Bounded weak solutions to elliptic PDE with data in Orlicz spaces

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Abstract: It is a classical result due to Trudinger that if Q is a uniformly elliptic matrix, and $f \in L^q(\Omega)$, $q > \frac{n}{2}$, then weak solutions u of the Dirichlet problem

$$\begin{cases} -\operatorname{Div}(Q\nabla u) &= f \quad \text{for } x \in \Omega \\ u &= 0 \quad \text{for } x \in \partial\Omega \end{cases}$$

are bounded functions and satisfy

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^q(\Omega)}.$$

This result is sharp in the sense that if $q = \frac{n}{2}$, then there exist $f \in L^{\frac{n}{2}}(\Omega)$ such that this inequality fails even for the Laplacian ($Q = I$).

In this talk we will show that the endpoint result can be improved by considering functions f in the Orlicz space $L^{\frac{n}{2}}(\log L)^q(\Omega)$. More precisely, we show that we can take $f \in L^A(\Omega)$, where L^A is the Orlicz space induced by the Young function $A(t) = t^{\frac{n}{2}}\log(e+t)^q$, where $q > \frac{n}{2}$. Moreover, we can sharpen the estimate on the L^∞ norm to get

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^{\frac{n}{2}}(\Omega)}\left(1 + \log\left(1 + \frac{\|f\|_{L^A(\Omega)}}{\|f\|_{L^{\frac{n}{2}}(\Omega)}}\right)\right).$$

We can give examples to show that this result is (almost) sharp.

Our main theorem is a generalization of this result that applies to a large class of degenerate matrices Q that satisfy an appropriate weighted Sobolev inequality with gain. We will discuss the main ideas of our proof, which uses a version of DeGeorgi iteration. If there is time we will discuss further generalizations to operators with lower order terms.

Results in this talk are joint work with Scott Rodney, Cape Breton University.