Global Schauder estimates for the p-Laplace system

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p-Laplacian system

Find a vector field $\mathbf{u} : \Omega \to \mathbb{R}^N$ satisfying the following

system of partial differential equations

$$\begin{cases} \operatorname{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\right) = \mathbf{f} = \operatorname{div}\mathbf{F} & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

- Assume $\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain and $p \in (1, \infty)$;
- Weak solution $\mathbf{u} \in W_0^{1,p}(\Omega)$ exists provided $\mathbf{F} \in L^{p'}(\Omega)$;
- How does the regularity of $\mathbf{F}$ transfers to $\mathbf{u}$?

Classical results

Regularity theory for p-harmonic functions

$$\operatorname{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\right) = 0 \quad \text{in} \quad \Omega.$$

- Ural'tseva ('68), $N = 1$: $u \in C^{1,\alpha}(\Omega)$ for some $\alpha < 1$;
- Uhlenbeck ('77), $N \geq 2$ and $p > 2$: $\mathbf{u} \in C^{1,\alpha}(\Omega)$;
- Acerbi-Fusco/DiBenedetto/Manfredi/Tolksdorff ('83-'87), $N \geq 2$ and $p < 2$: $\mathbf{u} \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$;
- Manfredi-Iwaniec ('89), $n = 2$: optimal values for α.

Local maximal regularity theory

For a ball B such that $2B \Subset \Omega$ and a function space X

$$\||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\|_{X(B)} \leq c \, \|\mathbf{F}\|_{X(2B)}$$

- $X = L^q$ with $q \geq p'$: Iwaniec ('83), DiBenedetto-Manfredi ('93), Kinnuen-Zhou ('99);
- $X = \mathrm{BMO}$: DiBenedetto-Manfredi ('93), Diening-Kaplický-Schwarzacher ('12);
- $X = C^\alpha$: Diening-Kaplický-Schwarzacher ('12).

Global maximal regularity theory

For a bounded domain Ω of class ?? and a function space X

$$\||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\|_{X(\Omega)} \leq c \|\mathbf{F}\|_{X(\Omega)}$$

- For smooth domains local estimates are expected to extend;
- $X = L^q$ with $q \geq p'$: Kinnuen-Zhou ('01) for $\partial\Omega \in C^{1,\beta}$, Byun-Wang ('08) for Reifenberg-flat domains;
- $X = C^\alpha$ and $\partial\Omega \in C^{1,\beta}$: Lieberman ('88), Chen-Di Benedetto ('89), non-divergence-form;
- Conjecture 1: $X = C^\alpha$ and $\partial\Omega \in C^{1,\alpha}$;
- Conjecture 2: $X = \mathrm{BMO}$ and $\partial\Omega \in$??.

Continuous gradients

Minimal assumption for bounded gradients

$$\operatorname{div}\left(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\right) = \mathbf{f},$$
$$\mathbf{f} \in L^{n,1} \quad \Rightarrow \quad \nabla \mathbf{u} \in C^0.$$

- Lorentz-space $L^{n,1}$ with $L^{n+\varepsilon} \subsetneq L^{n,1} \subsetneq L^n$;
- Stein ('81) $p = 2$, $\mathbf{f} \in L^{n,1}$ is optimal: Cianchi ('92);
- $p > 1$ by Cianchi-Maz'ya/Kuusi-Mingione ('14);
- Minimal boundary regularity $\partial\Omega \in W^2 L^{n-1,1}$: Cianchi-Maz'ya.

BMO and Campanato spaces (1)

What happens for $q = \infty$?

$$\mathcal{M}^{\sharp,s} f(x) := \sup_{B \ni x} \left(\fint_B |f - (f)_B|^s \, \mathrm{d}y \right)^{\frac{1}{s}},$$

$$f \in \mathrm{BMO} :\Leftrightarrow \|\mathcal{M}^{\sharp} f\|_\infty < \infty.$$

- We have $L^\infty \subsetneq \mathrm{BMO} \subsetneq \cap_q L^q$;
- Singular integrals are continuous on BMO;
- Campanato spaces C^α as weighted BMO_ω-spaces:

$$\mathcal{M}^{\sharp}_{\omega} f(x) := \sup_{B \ni x} \frac{1}{\omega(r_B)} \fint_B |f - (f)_B| \, \mathrm{d}y, \quad \omega(r) = r^\alpha.$$

BMO and Campanato spaces (2)

Campanato semi-norms defined by

$$\|\mathbf{f}\|_{\mathcal{L}^{\omega(\cdot)}(\Omega)} = \sup_{\substack{x \in \Omega \\ r > 0}} \frac{1}{\omega(r)} \fint_{\Omega \cap B_r(x)} \left|\mathbf{f} - \langle \mathbf{f} \rangle_{\Omega \cap B_r(x)}\right| \mathrm{d}y,$$

where $\omega(r) r^{-\beta_0}$ is almost decreasing.

- $C^{r^\alpha} = \mathcal{L}^{r^\alpha}$ for $\alpha > 0$, but in general $C^\omega \subsetneq \mathcal{L}^\omega$;
- We say that ω satisfies the Dini condition if $\int_0 \frac{\omega(r)}{r} \mathrm{d}r < \infty$;
- Scale of function spaces between BMO to C^α, separated through Dini condition.

Global Schauder estimate (1)

Theorem (Breit-Cianchi-Diening-Schwarzacher)

Let Ω be bounded s.t. $\partial\Omega \in W^1\mathcal{L}^{\sigma(\cdot)} \cap C^{0,1}$ for some σ s.t.

$$\sup_{r\in(0,1)} \frac{\sigma(r)}{\omega(r)} \int_r^1 \frac{\omega(\rho)}{\rho}\,\mathrm{d}\rho \ll 1,$$

$$\Rightarrow \quad \||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\|_{\mathcal{L}^{\omega(\cdot)}(\Omega)} \leq c\|\mathbf{F}\|_{\mathcal{L}^{\omega(\cdot)}(\Omega)}.$$

- $\omega = 1$: BMO estimate for $\sigma(r) \ll |\log(r)|^{-1}$;
- $\omega = |\log(r)|^{-1}$: $\mathcal{L}^{|\log(r)|^{-1}}$-estimate for $\sigma(r) \ll |\log(r)|^{-1}|\log(|\log(r)|)|^{-1}$;
- $\omega = r^\alpha$: C^α-estimate for $\sigma(r) \ll r^\alpha$.

Global Schauder estimate (2)

Theorem (Breit-Cianchi-Diening-Schwarzacher)

Let $\omega \in C^1(0,\infty)$ be any concave parameter function, s.t. $\int_0 \frac{\omega(r)}{r}\,\mathrm{d}r = \infty$. Then there exist a parameter function σ, $\Omega \subset \mathbb{R}^2$, and $\mathbf{F} \in \mathcal{L}^{\omega(\cdot)}(\Omega)$ such that, if $\Delta u = \operatorname{div}\mathbf{F}$, then:

$$\partial\Omega \in W^1\mathcal{L}^{\sigma(\cdot)} \cap C^{0,1}, \qquad \sup_{r\in(0,1)} \frac{\sigma(r)}{\omega(r)} \int_r^1 \frac{\omega(\rho)}{\rho}\,\mathrm{d}\rho < \infty,$$

$$\text{but } \nabla u \notin \mathcal{L}^{\omega(\cdot)}(\Omega).$$

- The smallness is necessary!
- Covers BMO and $\mathcal{L}^{|\log(r)|^{-1}}$.

Global Schauder estimate (3)

Theorem (Breit-Cianchi-Diening-Schwarzacher)

Let Ω be bounded s.t. $\partial\Omega \in W^1\mathcal{L}^{\omega(\cdot)}$ for some parameter function ω satisfying $\int_0 \frac{\omega(r)}{r}\,\mathrm{d}r < \infty$. Then

$$\||\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}\|_{C^{0,\underline{\omega}(\cdot)}(\Omega)} \le C\|\mathbf{F}\|_{\mathcal{L}^{\omega(\cdot)}(\Omega)}, \quad \underline{\omega}(r) = \int_0^r \frac{\omega(\rho)}{\rho}\,\mathrm{d}\rho.$$

- No smallness is needed!
- Covers $\mathcal{L}^{|\log(r)|^{-2}}$ and C^α (where $\omega(r) \sim \underline{\omega}(r) \sim r^\alpha$);
- In particular: $\mathbf{F} \in C^\alpha$ and $\partial\Omega \in C^{1,\alpha} \Rightarrow |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u} \in C^\alpha$;
- $\|\mathbf{F}\|_{\mathcal{L}^{\omega(\cdot)}(\Omega)}$ can be replaced with $\|\mathbf{F}\|_{C^{\omega(\cdot)}(\Omega)}$.

Pointwise estimate

Breit-Cianchi-Diening-Kuusi-Schwarzacher (2018):

$$\mathcal{M}^\sharp(|\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u})(x) \leq c\,\mathcal{M}^{\sharp,p'}(\mathbf{F})(x) \quad \text{for a.e. } x \in \mathbb{R}^n\,.$$

- Lower order term for bounded domains;
- Implies the known maximal regularity estimates;
- New estimates in Lorentz- and Orlicz spaces;
- Also for weighted $\mathcal{M}^\sharp_\omega$.

Flattening and reflection

After a change of coordinates in a small half ball

$$\mathrm{div}_z(|\nabla\bar{\mathbf{u}}|^{p-2}\nabla\bar{\mathbf{u}}) = \mathrm{div}_z\left(|\nabla\bar{\mathbf{u}}|^{p-2}\nabla\bar{\mathbf{u}}) - |\nabla\bar{\mathbf{u}}\mathbb{J}|^{p-2}\nabla\bar{\mathbf{u}}\mathbb{J}\mathbb{J}^T + \underline{\mathbf{F}}\right)$$

- Method due to Chen-Di Benedetto ('89);
- Local coordinates $\Psi(x', x_n) = (x', x_n - \psi(x'))$, $\mathbb{J} = \langle\nabla\Psi\rangle\nabla\Psi^{-1}$;
- Reflect at the boundary and apply local estimates;
- Control error between $\nabla\bar{\mathbf{u}}$ and $\nabla\bar{\mathbf{u}}\mathbb{J}$ by boundary regularity.

Boundary pointwise estimate (1)

There are $c > 0$ and $\theta \in (0,1)$ s.t.

$$\frac{1}{\omega(\theta s)}\left(\fint_{\Omega\cap B_{\theta s}(x)} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}_{\theta s}|^{\min\{2,p'\}}\,\mathrm{d}y\right)^{\frac{1}{\min\{2,p'\}}}$$

$$\leq \frac{c\sigma(s)}{\omega(s)}\fint_{\Omega\cap B_s(x)} |\mathbf{A}(\nabla \mathbf{u})|\,\mathrm{d}y + \frac{c}{\omega(s)}\left(\fint_{\Omega\cap B_s(x)} |\mathbf{F} - \mathbf{F}_0|^{p'q}\,\mathrm{d}y\right)^{\frac{1}{p'q}}$$

$$+ \frac{1}{2\omega(s)}\left(\fint_{\Omega\cap B_s(x)} |\mathbf{A}(\nabla \mathbf{u}) - \mathbf{A}_s|^{\min\{2,p'\}}\,\mathrm{d}y\right)^{\frac{1}{\min\{2,p'\}}}.$$

Boundary pointwise estimate (2)

- Proof by pointwise estimate for flat reflected system+ boundary Gehring estimate;
- Claim follows if we control $\frac{\sigma(s)}{\omega(s)} \fint_{\Omega \cap B_s(x)} |\mathbf{A}(\nabla \mathbf{u})| \, \mathrm{d}y$;
- Control $\fint_{\Omega \cap B_s(x)} |\mathbf{A}(\nabla \mathbf{u})| \, \mathrm{d}y$ by oscillation $+$ lower order term;
- If $\int_0 \frac{\omega(r)}{r} \, \mathrm{d}r < \infty$ we can prove boundedness of $\nabla \mathbf{u}$.

The domain

Given ω set $\sigma = \gamma\omega\big(\int_\cdot \frac{\omega(\rho)}{\rho}\,\mathrm{d}\rho\big)^{-1}$, $\psi(r) = \int_0^r \sigma(\rho)\,\mathrm{d}\rho$ and

$$\Omega = \big\{\xi = x_1 + ix_2 \,:\, |\xi| < \tfrac{1}{2}, x_2 > -\psi(|x_1|)\big\}$$
$$= \big\{\xi = i\rho\exp(i\theta) \,:\, \rho < \tfrac{1}{2}, |\theta| < \tfrac{\pi}{2} + \varphi(\rho)\big\}.$$

- We have $\partial\Omega \in W^1\mathcal{L}^\sigma$.
- Let $\zeta(\xi)$ denote the conformal map of Ω onto the half-disc $\mathbb{D}^+ = \{\zeta \,:\, \mathrm{Im}(\zeta) > 0, |\zeta| < 1\}$, with fixed point $\xi = 0$.
- Aim: describe behaviour of ζ by tools from complex analysis (Warschawski, '42).

The conformal mapping

Warschawski ('44): recall $\psi(r) = \int_0^r \sigma(\rho)\,\mathrm{d}\rho$ and $\varphi \sim \sigma$

$$|\zeta(\xi)| = c\,\exp\Big(-\pi\int_\rho^{\frac{1}{2}} \frac{\mathrm{d}r}{r(\pi + 2\varphi(r))} + o(1)\Big) \quad \text{as } \xi \to 0$$

- Analyse singularity of ζ at $\xi = 0$;
- Embedding $W^1\mathcal{L}^\omega \hookrightarrow C^v$, where $v = r\int_r^1 \frac{\omega(\rho)}{\rho}\,\mathrm{d}\rho$;
- Let $\zeta(\xi)$ denote the conformal map of Ω onto the
- Show $\zeta \notin C^v$ s.t. $\nabla\zeta \notin W^1\mathcal{L}^\omega$.

Conclusion

Warschawski with $\theta = 0$ and $|\xi| = \rho$ $+ W^1\mathcal{L}^\omega \hookrightarrow C^v$ gives

$$c^{-1}\rho\left(\int_\rho^1 \frac{\omega(r)}{r}\,\mathrm{d}r\right)^{\frac{2\gamma}{\kappa\pi}} \le |\zeta(\xi)| \le c\rho\left(\int_\rho^1 \frac{\omega(r)}{r}\,\mathrm{d}r\right) \quad \rho \to 0.$$

- No Dini condition $\Rightarrow \int_0 \frac{\omega(r)}{r}\,\mathrm{d}r = \infty$;
- Choosing $\gamma > \frac{\kappa\pi}{2}$ gives contradiction;
- $\nabla\zeta \notin W^1\mathcal{L}^\omega$.