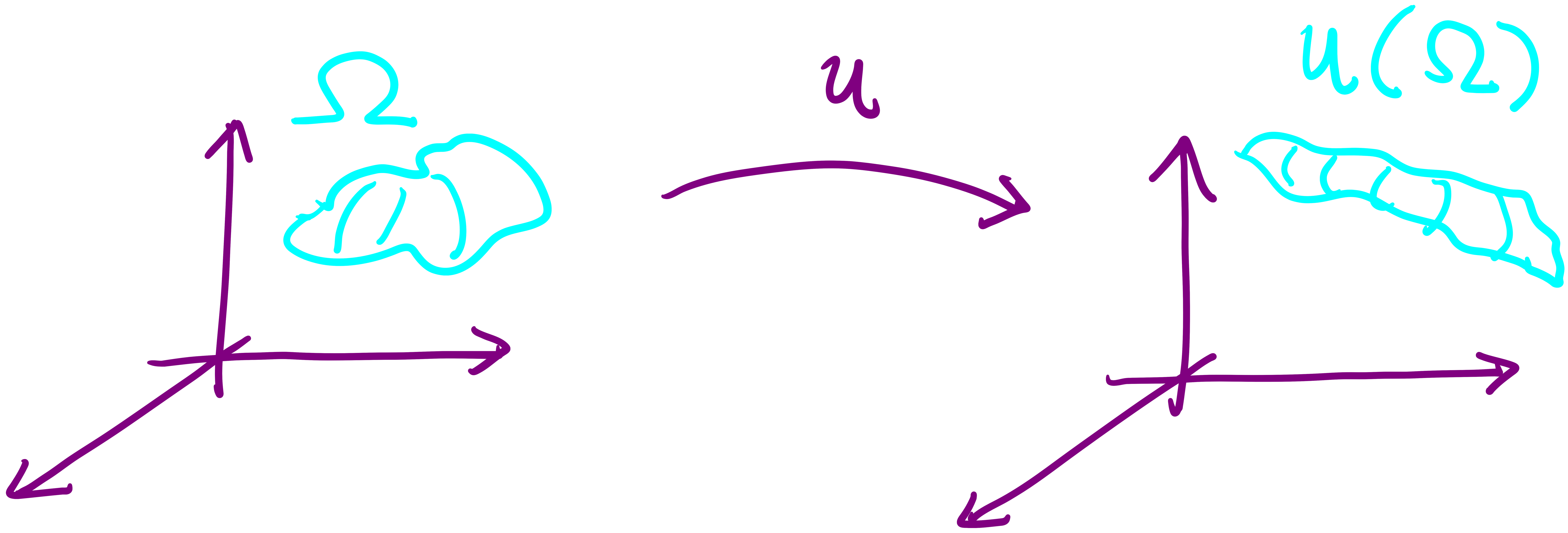
$$f\left(\frac{1}{2}Z + \frac{1}{2}\tilde{Z}\right) = 2\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^2 > 1 = \frac{1}{2}f(Z) + \frac{1}{2}f(\tilde{Z})$$

$$f(z) = |z|^2 + 4|\det z| = g(z, \det z)$$

$$g(z,t) = |z|^2 + 4|t|$$

$$(z,t) \longrightarrow |z|^2 + 4|t| \quad \text{convex}$$

$$u : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$



stored energy $\quad I(u) = \int_\Omega f(Du(x))\,dx$

$$Z = Du(x) \in \mathbb{R}^{3\times 3}$$

$$f(Z) = g(Z, \operatorname{adj}_2 Z, \det Z)$$

g convex

Z, $\operatorname{adj}_2 Z$, $\det Z$ govern deformations of line, surface, volume elements

Ball 1977

$$u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N$$

$$I(u) = \int_\Omega f(Du)\,dx$$

I weakly lower semicontinuous in $W^{1,p}$

f quasiconvex

$0 \le f(z) \le L|z|^p + c$

f convex

$0 \le f$

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f quasiconvex

$$\nu|z|^p - c \quad \le \quad f(z) \quad \le \quad L|z|^p + c$$

Weak convergence in $W^{1,p}$ of minimizing sequences

weak lower semicontinuity in $W^{1,p}$ for minimizing sequences

existence of minimizers for $I(u) = \int_\Omega f(Du)\,dx$

$0 \leq g$ convex

$v_k, v_\infty : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^M$

$v_k \longrightarrow v_\infty$ in $L^1(\Omega)$

$$\Downarrow$$

$$\int_\Omega g(v_\infty)\,dx \leq \liminf_k \int_\Omega g(v_k)\,dx$$

Polyconvex

$$u : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\int_\Omega g(Du, \operatorname{adj}_2 Du, \det Du)\, dx$$

g convex

$$|z|^p + |\operatorname{adj}_2 z|^q + |\det z|^r \leq g(z, \operatorname{adj}_2 z, \det z)$$

$1 < p, q, r$

$$\left.\begin{array}{l} u_k \longrightarrow u_\infty \text{ in } W^{1,p} \\ \operatorname{adj}_2 Du_k \longrightarrow a_\infty \text{ in } L^q \\ \det Du_k \longrightarrow d_\infty \text{ in } L^r \end{array}\right\} \Longrightarrow a_\infty = \operatorname{adj}_2 Du_\infty \quad \text{if } p \geq 2$$

$$\Downarrow$$

$$d_\infty = \det Du_\infty \quad \text{if } p \geq \frac{q}{q-1}$$

$$(Du_k, \operatorname{adj}_2 Du_k, \det Du_k) \xrightarrow{L^1} (Du_\infty, \operatorname{adj}_2 Du_\infty, \det Du_\infty)$$

g convex

$$\Downarrow$$

$$\int_\Omega g(Du_\infty, \operatorname{adj}_2 Du_\infty, \det Du_\infty)\,dx \le \liminf_k \int_\Omega g(Du_k, \operatorname{adj}_2 Du_k, \det Du_k)\,dx$$

$$\Downarrow$$

existence of minimizers

$$u : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$z = Du(x) \in \mathbb{R}^{3\times 3}$$

$$g(z, \operatorname{adj}_2 z, \det z) = |z|^p + |\operatorname{adj}_2 z|^q + |\det z|^r$$

$$2 \leq p < 3$$

$$2 = q$$

$$1 < r$$

$$|\operatorname{adj}_2 z| \leq c|z|^2$$

$$|\det z| \leq c\,|z|^3$$

$$|z|^p \leq g(z, \operatorname{adj}_2 z, \det z) \leq |z|^p + c|z|^{2q} + c|z|^{3r}$$

$$\frac{q}{q-1} = 2 \leq p < 3 < 3r$$

$$p < 4 = 2q$$

$\rightarrow$ non standard growth $\leftarrow$

Weak lower semicontinuity in $W^{1,p}$ for polyconvex integrals

Negative results in Ball-Murat 1984
Maly 1993

Positive results when u_k are "regular" in

Marcellini 1986

Dacorogna-Marcellini 1990

Maly 1993

Acerbi-Dal Maso 1994

Dal Maso-Sbordone 1995

Celada-Dal Maso 1994

Fusco-Hutchinson 1995

Focardi-Fusco-Leone-Marcellini-Mascolo-Verde 2014

Regularity for minimizers or equilibria of polyconvex integrals

Partial regularity in

Fusco-Hutchinson	1991
Fuchs-Seregin	1994
Fuchs-Reuling	1995
Passarelli	1999
Esposito-Mingione	2001
Hamburger	2003
Foss	2005
Carozza-Passarelli	2007

Everywhere regularity when $n=2$

Fusco - Hutchinson	1994
Dougherty	1997
Fuchs - Seregin	1997
Bevan	2017

Maximum principle in Leonetti 1991

$$u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad n = 2, 3$$

$$I(u) = \int_\Omega \left(|Du|^p + |\mathrm{adj}_2 Du|^q + |\det Du|^r \right) dx$$

$p > 0; \quad q, r \geq 0$

$u = (u^1, \dots, u^n)$ minimizes I

$$|u^\alpha| \leq L \text{ on } \partial\Omega \implies |u^\alpha| \leq L \text{ in } \Omega$$

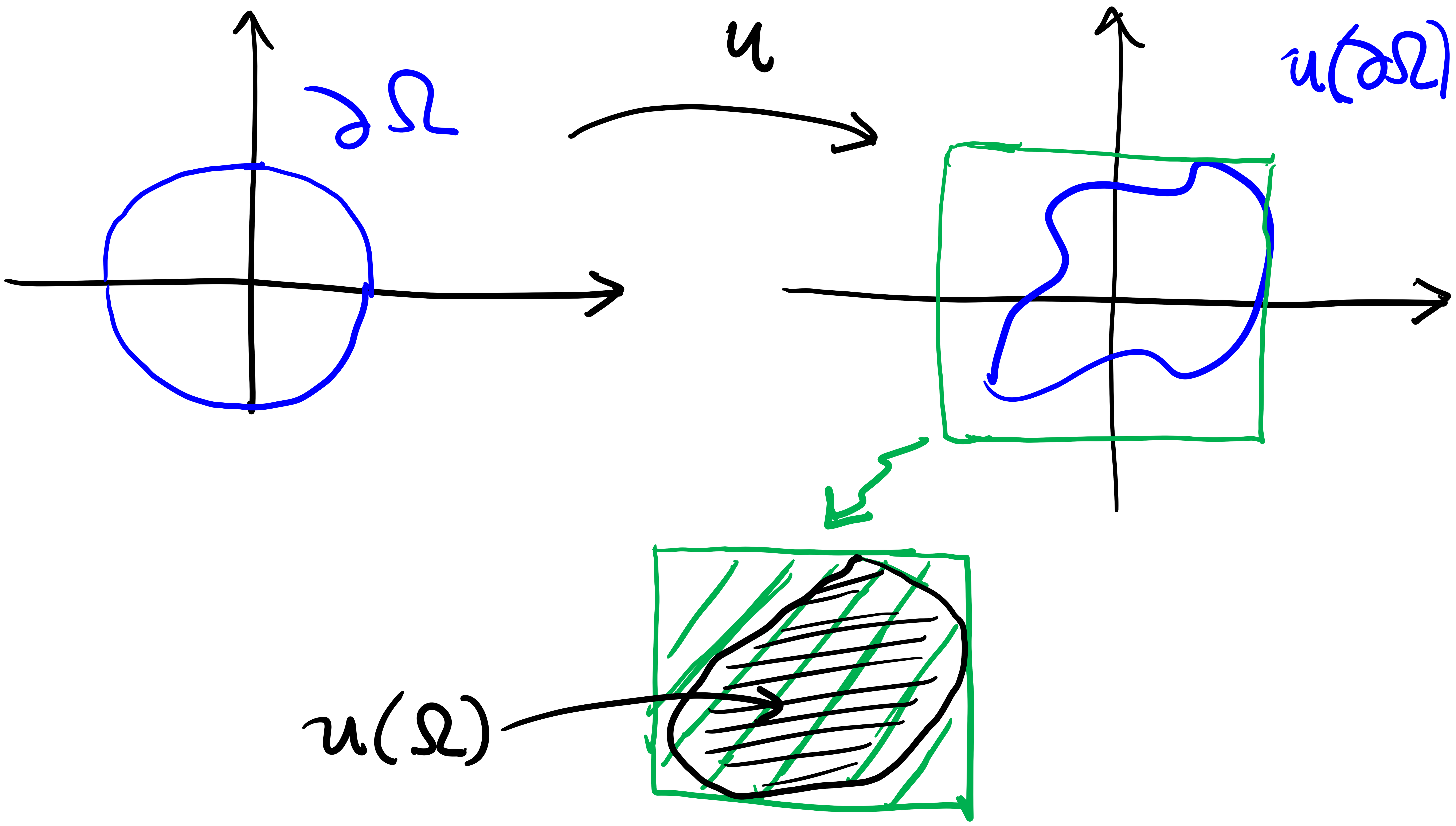
Also if $I(u) = \int_\Omega \Psi(|Du|, |\mathrm{adj}_2 Du|, |\det Du|)\, dx$

$|Du| \longrightarrow \Psi(|Du|, |\mathrm{adj}_2 Du|, |\det Du|) \nearrow$ strictly

$|\mathrm{adj}_2 Du| \longrightarrow \Psi(|Du|, |\mathrm{adj}_2 Du|, |\det Du|) \nearrow$

$|\det Du| \longrightarrow \Psi(|Du|, |\mathrm{adj}_2 Du|, |\det Du|) \nearrow$

$n = 2$



Importance of $|Du| \to \Psi(|Du|, \dots) \nearrow$ $\boxed{\text{strictly}}$

Sverak (personal communication, 1993)

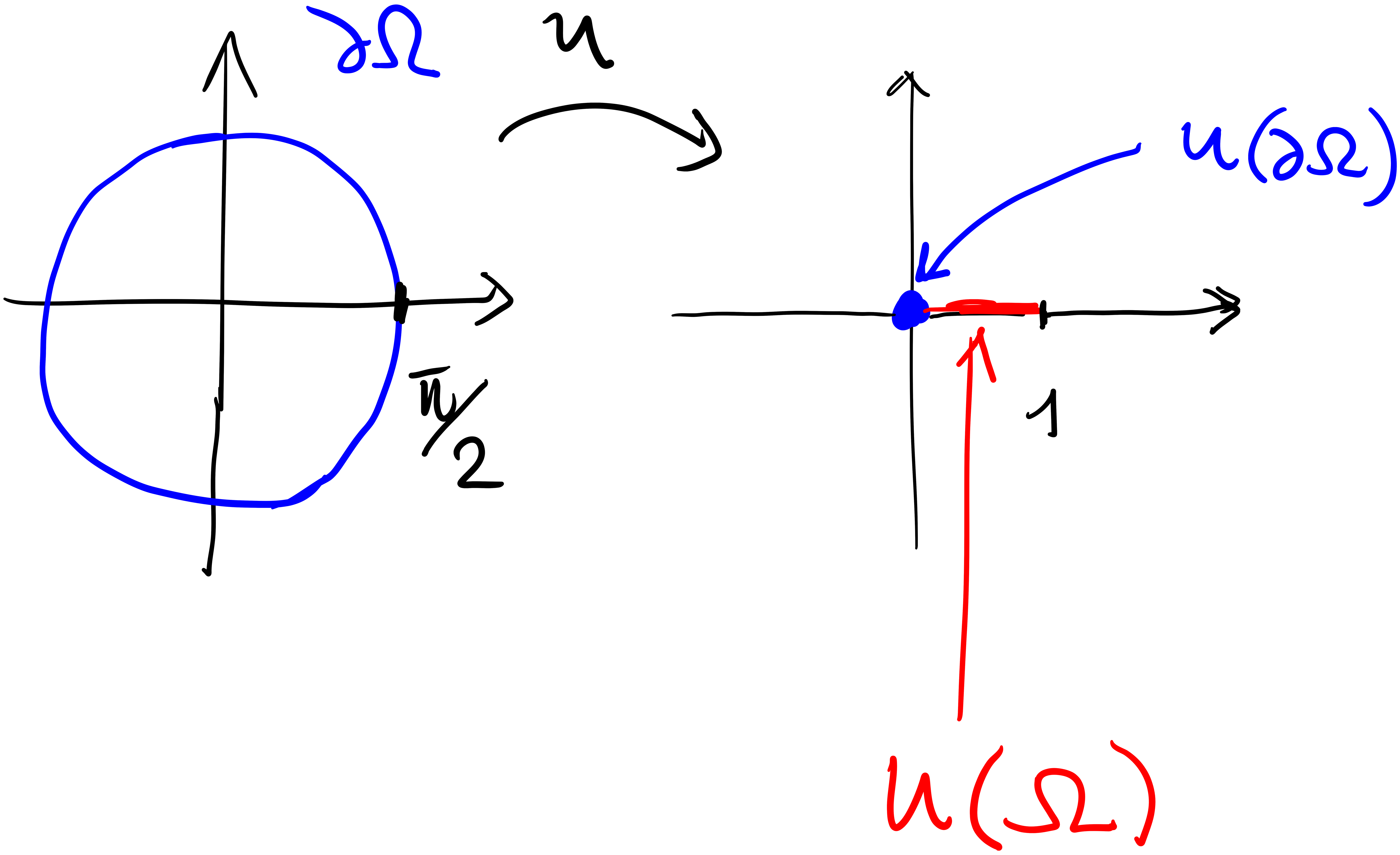
$$\Omega = B\left(0, \tfrac{\pi}{2}\right) \subset \mathbb{R}^2$$

$$u(x) = \begin{pmatrix} \cos|x| \\ 0 \end{pmatrix}$$

$$u^1\big|_{\Omega} > 0 = u^1\big|_{\partial\Omega}$$

$$Du(x) = \begin{pmatrix} D(\cos|x|) \\ 0 \end{pmatrix} \qquad \det Du = 0$$

u minimizes $\displaystyle\int_\Omega |\det Du|\, dx$



infinitely many minimizers

$$u_\lambda(x) = \begin{pmatrix} \lambda \cos|x| \\ 0 \end{pmatrix}$$

At least one minimizer enjoys maximum principle

Leonetti–Siepe J Conv. Anal. 2005

$$I(u) = \int_\Omega \Psi(|Du|, |\mathrm{adj}_2 Du|, |\det Du|)\,dx$$

$t_i \longrightarrow \Psi(t_1, t_2, t_3) \nearrow \quad i = 1,2,3$

u minimizes I

$\sup_{\partial\Omega} u^1 \le L \implies \underbrace{(u^1 \wedge L, u^2, \dots, u^n)}_{\text{minimizes } I}$

Anisotropic case

$$u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$I(u) = \int_\Omega \left(|D_1 u|^{p_1} + \cdots + |D_n u|^{p_n} + |\mathrm{adj}_2 Du|^{q_2} + \cdots + |\det Du|^{q_n} \right) dx$$

$p_i > 0$

$q_i \geq 0$

u minimizes I

$$\Downarrow$$

$$\inf_{\partial\Omega} u^\alpha \leq \inf_{\Omega} u^\alpha \leq \sup_{\Omega} u^\alpha \leq \sup_{\partial\Omega} u^\alpha$$

D'Ottavio - Leonetti - Musciano 1998

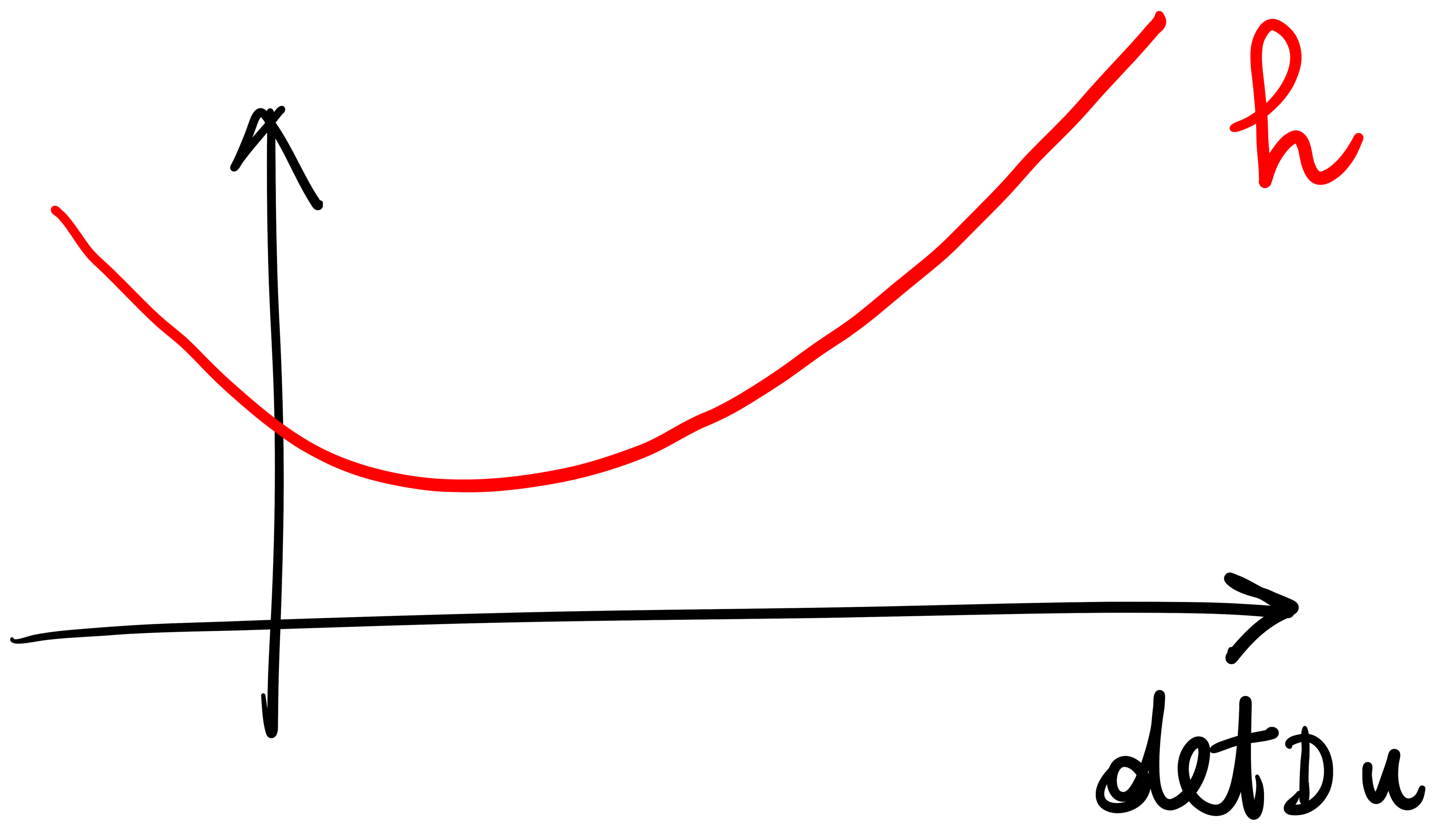
A model functional

$$u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$I(u) = \int_\Omega \left(|Du|^p + h(\det Du) \right) dx$$

$p \geq 2$

$h : \mathbb{R} \longrightarrow [0, +\infty)$ convex



u minimizes I

$$\Downarrow$$

$$-c(n,p,\Omega)\left[h(0) - \inf_{\mathbb{R}} h\right]^{1/p} + \inf_{\partial\Omega} u^\alpha \leq u^\alpha \leq \sup_{\partial\Omega} u^\alpha + c(n,p,\Omega)\left[h(0) - \inf_{\mathbb{R}} h\right]^{1/p}$$

Leonetti - Siepe Ric. Mat. 2005

$$p \geq 2$$

fix α

$$|Du|^2 = |Du^\alpha|^2 + \sum_{\beta \neq \alpha} |Du^\beta|^2$$

$$\frac{p}{2} \geqslant 1 \qquad \Downarrow \qquad (A+B)^{p/2} \geq A^{p/2} + B^{p/2}$$

$$|Du|^p \geq |Du^\alpha|^p + \Big(\sum_{\beta \neq \alpha} |Du^\beta|^2\Big)^{p/2}$$

$$1 < p$$

Carozza - Esposito - Leonetti

to appear

L^∞ estimate $\Longrightarrow$ $\begin{cases} W^{1+\theta,2} \text{ estimate} \\ 0<\theta<1 \end{cases}$

Esposito - Leonetti - Mingione NoDEA 1999

$$|z|^2 \leq f(z) \leq L(|z|^q + 1)$$

$$2 < q < 4$$

$$|\lambda|^2 \leq \langle DDf(z)\lambda, \lambda \rangle \leq L(|z|^{q-2} + 1)|\lambda|^2$$

Euler equation

$$\int_\Omega Df(Du)D\varphi = 0$$

$$\tau_{s,h} g(x) = g(x + h e_s) - g(x)$$

$$e_s = (0, \dots, 0, 1, 0, \dots, 0)$$

s^{th} place

$$\varphi = \tau_{s,-h}\left(\eta^2 \tau_{s,h} u\right)$$

cut off function

Caccioppoli inequality

$$\int_{B_\rho} |\tau_{s,h} Du|^2 \leq c \int_{B_R} \left(1 + |Du| + |\tau_{s,h} Du|\right)^{q-2} |\tau_{s,h} u|^2$$

$2 < q < 4$

$$\int_{B_R} (1+|Du|+|\tau_{s,h}Du|)^{q-2} |\tau_{s,h}u|^2 \le$$

$$\le \left(\int_{B_R} (1+|Du|+|\tau_{s,h}Du|)^2 \right)^{\frac{q-2}{2}} \left(\int_{B_R} \boxed{|\tau_{s,h}u|^{\frac{4}{4-q}}} \right)^{\frac{4-q}{2}}$$

$\boxed{\text{Holder with } \frac{2}{q-2}, \frac{2}{4-q}}$

$$= |\tau_{s,h}u|^2 \underbrace{|\tau_{s,h}u|}_{\le 2\|u\|_{L^\infty}}^{\frac{4}{4-q}-2}$$

$$2 < \frac{4}{4-q}$$

$$\int_{B_\rho} |\tau_{s,h} Du|^2 \leq \tilde{C}\, |h|^{4-q}$$

$$\Downarrow \qquad 0 < 4-q < 2$$

$$Du \in W^{\theta,2}$$

$$0 < \theta < \frac{4-q}{2} < 1$$

$$Du \in W^{\theta,2} \Rightarrow Du \in L^{2+\sigma}$$

Sometimes it can be iterated

Campanato - Cannarsa 1981

Bhattacharya - Leonetti 1993

Esposito - Leonetti - Mingione JDE 1999

Carozza - Leonetti - Passarelli 2009

Fractional Differentiability

Leonetti - Musciano 1995

Cavaliere - D'Ottavio - Leonetti - Musciano 1996

Mingione 2003

Esposito - Leonetti - Mingione 2004 ← Double phase

Diening - Ettwein 2008

Cupini - Leonetti - Mascolo 2015

$u \in L^\infty$ sometimes helps

Choe 1992

D'Ottavio 1995

Canale - D'Ottavio - Leonetti - Longobardi 2001

Bildhauer - Fuchs 2007

Colombo - Mingione 2015

Softova 2017

Giova - Passarelli 2017

Bulicek - Cupini - Stroffolini - Verde 2018

Cupini - Giannetti - Giova - Passarelli 2018

Some elliptic systems

$$-\operatorname{div}\left(A(x, Du(x))\right) = 0$$

$$u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N$$

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(A_i^\alpha(x, Du(x))\right) = 0 \qquad \alpha = 1, \dots, N$$

$$\left|A_i^\alpha(x, z)\right| \leq c\left(|z|^{p-1} + 1\right)$$

$$\nu |z^\alpha|^p - M \leq \sum_{i=1}^{n} A_i^\alpha(x, z)\, z_i^\alpha \qquad \alpha = 1, \dots, N$$

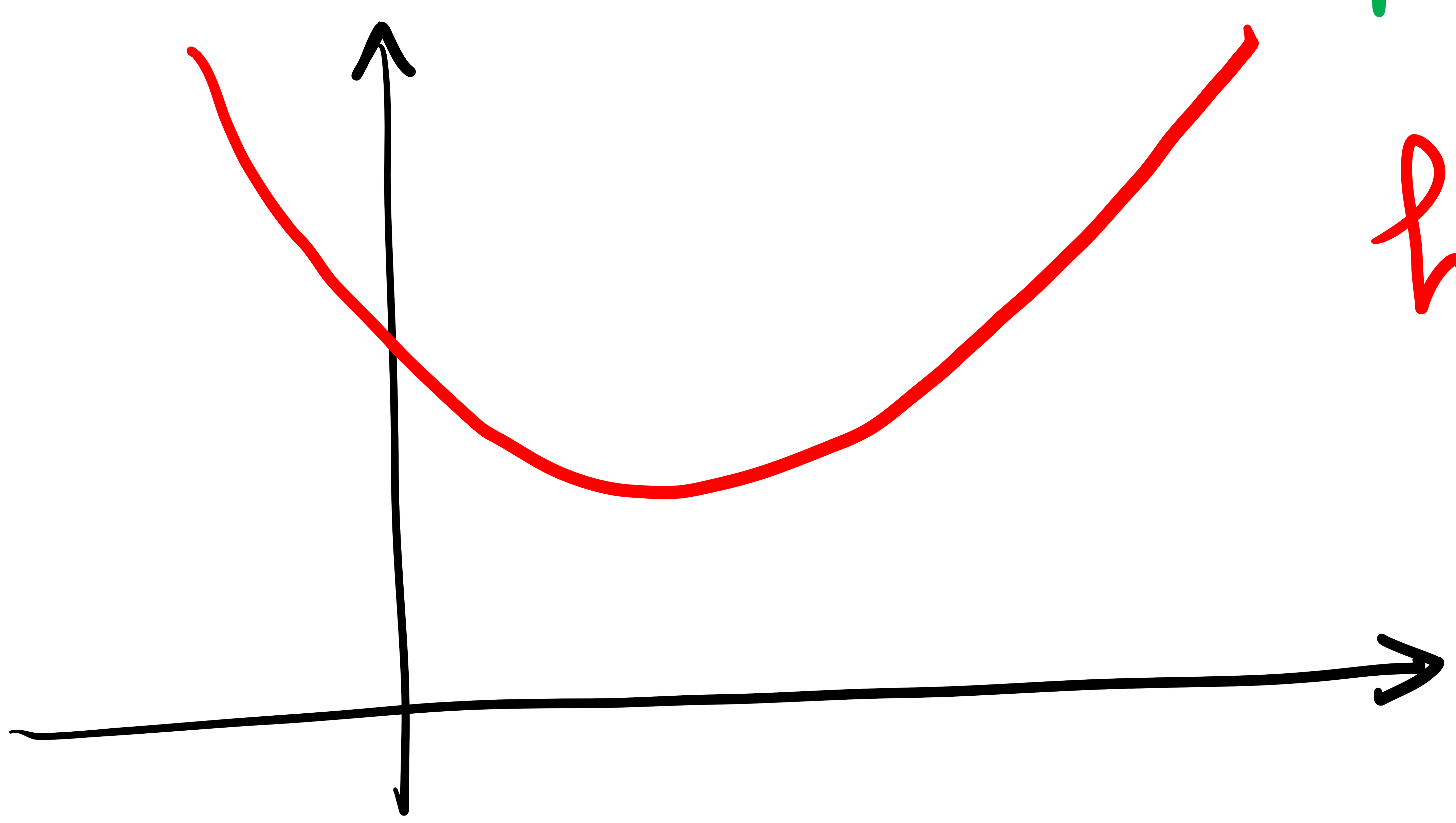
$$\Downarrow$$

$$\sup_{\Omega} u^\alpha \leq \sup_{\partial\Omega} u^\alpha + \tilde{c}\, M^{\frac{1}{p}} \qquad \alpha = 1, \dots, N$$

Leonetti - Petricca 2011

Example $N = n$ $\quad u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$

Euler equation of $\displaystyle\int_\Omega \left(|Du|^p + h(\det Du) \right) dx$



$p \geq 2$

h convex ≥ 0

$$A_i^\alpha(z) = \frac{\partial}{\partial z_i^\alpha}\left(|z|^p + h(\det z) \right) =$$

$$= p\,|z|^{p-2} z_i^\alpha + h'(\det z)\,\frac{\partial}{\partial z_i^\alpha}(\det z)$$

$$\det z = \sum_{j=1}^{n} z_j^\alpha \operatorname{Cof}_j^\alpha(z)\,; \qquad \frac{\partial}{\partial z_i^\alpha}(\det z) = \operatorname{Cof}_i^\alpha(z)$$

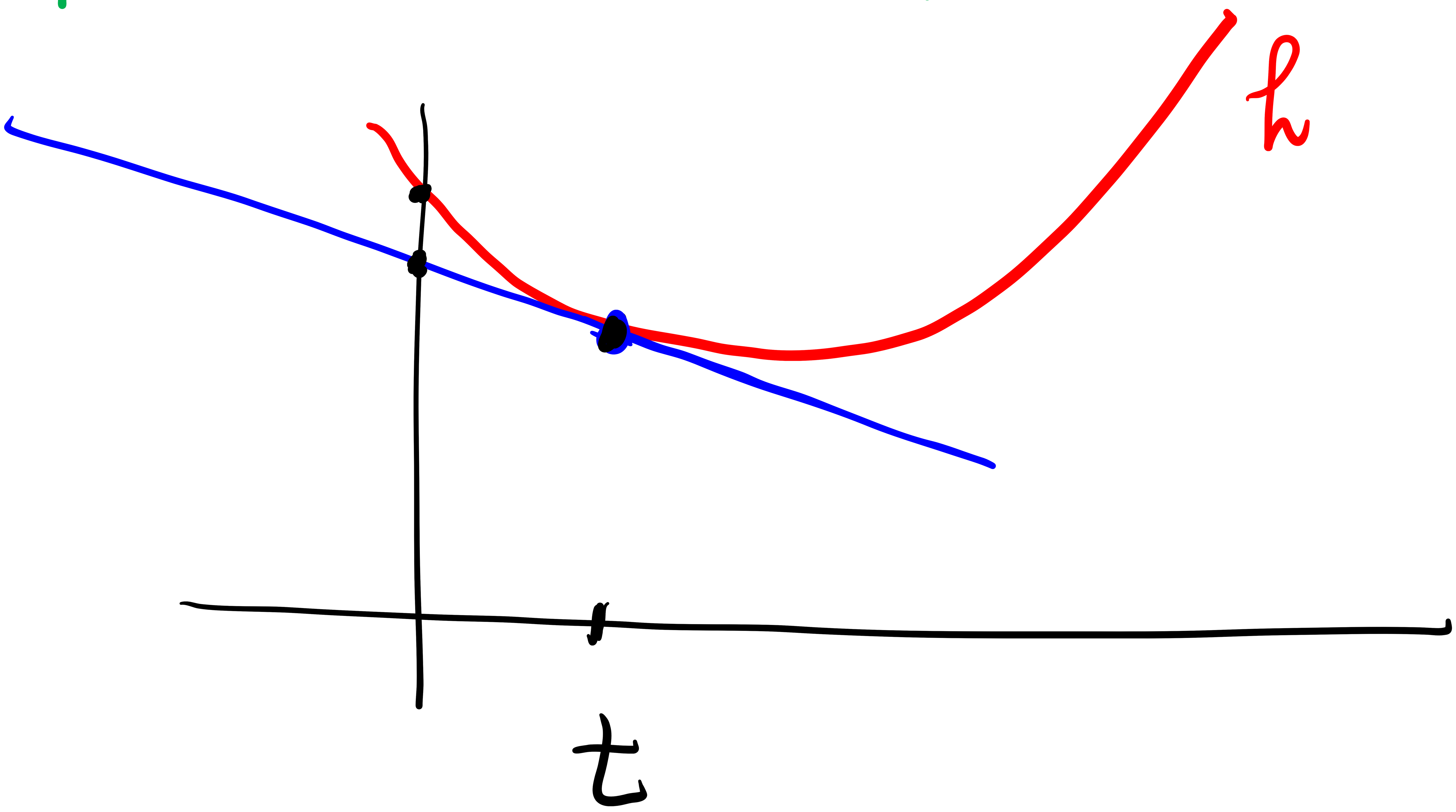
$Z \in \mathbb{R}^{n \times n}$

$$A_i^\alpha(Z) = p|Z|^{p-2} Z_i^\alpha + h'(\det Z)\, \mathrm{Cof}_i^\alpha(Z)$$

$$\sum_{i=1}^{n} A_i^\alpha(Z) Z_i^\alpha = p|Z|^{p-2} \underbrace{\sum_i Z_i^\alpha Z_i^\alpha}_{=|Z^\alpha|^2} + h'(\det Z) \underbrace{\sum_i \mathrm{Cof}_i^\alpha(Z) Z_i^\alpha}_{=\det Z} =$$

$$= p|Z|^{p-2}|Z^\alpha|^2 + h'(\det Z)\det Z$$

$p \geq 2 \Rightarrow |Z|^{p-2} \geq |Z^\alpha|^{p-2}$



h convex

$$h(0) \geq h(t) + h'(t)(0-t)$$

$$h'(t)\,t \geq -h(0) + h(t) \geq$$

$$\geq -h(0) + \inf_{\mathbb{R}} h =$$

$$= -\left[h(0) - \inf_{\mathbb{R}} h\right]$$

$$\boxed{\sum_i A_i^\alpha(z) z_i^\alpha \geq p|z^\alpha|^p - \left[h(0) - \inf_{\mathbb{R}} h\right]}$$

$$\sum_i A_i^\alpha(z) z_i^\alpha = \cdots + h'(\det z) \underbrace{\sum_i \mathrm{Cof}_i^\alpha(z) z_i^\alpha}_{= \det z}$$

$$\det z = \sum_i \underbrace{\mathrm{Cof}_i^\alpha(z)}_{= \frac{\partial}{\partial z_i^\alpha}(\det z)} z_i^\alpha = \sum_i \left(\frac{\partial}{\partial z_i^\alpha} (\det z) \right) z_i^\alpha$$

$z^\alpha \longrightarrow \det(z)$ linear $\Longrightarrow$ homogeneous with degree 1

$$\sum_i \left[\frac{\partial}{\partial z_i^\alpha} (\det z) \right] z_i^\alpha = 1 \det z$$

Example $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N$

$$z = Du(x) \in \mathbb{R}^{N \times n}$$

$$g : \mathbb{R}^{N \times n} \longrightarrow \mathbb{R}$$

$z^\alpha \longrightarrow g(z)$ homogeneous of degree $d_\alpha \geq 0$

$$\int_\Omega \left(|Du|^p + h(g(Du)) \right) dx$$

$p \geq 2$ $\quad h$ convex ≥ 0

Euler equation

$$A_i^\alpha(z) = p|z|^{p-2} z_i^\alpha + h'(g(z)) \frac{\partial g}{\partial z_i^\alpha}(z)$$

$$\sum_i A_i^\alpha(z)\, z_i^\alpha = p|z|^{p-2}\sum_i z_i^\alpha z_i^\alpha + h'(g(z)) \underbrace{\sum_i \frac{\partial g}{\partial z_i^\alpha}(z)\, z_i^\alpha}_{=\, d_\alpha\, g(z)}$$

$$\underbrace{h'(g(z))\, g(z)}_{\geq -\left[h(0)-\inf_{\mathbb{R}} h\right]}\ \underbrace{d_\alpha}_{\geq 0} \geq -\left[h(0)-\inf_{\mathbb{R}} h\right] d_\alpha$$

Example for g

fix $\beta, \gamma \in \{1, \dots, N\}$ with $\beta \neq \gamma$

take $g(z) = \sum_{j=1}^{n} z_j^{\beta} z_j^{\gamma}$

then

$z^{\alpha} \longrightarrow g(z)$ is homogeneous with degree d_α

$$d_\alpha = \begin{cases} 1 & \text{if } \alpha \in \{\beta; \gamma\} \\ 0 & \text{if } \alpha \notin \{\beta; \gamma\} \end{cases}$$

$\boxed{\text{local}}$ L^∞ regularity

$$u : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$I(u) = \int_\Omega g(Du, \operatorname{adj}_2 Du, \det Du)$$

$$g = |Du^1|^p + |Du^2|^p + |Du^3|^p + |\operatorname{adj}_2 Du|^q + |\det Du|^r$$

Cupini - Leonetti - Mascolo 2017

or

$$g = |Du|^p + |\operatorname{adj}_2 Du|^q + |\det Du|^r$$

Carozza - Gao - Giova - Leonetti 2018

$$\boxed{p = \frac{14}{5} \qquad q = 2 \qquad r = \frac{3}{2}}$$

$$u = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}$$

to prove that $u^1 \in L^\infty_{loc}$

test minimality with

$$v = \begin{pmatrix} u^1 - \eta^\square [(u^1 - L) \vee 0] \\ u^2 \\ u^3 \end{pmatrix}$$

η = cut-off function

$\square$ = suitable exponent

on $\{u^1 \leq L\} \cup \{\eta = 0\}$ $\quad v = u$

on $\{u^1 > L\} \cap \{\eta > 0\}$

$$Dv = \begin{pmatrix} (1-\eta^{\square}) Du^1 + \eta^{\square}\left(\frac{-\square}{\eta}\right)(D\eta)(u^1-L) \\ Du^2 \\ Du^3 \end{pmatrix}$$

linearity of determinant with respect to first row

$$\det Dv = (1-\eta^{\square}) \det \underbrace{\begin{pmatrix} Du^1 \\ Du^2 \\ Du^3 \end{pmatrix}}_{= Du} + \eta^{\square} \det \underbrace{\begin{pmatrix} \frac{-\square}{\eta}(D\eta)(u^1-L) \\ Du^2 \\ Du^3 \end{pmatrix}}_{= A}$$

$$\det Dv = (1-\eta^{\square})\det Du + \eta^{\square}\det A$$

$$\operatorname{adj}_2 Dv = (1-\eta^{\square})\operatorname{adj}_2 Du + \eta^{\square}\operatorname{adj}_2 A$$

$$Dv = (1-\eta^{\square})\, Du + \eta^{\square} A$$

$$g(Dv, \operatorname{adj}_2 Dv, \det Dv) =$$

$$= g\big((1-\eta^{\square})Du+\eta^{\square}A,\ (1-\eta^{\square})\operatorname{adj}_2 Du+\eta^{\square}\operatorname{adj}_2 A,\ (1-\eta^{\square})\det Du+\eta^{\square}\det A\big) \leq$$

$$\leq (1-\eta^{\square})\, g(Du, \operatorname{adj}_2 Du, \det Du) + \eta^{\square} g(A, \operatorname{adj}_2 A, \det A)$$

g convex

$$\left|\det A\right| = \left|\det\begin{pmatrix} \frac{-\square}{\eta}(D\eta)(u^1 - L) \\ Du^2 \\ Du^3 \end{pmatrix}\right| \leq$$

$$\leq \frac{C}{\eta}\,|D\eta|\,|u^1 - L|\,\left|\operatorname{adj}_2 Du\right|$$

$$\int_{\{u^1 > L\}} |u^1 - L|\,|\operatorname{adj}_2 Du| \leq \left(\int_{\{u^1 > L\}} |u^1 - L|^{p^*}\right)^{\frac{1}{p^*}} \left(\int_{\{u^1 > L\}} |\operatorname{adj}_2 Du|^{(p^*)'}\right)^{\frac{1}{(p^*)'}}$$

p close to $n = 3 \Rightarrow p^*$ close to $\infty \Rightarrow (p^*)'$ close to 1

$$\left(\int_{\{u^1 > L\}} |\operatorname{adj}_2 Du|^{(p^*)'}\right)^{\frac{1}{(p^*)'}} \leq \left(\int_{\{u^1 > L\}} |\operatorname{adj}_2 Du|^{2}\right)^{\frac{1}{2}} \left\|\{u^1 > L\}\right|^{\cdots}$$

$$\int_{B_\rho\cap\{u^1>L\}} |Du^1|^{p}\,dx \le c\int_{B_R\cap\{u^1>L\}} \left(\frac{u^1-L}{R-\rho}\right)^{p^*} dx + c\,\left|B_R\cap\{u^1>L\}\right|^{(\cdots)}$$

$$\Downarrow$$

$$\sup_{B_s} u^1 < +\infty$$

Moscariello - Nania 1991

Fusco - Sbordone 1993

Cianchi 2000

Cupini - Marcellini - Mascolo 2017

Granucci - Randolfi 2019

Karpinen - Lee (Lee's seminar)

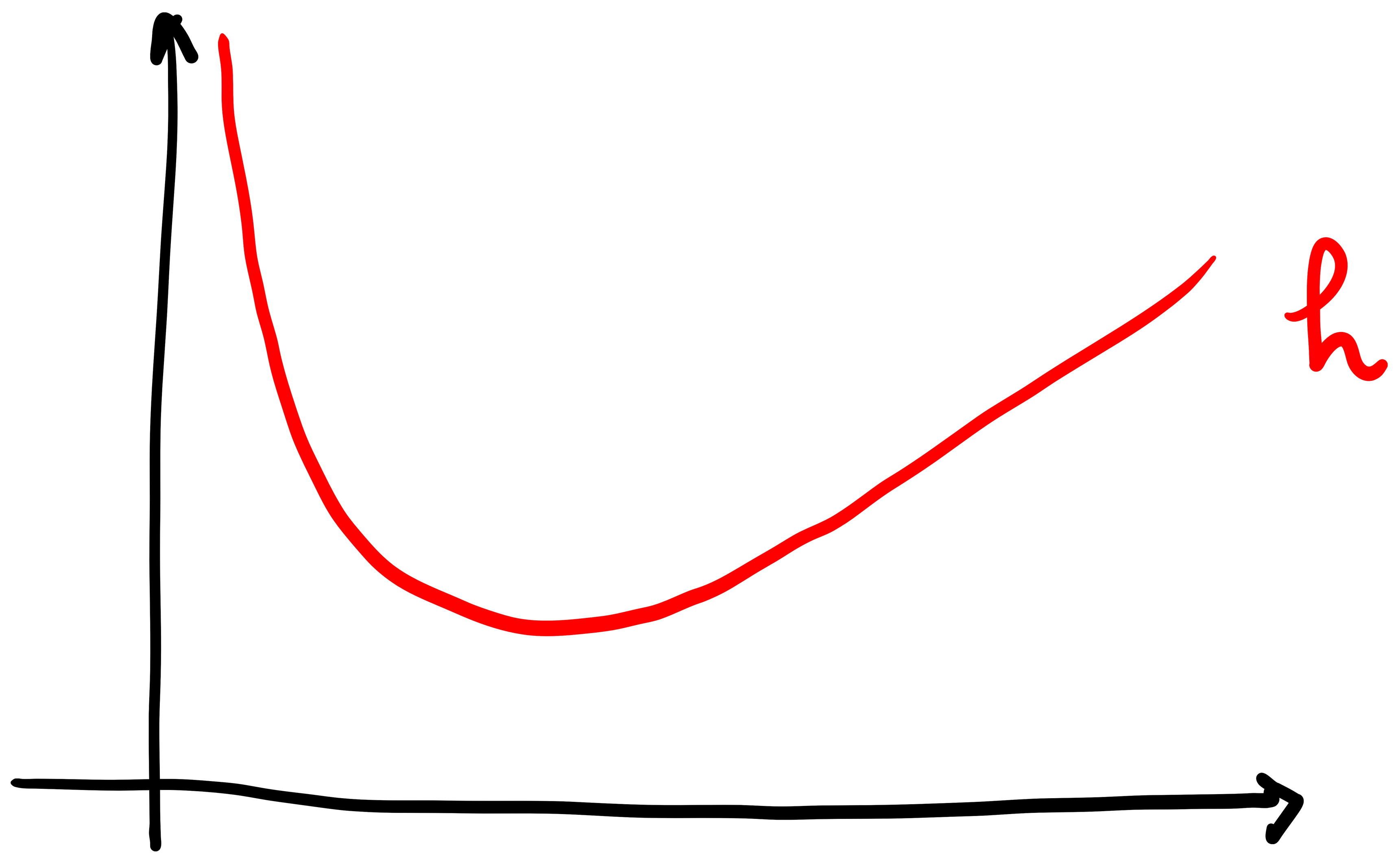
Hirsch - Schaffner 2020

Models from nonlinear elasticity

$$u : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$I(u) = \int_\Omega \left(|Du|^p + b\,|\mathrm{adj}_2 Du|^2 + h(\det Du) \right) dx$$

$2 \le p < 3$



$$\left.\begin{array}{l} h(t) = |\ln t| \\ \text{or} \\ h(t) = |\ln t|^2 \end{array}\right\} t \in (0, t_0)$$

$u = (u^1, u^2, u^3)$ minimizes I

$$\Downarrow$$

$$\inf_{\partial\Omega} u^\alpha - C \leq u^\alpha(x) \leq \sup_{\partial\Omega} u^\alpha + C$$

$$\underset{t \in (0, t_0)}{h(t) = |\ln t|} \rightsquigarrow C = C(p, \Omega, t_0)$$

$$h(t) = |\ln t|^2 \rightsquigarrow C = C(p, \Omega, t_0, I(u))$$

Leonetti 2006

Gao - Leonetti - Macrì - Petricca 2020

Thank you!