The streamlines of ∞-harmonic potentials

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Outline

1. Introduction to Δ_∞
2. Main results
3. Ideas of the proofs
   - Speed
   - The fundamental inequality
   - The Quadrilateral rule
   - The main theorem

Δ_∞: The infinity Laplacian

The infinity Laplacian

$$\Delta_\infty u := \langle \nabla u, D^2 u \nabla u \rangle = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

Solutions of

$$\Delta_\infty u = 0$$

are called ∞-harmonic functions.

Discovered by Gunnar Aronsson in the 60's in connection to Lipschitz extensions.

Motivation via Δ_p

If u_p minimizes

$$\int_\Omega |\nabla u|^p, \quad \text{among functions coinciding on } \partial\Omega,$$

then

$$\Delta_p u_p = \operatorname{div}(|\nabla u_p|^{p-2}\nabla u_p) = 0 \quad \text{in } \Omega$$

Solutions are called p-harmonic functions.

Note: For $p = 2$ we get the usual Laplace equation.

Δ_∞ via the p-Laplacian

As $p \to \infty$

$$\|\nabla u\|_{L^p(\Omega)} \to \|\nabla u\|_{L^\infty(\Omega)},$$

$$\Delta_p u = |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}\Delta_\infty u \to \Delta_\infty u$$

Reasonable that $u_p \to u$ where u minimizes

$$\|\nabla u\|_{L^\infty(\Omega)}, \quad \text{among functions coinciding on } \partial\Omega$$

and solves $\Delta_\infty u = 0$ in Ω.

Aronsson 66. Bhattacharya, DiBenedetto and Manfredi 89.

Lipschitz extensions and Δ_∞

Let Ω be open and bounded, $g : \partial\Omega \to \mathbb{R}$ be Lipschitz and

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Then

$$\sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|} = \sup_{x,y\in\partial\Omega} \frac{|g(x)-g(y)|}{|x-y|}.$$

This equality also holds in any open $U \subset \Omega$ (AMLE).

This was first proved by Aronsson for C^2 functions.

Properties of the ∞-Laplace equation

- Classical solutions is not a good notion of solutions. Instead: *viscosity solutions*.
- Existence and uniqueness of solutions of the Dirichlet problem on bounded domains, Aronsson 67, Jensen 93.
- Solutions $\Leftrightarrow$ AMLE $\Leftrightarrow$ minimizers of $\|\nabla u\|$, Aronsson 67, Jensen 93, Crandall-Evans-Gariepy 2001
- Differentiability in any dimension, Evans-Smart 2011
- $C^{1,\alpha}$-regularity in the plane, Savin-Evans 2008
- $C^2 + u_{xx}u_{yy} - u_{xy}^2 \neq 0 \Rightarrow C^\infty$, Aronsson 67.

Some ∞-harmonic functions

- Cones: $|x - x_0|$ for $x \neq x_0$
- Aronsson's function $x^{\frac{4}{3}} - y^{\frac{4}{3}}$. It is merely $C^{1,1/3}$ which is believed to be the optimal regularity of solutions.
- Any C^1 solution of the eikonal equation $|\nabla u| =$constant. Note that
$$\Delta_\infty u = \frac{1}{2}\langle \nabla u, \nabla |\nabla u|^2 \rangle.$$
- The distance function to a set is ∞-harmonic wherever it is C^1.

Streamlines for smooth solutions

A streamline $\alpha = \alpha(t)$, is a solution of

$$\frac{d\alpha}{dt} = \nabla u(\alpha(t)).$$

If u is ∞-harmonic

$$\frac{d}{dt}|\nabla u(\alpha(t))|^2 = 2\Delta_\infty u(\alpha(t)) = 0.$$

Hence,

$$|\nabla u(\alpha(t))| = \text{the speed} = \text{constant.}$$

Requires second order derivatives! We will see that this may fail quite often in general for non-smooth solutions.

Streamlines cont'd

In general, the solution of

$$\frac{d\alpha}{dt} = \nabla u(\alpha(t))$$

may not be unique, even if ∇u is Hölder continuous. For instance, the Picard–Lindelöf theorem requires ∇u to be Lipschitz.

A sufficient condition for uniqueness is that u is semiconcave.

Streamlines cont'd

We will discuss the *ascending* streamlines:

$$\frac{d\alpha}{dt} = +\nabla u(\alpha(t))$$

and the *descending* streamlines:

$$\frac{d\alpha}{dt} = -\nabla u(\alpha(t))$$

In our setup, the ascending streamlines are unique but the descending ones are not in general. We will characterize certain *attracting* streamlines which are the special streamlines on which streamlines may meet.

Our setup

Ω - convex bounded domain in $\mathbb{R}^2$
K - a closed convex set in Ω (possibly a single point)

The Dirichlet boundary value problem in the convex ring $\Omega \setminus K$:

$$\begin{cases} \Delta_\infty u = 0 & \text{in} \quad \Omega \setminus K \\ u = 0 & \text{on} \quad \partial\Omega \\ u = 1 & \text{on} \quad \partial K \end{cases}$$

Basic properties

The solution of the Dirichlet problem satisfies

- There is a unique (viscosity) solution.
- The boundary values are in the classical sense.
- The solution is locally $C^{1,\alpha}$, i.e., the gradient is Hölder continuous.
- The gradient does not vanish and

$$\|\nabla u\|_{L^\infty(\Omega)} = \frac{1}{\operatorname{dist}(\partial K, \partial\Omega)}.$$

- The superlevel sets are convex.

Explicit solutions

1. If Ω is the unit ball and K the origin, then
$$u(x) = 1 - |x|.$$
2. Suppose Ω is a "stadium": the distance function is differentiable everywhere except at its maximum point. If K is the set of maximum points of the distance function, then u is the distance function:
$$u(x) = \text{dist}(x, \partial\Omega).$$

The set where the distance function attains its max is called the *high ridge*.

Our results I

- The *ascending* gradient flow is unique and terminates at ∂K.
- The *descending* gradient flow is in general not unique and we have a device for detecting these situations.
- Suppose $\partial K = K$ is a subset of the high ridge of Ω. Unless Ω is a stadium and ∂K its *high ridge*, there are streamlines that meet. In particular, u is not even locally semiconvex. In particular, if ∂K is a single point, then either Ω is a ball or there are streamlines that meet. **Instability!**

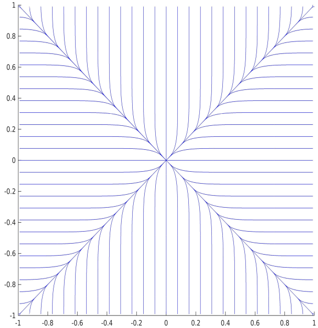
Main theorem

Structure: The streamlines starting in the corners constitutes the attracting streamlines. $|\nabla u|$ is constant along the streamline α from the initial point on until it meets one of the attracting streamlines, after which the speed is non-decreasing. It cannot meet any other streamline before it meets an attracting one.

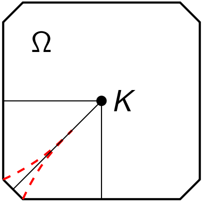
Holds also in more general convex domains under certain assumptions.

Example 1: The square

Let Ω be a square and K the center. The attracting streamlines are the four half-diagonals. All streamlines meet at a diagonal, except the four segments along the coordinate axes.



Example 2: A truncated square



The attracting streamlines are in red. The only streamlines that do not meet any other before reaching origin, are the medians. The streamline starting in the middle of the truncated corner will be a straight line to the origin and will be joined by the attracting streamlines from both sides before terminating at the origin.

The speed

We will prove that the speed along a streamline α

$$\left|\frac{d\alpha(t)}{dt}\right| = |\nabla u_\infty(\alpha(t))|$$

is non-decreasing.

p-harmonic approximation

We use

$$\begin{cases} \Delta_p u_p = 0 & \text{in} \quad \Omega \setminus K, \\ u_p = 0 & \text{on} \quad \partial\Omega, \\ u_p = 1 & \text{on} \quad \partial K, \end{cases}$$

for $p > 2$. It is known that $u_p \in C(\Omega \setminus K)$ and it takes the correct values (in the classical sense) at each boundary point. Also:

1. $u_p \nearrow u_\infty$ uniformly in $\Omega \setminus K$,
2. u_p is real analytic in $\Omega \setminus K$,
3. $\Delta u_p \leq 0$.

Convexity

Let $F_p(t) = u_p(\alpha_p(t))$. Then

$$\frac{dF_p(t)}{dt} = \left\langle \nabla u_p(\alpha_p(t)), \frac{d\alpha_p(t)}{dt} \right\rangle = |\nabla u_p(\alpha_p(t))|^2$$

and

$$\frac{d^2 F_p(t)}{dt^2} = 2\,\Delta_\infty u_p(\alpha_p(t)) = -\frac{2}{p-2}\,\Delta u_p(\alpha_p(t))\,|\nabla u_p(\alpha_p(t))|^2.$$

Recall that $\Delta u_p \leq 0$. Thus,

$$\frac{d^2 F_p(t)}{dt^2} \geq 0$$

and so $F_p(t)$ *is convex*.

Koch-Zhang-Zhou

To pass to the limit we need some convergence. But in general we don't know if ∇u_p converges to ∇u in a pointwise sense.

However, from the recent pathbreaking work by H. Koch, Y. R-Y. Zhang and Y. Zhou, one can extract that

$$|\nabla u_p| \to |\nabla u|$$

in a pointwise sense. This is enough to conclude that the speed is non-decreasing.

A fundamental inequality

Assume that $D \subset\subset \Omega \setminus \partial K$ has a Lipschitz boundary and $p > 2$. Then

$$\boxed{\oint_{\partial D} |\nabla u|^{p-2} \langle \nabla u, \mathbf{n} \rangle \, ds \;\leq\; 0}$$

where **n** is the outer normal.

If u is C^2, then the inequality follows from that:

$$\oint_{\partial D} |\nabla u|^{p-2} \langle \nabla u, \mathbf{n} \rangle \, ds = \int_D \Delta_p u \, dx \leq 0,$$

since $\Delta_p u \leq 0$.

The proof of the fundamental inequality

Based on an idea of Juhlin-Juutinen which uses two regularizations:
1) the *infimal* convolution

$$u_\varepsilon(x) = \inf_y \left(u(y) + \frac{|x-y|^2}{2\varepsilon} \right),$$

and 2) a mollification:

$$u_{\varepsilon,j} = u_\varepsilon \star \phi_j.$$

The proof requires C^1-estimates, therefore it only works in the plane.

Uniqueness of ascending streamlines

1. Take two streamlines $\alpha_1(t)$ and $\alpha_2(t)$ that separate at a point x^*.
2. Suppose they intersect some level curve at the points y_1 and y_2 and $y_1 \neq y_2$.
3. The fundamental inequality implies

$$0 \geq \oint_{\partial D} |\nabla u|^{p-2} \langle \nabla u, \mathbf{n} \rangle \, ds = \int_{y_1}^{y_2} |\nabla u|^{p-1} \, ds$$

4. Since $|\nabla u| \neq 0$ this is a contradiction.

A device for detecting bifurcations

The gradient is non-increasing along streamlines as long as they do not meet.

In other words, if x is on a higher level curve $u = b$ and y on a higher level $u = a$ and

$$|\nabla u(x)| > |\nabla u(y)|,$$

then any pair of stream lines connecting a neighborhood of ξ_0 with a neighborhood of ξ_1 must meet.

A device for detecting bifurcations cont.

Sketch of proof:
Assume that there are no streamlines meeting and

- the points x_1 and x_2 are on the same level curve $u = a$,
- the points y_1 and y_2 both are on the higher level curve $u = b > a$,
- ascending streamlines join x_1 with y_1 and x_2 with y_2.

Then

$$\|\nabla u\|_{\infty,\overline{y_1 y_2}} \leq \|\nabla u\|_{\infty,\overline{x_1 x_2}},$$

that is, the lower level curve has the larger gradient.

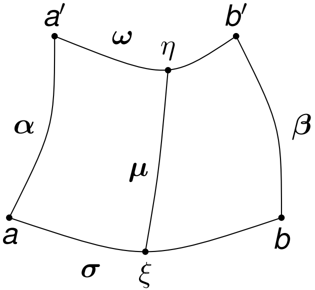
Quadrilateral rule

Recap so far: The speed is non-decreasing. Sometimes, when the streamlines do not meet, it is also non-increasing, so that it is constant along suitable arcs of streamlines.

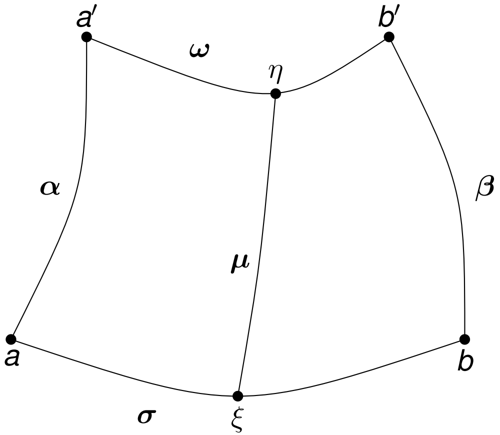
This is the idea behind the **Quadrilateral rule**.

A quadrilateral

Quadrilateral rule: Suppose $|\nabla u(\xi)| = M = \max_\sigma |\nabla u|$. If $|\nabla u(\beta(t))|$ is strictly monotone on the arcs $\overline{a\xi}$ and $\overline{\xi b}$ of the level curve σ, then no streamlines can meet inside the quadrilateral. A streamline with initial point on the arc $\overline{ab}$ (but not a or b) has constant speed till it meets α, β or reaches ω.



The proof of the quadrilateral rule



Proof of the main theorem

Recall, Ω is a convex polygon with vertices $P_1, P_2, \ldots, P_N$.

$$|\nabla u(P_j)| = 0, \quad j = 1, 2, \ldots, N.$$

From each vertex P_j, there is a unique streamline γ_j that terminates on K. They are the attracting streamlines.

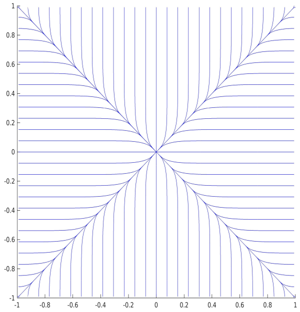
Let M_j denote a point on the edge $\overline{P_j P_{j+1}}$ at which $|\nabla u|$ attains its maximum and μ_j be the streamline starting at the point M_j.

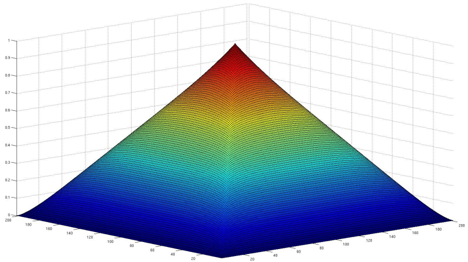
From the convexity of the boundary it follows that the normal derivative $(= |\nabla u|)$ is monotone along the half-edges $\overline{P_j M_j}$.

Proof of the main theorem cont'd

Consider the region bounded by $\overline{P_1P_2}, \gamma_1, \gamma_2$ and by ∂K. The quadrilateral rule imply that no streamlines can meet (on either side of μ_1) and that they have constant speed until they meet γ_1 or γ_2, or hit ∂K.

The potential and its streamlines in the square.





Things I would like to understand:

- Convergence $\nabla u_p \to \nabla u$?
- Is the solution convex or concave in anyway?
- Classification of singularities? Maybe where the speed is constant, the function is more regular?
- Detect bifurcation or singularities in other geometries? Detect regularity?
- Can we without the convexity assumption prove something similar?
- Higher dimensions?

Thank you for listening!

Some references:

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