Two recent results on the double phase equations

Chao Zhang
School of Mathematics & IASM
Harbin Institute of Technology

哈爾濱工業大學

Monday's Nonstandard Seminar 2020/21

University of Warsaw

March 15, 2021

Outline

The talk is based on the following papers:

1. Y. Fang and C. Zhang, **Equivalence between viscosity and distributional solutions for the double-phase equation**, *Adv. Calc. Var.*, 2020, https://doi.org/10.1515/acv-2020-0059.
2. Y. Fang, V. Rădulescu, C. Zhang and X. Zhang, **Gradient estimates for multi-phase problems in Campanato spaces**, *Indiana Univ. Math. J.*, 2021, to appear.

A quick review of double phase problems

Double phase functional

- Consider the functional

$$\mathcal{F}(u) := \int_{\Omega} (|Du|^p + a(x)|Du|^q)\,dx,$$

  where $a(x) \geq 0, 1 < p \leq q$.

- In the previous talks, there have been many good introduction for such kind problems.

- It was initially introduced by Zhikov [1] [2] to model the strongly anisotropic materials.

---

[1] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, Izv. Akad. Nauk SSSR Ser. Mat. 50 (4) (1986), 675–710.

[2] V.V. Zhikov, On Lavrentiev's phenomenon, Russ. J. Math. Phys. 3 (2) (1995), 249–269.

Non-standard growth

- $a(x) \equiv 1$ and $p = q$: Standard growth.
- $a(x) \equiv 1$ and $p < q$: Non-standard growth. See the pioneering works by Marcellini [3] [4] [5].
- If p and q are not too far away, i.e.,

$$\frac{q}{p} < 1 + o(n), \quad o(n) \approx \frac{1}{n},$$

the minimizers are regular (due to Marcellini and Giaquinta).

[3] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions, Arch. Ration. Mech. Anal. 105 (1989), 267–284.

[4] P. Marcellini, Regularity and existence of solutions of elliptic equations with p, q-growth conditions, J. Differential Equations 90 (1991), 1–30.

[5] P. Marcellini, Regularity for elliptic equations with general growth conditions, J. Differential Equations 105 (1993), 296–333.

Non-autonomous case

- $a(x) \geq 0$, $a(x) \not\equiv 0$ and $p < q$: Non-autonomous case. New phenomena appear in this situation, and the presence of x *is not any longer a perturbation*.
- Indeed, even basic regularity issues for these double phase problems have remained unsolved for several decades. The first contribution was due to Colombo and Mingione [6] [7].

[6] M. Colombo, G. Mingione, Regularity for double phase variational problems, Arch. Ration. Mech. Anal. 215 (2015), 443–496.

[7] M. Colombo, G. Mingione, Bounded minimisers of double phase variational integrals, Arch. Ration. Mech. Anal. 218 (2015), 219–273.

Known results

Theorem (Colombo-Mingione, ARMA 2015)

Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the functional

$$u \longmapsto \int_\Omega (|Du|^p + a(x)|Du|^q)\, dx$$

and assume that

$$0 \leq a(\cdot) \in C^\alpha(\Omega), \quad \frac{q}{p} < 1 + \frac{\alpha}{n}$$

then

Du is locally Hölder continuous.

Known results

Theorem (Colombo-Mingione, ARMA 2015)

Let $u \in W^{1,p}(\Omega)$ be a bounded local minimizer of the functional

$$u \longmapsto \int_\Omega (|Du|^p + a(x)|Du|^q)\, dx$$

and assume that

$$0 \leq a(\cdot) \in C^\alpha(\Omega), \quad q \leq p + \alpha$$

then

Du is locally Hölder continuous.

Known results

Chao Zhang

Theorem (Baroni-Colombo-Mingione, Calc. Var. 2018)

Let $u \in W^{1,p}(\Omega)$ be a local minimizer of the functional

$$u \longmapsto \int_\Omega (|Du|^p + a(x)|Du|^q)\, dx, \quad 0 \le a(\cdot) \in C^\alpha(\Omega)$$

and assume that one of the following assumptions holds:

- $q/p \le 1 + \alpha/n$
- $u \in L^\infty$ *and* $q \le p + \alpha$
- $u \in C^\gamma$ *and* $q < p + \frac{\alpha}{1-\gamma}$, $\gamma \in (0,1)$

then

Du is locally Hölder continuous.

Known results

Theorem (Colombo-Mingione, JFA 2016; De Filippis-Mingione, St. Petersburg Math. J. 2020)

Let $u \in W^{1,p}(\Omega)$ *be a weak solution to*

$$\begin{aligned}\operatorname{div}(|Du|^{p-2}Du + a(x)|Du|^{q-2}Du) \\ = \operatorname{div}(|F|^{p-2}F + a(x)|F|^{q-2}F)\end{aligned}$$

and assume that

$$0 \le a(\cdot) \in C^{\alpha}(\Omega), \quad \frac{q}{p} \le 1 + \frac{\alpha}{n},$$

then for every $\gamma \ge 1$,

$$(|F|^p + a(x)|F|^q) \in L^{\gamma}_{\mathrm{loc}} \Longrightarrow (|Du|^p + a(x)|Du|^q) \in L^{\gamma}_{\mathrm{loc}}.$$

Global case: Byun-Oh, Calc. Var. 2017

More results (just mention a few...)

- De Filippis-Mingione [JGA 2020, JGA 2020, arXiv 2020, 2021]: **Manifold constrained problems and several optimal regularity results for a large class of non-autonomous problems.**
- Baasandorj-Byun-Oh [JFA 2020], Byun-Oh [Anal. PDE 2020]: **Generalized double phase problem.**
- Chlebicka-De Filippis [AMPA 2020]: **Removability of the singularities; obstacle problems.**
- De Filippis-Oh [JDE 2019], Baasandorj-Byun-Oh [Calc. Var. 2021]: **Multi-phase problems.**
- De Filippis-Palatucci [JDE 2019], Scott-Mengesha [arXiv 2020]: **Nonlocal double phase problems.**

More results (just mention a few...)

- Ragusa-Tachikawa [Adv. Nonlinear Anal. 2020], Tachikawa [JMAA 2020], Byun-Lee [Quart. J. Math. 2021]: **Double phase problems with variable exponents.**
- Hästö-Ok [JEMS 2020]: **Maximal regularity for local minimizers of non-autonomous functionals.**
- Balci-Diening-Surnachev [Calc. Var. 2020]: **Lavrentiev gap without the dimension threshold for the double phase functionals.**
- De Filippis [Proc. Roy. Soc. Edinburgh Sect. A 2021], Da Silva-Ricarte [Calc. Var. 2020]: **Fully nonlinear problems with double phase type degeneracies.**

More results (just mention a few...)

- **Bögelein-Duzaar-Marcellini-Scheven [ARMA 2018]: Variational solutions to doubly nonlinear parabolic equations.**
- **Chlebicka-Gwiazda-Zatorska-Goldstein [JDE 2019; Ann. Inst. H. Poincaré Anal. Non Linéaire 2019]: Renormalized solutions and weak solutions to a family of general parabolic equations.**
- **De Filippis [Calc. Var. 2020]: Gradient bounds for solutions to parabolic double phase equations.**
- **Radulescu-Zhang [JMPA 2018], Papageorgiou-Radulescu-Repovs [Proc. AMS 2019, ZAMP 2019]: Double phase variational problems.**

More results (just mention a few...)

- **Bahrouni-Radulescu-Repovs [Nonlinearity 2019]: Double phase transonic flow problems with variable growth.**
- **Mascolo, Gupini, Esposito, Leonetti, Fonseca, Malý, Eleuteri, Passarelli di Napoli, Harjulehto, Karppinen, Gasinski, Lee, Winkert, Fiscella, Zheng, Dai, and many many others...**

Our results

Chao Zhang

Main results

Theorem 1. [Fang–Z., Adv. Calc. Var. 2020]

Consider

$$-\operatorname{div}(|Du|^{p-2}Du + a(x)|Du|^{q-2}Du) = 0 \quad \text{in } \Omega, \qquad (1)$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $1 < p \leq q < \infty$.

- If $0 \leq a(\cdot) \in C^{\alpha}(\Omega), \alpha \in (0,1]$ and $\frac{q}{p} \leq 1 + \frac{\alpha}{n}$, then the weak solutions coincide with $\mathcal{A}_{H(\cdot)}$-harmonic functions.
- If $0 < a(x) \in C^1(\Omega)$ and $\frac{q}{p} \leq 1 + \frac{1}{n}$, then the viscosity solutions coincide with $\mathcal{A}_{H(\cdot)}$-harmonic functions.

Notions of solutions

Define

$$A(x,\xi) := |\xi|^{p-2}\xi + a(x)|\xi|^{q-2}\xi,$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

Definition (weak solution)

A function $u \in W^{1,H(\cdot)}_{\mathrm{loc}}(\Omega)$ is called a weak supersolution, if

$$\int_\Omega \langle A(x,Du), D\eta\rangle \, dx \geq 0$$

for every nonnegative function $\eta \in W^{1,H(\cdot)}_0(\Omega)$. The inequality is converse for weak subsolution.

Notions of solutions

Definition ($\mathcal{A}_{H(\cdot)}$-harmonic function)

We say that $u : \Omega \to (-\infty, \infty]$ is a $\mathcal{A}_{H(\cdot)}$-superharmonic function in Ω, if

1. u is lower semicontinuous in Ω;
2. u is finite a.e. in Ω;
3. for any subdomain $D \subset\subset \Omega$ the comparison principle holds: when $h \in C(\overline{D})$ is a weak solution to (1), and $u \geq h$ on ∂D, then

$$u \geq h \quad \text{in } D.$$

If $-u$ is $\mathcal{A}_{H(\cdot)}$-superharmonic, then $u : \Omega \to [-\infty, \infty)$ is called $\mathcal{A}_{H(\cdot)}$-subharmonic function.

Notions of solutions

Chao Zhang

Definition (viscosity solution)

A lower semicontinuous function $u : \Omega \to (-\infty, \infty]$ is a viscosity supersolution in Ω, if u is finite a.e. in Ω and for each $\varphi \in C^2(\Omega)$ such that

$$\begin{cases} \varphi(x_0) = u(x_0) & x_0 \in \Omega, \\ \varphi(x) < u(x) & x \neq x_0, \\ D\varphi(x_0) \neq 0, \end{cases}$$

there holds

$$-\operatorname{div} A(x_0, D\varphi(x_0)) \geq 0.$$

A function u is viscosity subsolution, when $-u$ is a viscosity supersolution.

Remark

- We revisit the methods developed by Juutinen-Lindqvist-Manfredi [SIAM J. Math. Anal. 2001] and Juutinen-Lukkari-Parviainen [Ann. Inst. H. Poincaré Anal. Non Linéaire 2010].
- Our proof relies on: different comparison principle, the approximation technique and 'the theorem of sums' in [Crandall-Ishii-Lions 1992].
- The double-phase problem is not translation invariant and possesses two diverse growth terms, which generates more delicate difficulties than equations of p-Laplace type.

Related paper

- I. Chlebicka, A. Zatorska-Goldstein, Generalized superharmonic functions with strongly nonlinear operator, arXiv:2005.00118
  - The properties of $\mathcal{A}$-harmonic ($\mathcal{A}$-superharmonic) functions involving an operator having generalized Orlicz growth were studied.
  - The framework embraces reflexive Orlicz spaces, as well as natural variants of variable exponent and double-phase spaces.

Work in progress

Consider the following nonlocal double phase equation:

$$\mathcal{L}u(x) := \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\,dy + \int_{\mathbb{R}^n} a(x,y)\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+tq}}\,dy = 0,$$

where $p \le q$ and $a(\cdot,\cdot) \ge 0$.

Work in progress

Our aim:

- [Fang-Z., in preparation] To prove that the (bounded) viscosity solutions are the (bounded) fractional harmonic functions and/or (bounded) weak solutions, and vice versa (proposed by De Filippis-Palatucci [JDE 2019]).

Based on the papers:

- Korvenpää-Kuusi-Palatucci [Calc. Var. 2016, Math. Ann. 2017].
- Korvenpää-Kuusi-Lindgren [JMPA 2019].
- Di Castro-Kuusi-Palatucci [Ann. Inst. H. Poincaré Anal. Non Linéaire 2016].

Main results

Theorem 2. [Fang–Radulescu–Z.–Zhang, Indiana Univ. Math. J. 2021]

Consider the multi-phase equation

$$\operatorname{div}\left(|Du|^{p-2}Du + a(x)|Du|^{q-2}Du + b(x)|Du|^{s-2}Du\right) = \operatorname{div}\left(|F|^{p-2}F + a(x)|F|^{q-2}F + b(x)|F|^{s-2}F\right) \quad \text{in } \Omega.$$

Under the assumptions:

- $a \in C^{\alpha}_{\mathrm{loc}}(\Omega), b \in C^{\beta}_{\mathrm{loc}}(\Omega), \alpha, \beta \in (0,1], a, b \geq 0$
- $1 < p < q \leq s < \infty, \frac{q}{p} \leq 1 + \frac{\alpha}{n}, \frac{s}{p} \leq 1 + \frac{\beta}{n}$

we have

$$A(x,F) \in \mathcal{C} \implies Du \in \mathcal{C}',$$

where $A(x,z) = |z|^{p-2}z + a(x)|z|^{q-2}z + b(x)|z|^{s-2}z$ and $\mathcal{C}, \mathcal{C}'$ are Campanato spaces.

Main results

Remark: This result recovers the BMO type estimates for p-Laplace equation.

- DiBenedetto-Manfredi [Amer. J. Math. 1993] established the BMO estimates for the weak solution of

$$\operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F),$$

  which states that if $p > 2$ and $|F|^{p-2}F \in \mathrm{BMO}(\mathbb{R}^n)$, then $Du \in \mathrm{BMO}(\mathbb{R}^n)$. Meanwhile, the local counterpart

$$|F|^{p-2}F \in \mathrm{BMO}_{\mathrm{loc}}(\Omega) \Rightarrow Du \in \mathrm{BMO}_{\mathrm{loc}}(\Omega)$$

  is obtained simultaneously for $p > 2$.
- Diening-Kaplický-Schwarzacher [Nonlinear Anal. 2012]: BMO estimates for the full case $1 < p < \infty$ (even more general growth).

Sketch of proof

Theorem 2

Notations

Consider

$$\operatorname{div}\left(|Du|^{p-2}Du + a(x)|Du|^{q-2}Du + b(x)|Du|^{s-2}Du\right) = \operatorname{div}\left(|F|^{p-2}F + a(x)|F|^{q-2}F + b(x)|F|^{s-2}F\right) \quad \text{in } \Omega.$$

Assume that

$$a \in C^{\alpha}_{\mathrm{loc}}(\Omega), b \in C^{\beta}_{\mathrm{loc}}(\Omega), \alpha, \beta \in (0,1], a, b \geq 0 \qquad (2)$$

and

$$1 < p < q \leq s < \infty, \frac{q}{p} \leq 1 + \frac{\alpha}{n}, \frac{s}{p} \leq 1 + \frac{\beta}{n}. \qquad (3)$$

We shall use the notation

$$A(x,z) = |z|^{p-2}z + a(x)|z|^{q-2}z + b(x)|z|^{s-2}s$$

and

$$H(x,z) = |z|^p + a(x)|z|^q + b(x)|z|^s$$

whenever $x \in \Omega$ and $z \in \mathbb{R}^n$.

Campanato space

For any $x \in \Omega$, $\rho > 0$, let $\Omega(x,\rho) := \Omega \cap B(x,\rho)$.

Definition

Let $s \geq 1$, $\mu \geq 0$. The Campanato space $\mathcal{L}^{s,\mu}(\Omega)$ is the class of all functions $u \in L^s(\Omega)$ such that

$$[u]_{s,\mu;\Omega} := \sup_{x\in\Omega,\, 0<\rho<\operatorname{diam}\Omega} \left(\rho^{-\mu} \int_{\Omega(x,\rho)} |u(z) - u_{x,\rho}|^s \, dz \right)^{\frac{1}{s}}$$

is finite, where

$$u_{x,\rho} = \frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} u(z)\, dz$$

and $|\Omega(x,\rho)|$ is the Lebesgue measure of $\Omega(x,\rho)$.

Main theorem

Chao Zhang

Theorem

Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local weak solution with

$$H(x, Du) \in L^1_{\mathrm{loc}}(\Omega), \quad H(x, F) \in L^{1+\sigma}_{\mathrm{loc}}(\Omega)$$

for some $\sigma > 0$. Under the assumptions (2) and (3), if $A(x, F) \in \mathcal{L}^{\frac{p}{p-1},\mu}_{\mathrm{loc}}(\Omega)$, where $0 < \mu < n$, we have

$$Du \in \mathcal{L}^{p,\widetilde{\mu}}_{\mathrm{loc}}(\Omega).$$

Here the constant $\widetilde{\mu}$ is defined by

$$\widetilde{\mu} = \begin{cases} \mu & \text{if } 2 \le p < q \le s, \\ (p-1)\mu & \text{others.} \end{cases}$$

Remark and main tools

- It should be mentioned that the higher integrability assumption $H(\cdot, F) \in L^{1+\sigma}_{\mathrm{loc}}(\Omega)$ is needed only for treating the borderline case

$$\frac{q}{p} = 1 + \frac{\alpha}{n}, \quad \frac{s}{p} = 1 + \frac{\beta}{n}.$$

- The technical approach is based on the different comparison estimates along with the good properties of homogeneous problems and the appropriate localization method.

Key steps

Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local weak solution. For any bounded domain $\Omega_0 \subset\subset \widetilde{\Omega} \subset\subset \Omega$ and $x^0 \in \Omega_0$, we first consider the following homogeneous problem:

$$\begin{cases} \operatorname{div} A(x, Dw) = 0 \quad \text{in } B_{2R}(x^0), \\ w \in u + W_0^{1,p}(B_{2R}(x^0)), \quad R > 0. \end{cases} \tag{4}$$

In the following, we suppose

$$x^0 = 0, \quad R \leq \frac{1}{2}\mathrm{dist}(\Omega_0, \partial\widetilde{\Omega}) := R_0 \leq 1$$

and set

$$K := \int_{B_{2R_0}} H(x, Du)\, dx.$$

Key steps

Lemma

There exists a unique weak solution $w \in u + W_0^{1,p}(B_{2R})$ to (4) such that $H(x, Dw) \in L^1(B_{2R})$. Moreover,

(1) *for* $1 < p < q \leq s$

$$\int_{B_{2R}} H(x, Dw)\, dx \leq C \int_{B_{2R}} H(x, Du)\, dx,$$

where C depends on p, q, s;

(2) *for* $2 \leq p < q \leq s$

$$\int_{B_{2R}} H(x, D(u - w))\, dx \leq CR^{\mu}[A(x, F)]^{\frac{p}{p-1}}_{\frac{p}{p-1},\mu;\widetilde{\Omega}},$$

where C depends on p, q, s, n;

Key steps

(3) for $1 < p < q \le s < 2$

$$\int_{B_{2R}} H(x, D(u-w))\,dx \le CR^{\mu(p-1)}\left([A(x,F)]_{\frac{p}{p-1},\mu;\widetilde{\Omega}} + 1\right)^{\frac{ps}{2p-s}},$$

where C depends on p, q, s, n, K;

(4) for $1 < p < 2 \le q \le s$ or $1 < p < q < 2 \le s$

$$\int_{B_{2R}} H(x, D(u-w))\,dx \le CR^{\mu(p-1)}\left([A(x,F)]_{\frac{p}{p-1},\mu;\widetilde{\Omega}} + 1\right)^{\frac{p}{p-1}},$$

where C depends on p, q, s, n, K.

Key steps

Lemma

For $0 < \delta < n$ and $R \leq R_0$, we have

$$\int_{B_R} H(x, Dw)\, dx \leq CKR^{n-\delta},$$

where the constant C depends on δ, R_0 and data.

Proof: Similarly to the arguments of Theorem 2 in [De Filippis-Oh, JDE 2019], we could obtain the following Morrey type estimate. For any $0 < \delta < n$ and $0 < \rho \leq R$, we have

$$\int_{B_\rho} H(x, Dw)\, dx \leq C \left(\frac{\rho}{R}\right)^{n-\delta} \int_{B_R} H(x, Dw)\, dx. \tag{5}$$

Recalling that

$$\int_{B_\rho} H(x, Dw)\,dx \le C\left(\frac{\rho}{R}\right)^{n-\delta} \int_{B_{2R}} H(x, Du)\,dx,$$

then

$$\begin{aligned}\int_{B_\rho} H(x, Dw)\,dx &\le C\rho^{n-\delta} R_0^{\delta-n} \int_{B_{2R_0}} H(x, Du)\,dx \\ &\le CK\rho^{n-\delta}\end{aligned}$$

by choosing $R = R_0$.

Key steps

Next, we introduce further comparison problems. Denote

$$a_0(R) = \inf_{x\in B_R} a(x) \quad \text{and} \quad b_0(R) = \inf_{x\in B_R} b(x).$$

Let $v \in W^{1,1}(B_R)$ be the weak solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}(|Dv|^{p-2}Dv + a_0(R)|Dv|^{q-2}Dv \\ \qquad +b_0(R)|Dv|^{s-2}Dv) = 0 \quad \text{in } B_R, \\ v \in w + W_0^{1,p}(B_R). \end{cases}$$

Key steps

Denote

$$\begin{cases} A_0(z) := |z|^{p-2}z + a_0(R)|z|^{q-2}z + b_0(R)|z|^{s-2}z, \\ H_0(z) := |z|^p + a_0(R)|z|^q + b_0(R)|z|^s. \end{cases}$$

Lemma

Let $v \in W^{1,1}(B_R)$ be a weak solution such that $H_0(Dv) \in L^1(B_R)$, then

$$\int_{B_R} H_0(Dv)\,dx \leq C \int_{B_R} H_0(Dw)\,dx,$$

where the constant C depends on p, q, s.

Key steps

Lemma

For any $0 < R \leq R_0$, *we have*

$$\int_{B_R} H_0(Dw)\,dx \leq CKR^{n-\delta}$$

and

$$\int_{B_R} H_0(Dv)\,dx \leq CKR^{n-\delta},$$

where C *depends on* δ, R_0 *and* data.

Key steps

Finally, we give the following key result.

Theorem

Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local weak solution. For any $0 < \rho \leq \frac{R_0}{2}$, $\delta < n - \mu$, $x \in \Omega$, we have

$$\rho^{-\widetilde{\mu}} \int_{B_\rho(x)} |Du - (Du)_{B_\rho(x)}|^p \, dy \leq C, \tag{6}$$

where

$$\widetilde{\mu} = \begin{cases} \mu & \text{if } 2 \leq p < q \leq s, \\ (p-1)\mu & \text{others} \end{cases}$$

and the constant C depends on $R_0, \delta, [A(x, F)]_{\frac{p}{p-1}, \mu; \widetilde{\Omega}}$, data.

Key steps

Proof. From the work of [Lieberman, CPDE 1991], there exists $\gamma \in (0,1)$ such that for any $0 < \rho < \frac{R}{2}$,

$$\begin{aligned}
&\fint_{B_\rho} |Dv - (Dv)_{B_\rho}|^p + a_0(R)|Dv - (Dv)_{B_\rho}|^q \\
&\qquad + b_0(R)|Dv - (Dv)_{B_\rho}|^s \, dx \\
&\le C\left(\frac{\rho}{R}\right)^\gamma \fint_{B_R} |Dv - (Dv)_{B_R}|^p + a_0(R)|Dv - (Dv)_{B_R}|^q \\
&\qquad + b_0(R)|Dv - (Dv)_{B_R}|^s \, dx.
\end{aligned}$$

Key steps

Chao Zhang

Observe that

$$\fint_{B_\rho} |Du-(Du)_{B_\rho}|^p\,dx \le C\fint_{B_\rho} |Du-(Dv)_{B_\rho}|^p\,dx$$

$$\le C\fint_{B_\rho} |Du-Dw|^p\,dx + C\fint_{B_\rho} |Dw-Dv|^p\,dx$$

$$+ C\fint_{B_\rho} |Dv-(Dv)_{B_\rho}|^p\,dx$$

$$\le C\left(\frac{R}{\rho}\right)^n \fint_{B_R} |Du-Dw|^p\,dx$$

$$+ C\left(\frac{R}{\rho}\right)^n \fint_{B_R} |Dw-Dv|^p\,dx$$

$$+ C\left(\frac{\rho}{R}\right)^\gamma \Big[\fint_{B_R} |Dv-(Dv)_{B_R}|^p dx +$$

$$a_0(R)\fint_{B_R} |Dv-(Dv)_{B_R}|^q\,dx + b_0(R)\fint_{B_R} |Dv-(Dv)_{B_R}|^s\,dx\Big].$$

Key steps

Note that

$$
\begin{aligned}
&\fint_{B_R} |Dv - (Dv)_{B_R}|^p \, dx \\
&\leq C \fint_{B_R} |Dv - (Du)_{B_R}|^p \, dx \\
&\leq C \fint_{B_R} |Dv - Dw|^p \, dx + C \fint_{B_R} |Dw - (Dw)_{B_R}|^p dx \\
&\quad + C \fint_{B_R} |(Dw)_{B_R} - (Du)_{B_R}|^p \, dx \\
&\leq C \fint_{B_R} |Dv - Dw|^p \, dx + C \fint_{B_R} |Du - Dw|^p \, dx \\
&\quad + C \fint_{B_R} |Du - (Du)_{B_R}|^p \, dx.
\end{aligned}
$$

Key steps

Thus

$$
\begin{aligned}
&\fint_{B_\rho} |Du - (Du)_{B_\rho}|^p \, dx \\
\le\, & C\left[\left(\frac{R}{\rho}\right)^n + \left(\frac{\rho}{R}\right)^\gamma\right] \fint_{B_R} |Du - Dw|^p \, dx \\
&+ C\left[\left(\frac{R}{\rho}\right)^n + \left(\frac{\rho}{R}\right)^\gamma\right] \fint_{B_R} |Dw - Dv|^p \, dx \\
&+ C\left(\frac{\rho}{R}\right)^\gamma \fint_{B_R} a_0(R)|Dv - (Dv)_{B_R}|^q \\
&\qquad\qquad + b_0(R)|Dv - (Dv)_{B_R}|^s \, dx \\
&+ C\left(\frac{\rho}{R}\right)^\gamma \fint_{B_R} |Du - (Du)_{B_R}|^p \, dx.
\end{aligned}
$$

Key steps

Then combining these previous inequalities arrives at

$$
\begin{aligned}
&\fint_{B_\rho} |Du-(Du)_{B_\rho}|^p\,dx \\
&\le C\left(\frac{\rho}{R}\right)^\gamma \fint_{B_R} |Du-(Du)_{B_R}|^p\,dx \\
&\quad + C\left[\left(\frac{R}{\rho}\right)^n+\left(\frac{\rho}{R}\right)^\gamma\right]\fint_{B_R}|Du-Dw|^p\,dx \\
&+ C\left[\left(\frac{R}{\rho}\right)^n+\left(\frac{\rho}{R}\right)^\gamma\right]\left(\fint_{B_R} H_0(Dv)\,dx+\fint_{B_R} H_0(Dw)\,dx\right).
\end{aligned}
$$

Key steps

For any $R \le \frac{R_0}{2}$, we could derive the following results:

Case 1. When $2 \le p < q \le s < \infty$, we get

$$
\begin{aligned}
&\fint_{B_\rho} |Du - (Du)_{B_\rho}|^p \, dx \\
\le\ & C \left(\frac{\rho}{R}\right)^\gamma \fint_{B_R} |Du - (Du)_{B_R}|^p \, dx \\
&+ C \left[\left(\frac{R}{\rho}\right)^n + \left(\frac{\rho}{R}\right)^\gamma \right] R^{\mu - n} [A(x, F)]^{\frac{p}{p-1}}_{\frac{p}{p-1}, \mu; \widetilde{\Omega}} \\
&+ C \left[\left(\frac{R}{\rho}\right)^n + \left(\frac{\rho}{R}\right)^\gamma \right] R^{-\delta}.
\end{aligned}
$$

Denote

$$\phi(\rho) = \int_{B_\rho} |Du - (Du)_{B_\rho}|^p \, dx.$$

Key steps

Taking $\delta < n - \mu$, we have

$$\begin{aligned}\phi(\rho) &\le C\left[R^n + \left(\frac{\rho}{R}\right)^{\gamma}\rho^n\right]R^{\mu-n}[A(x,F)]^{\frac{p}{p-1}}_{\frac{p}{p-1},\mu;\widetilde{\Omega}} \\ &\quad + C\left(\frac{\rho}{R}\right)^{\gamma+n}\phi(R) + C\left[R^n + \left(\frac{\rho}{R}\right)^{\gamma}\rho^n\right]R^{-\delta} \\ &\le C\left(\frac{\rho}{R}\right)^{\gamma+n}\phi(R) + CR^{n-\delta} + CR^{\mu} \\ &\le C\left(\frac{\rho}{R}\right)^{\gamma+n}\phi(R) + CR^{\mu},\end{aligned}$$

for any $0 < \rho \le R \le \frac{R_0}{2}$ and $0 < \mu < n$.

Key steps

By using the iteration lemma [8], we get

$$\phi(\rho) \le C\left[\left(\frac{\rho}{R}\right)^{\mu}\phi(R) + \rho^{\mu}\right] \le C\left[\rho^{\mu}R_0^{-\mu}\phi\left(\frac{R_0}{2}\right) + \rho^{\mu}\right].$$

Then

$$\rho^{-\mu}\int_{B_\rho} |Du - (Du)_{B_\rho}|^p dx \le C.$$

[8] E. Giusti, Direct Methods in the Calculus of Variations, World Scientific Publishing Co, Inc., River Edge, NJ, 2003.

Key steps

Case 2. If $1 < p < q \leq s < 2$, then we have

$$
\begin{aligned}
&\fint_{B_\rho} |Du - (Du)_{B_\rho}|^p \, dx \\
\leq\, & C\left[\left(\frac{R}{\rho}\right)^n + \left(\frac{\rho}{R}\right)^\gamma\right] R^{\mu(p-1)-n}\left(1 + [A(x,F)]_{\frac{p}{p-1},\mu;\widetilde{\Omega}}\right)^{\frac{ps}{2p-s}} \\
&+ C\left[\left(\frac{R}{\rho}\right)^n + \left(\frac{\rho}{R}\right)^\gamma\right] R^{-\delta} \\
&+ C\left(\frac{\rho}{R}\right)^\gamma \fint_{B_R} |Du - (Du)_{B_R}|^p \, dx.
\end{aligned}
$$

Key steps

By choosing $\delta < n - \mu(p-1)$, after some calculations we arrive at

$$\begin{aligned}\phi(\rho) &\le C\left(\frac{\rho}{R}\right)^{\gamma+n}\phi(R) + CR^{n-\delta} + CR^{\mu(p-1)}\\ &\le C\left(\frac{\rho}{R}\right)^{\gamma+n}\phi(R) + CR^{\mu(p-1)},\end{aligned}$$

for any $0 < \rho \le R \le \frac{R_0}{2}$. Note that $\mu(p-1) < \mu < n$. Again utilizing the iteration lemma, we get

$$\begin{aligned}\phi(\rho) &\le C\left[\left(\frac{\rho}{R}\right)^{\mu(p-1)}\phi(R) + \rho^{\mu(p-1)}\right]\\ &\le C\left[\rho^{\mu(p-1)}R_0^{-\mu(p-1)}\phi\left(\frac{R_0}{2}\right) + \rho^{\mu(p-1)}\right].\end{aligned}$$

Key steps

Therefore, we find

$$\rho^{-\mu(p-1)} \int_{B_\rho} |Du - (Du)_{B_\rho}|^p dx \leq C.$$

Case 3. For the case $1 < p < 2 \leq q \leq s$ or $1 < p < q < 2 \leq s$, we also derive that

$$\rho^{-\mu(p-1)} \int_{B_\rho} |Du - (Du)_{B_\rho}|^p dx \leq C.$$

Key steps

Consequently, when $\delta < n - \mu$, we could verify that for any $x \in \Omega$,

$$\rho^{-\widetilde{\mu}} \int_{B_\rho(x)} |Du - (Du)_{B_\rho(x)}|^p \, dy \le C \int_{B_{\frac{R_0}{2}}(x)} |Du - (Du)_{B_{\frac{R_0}{2}}(x)}|^p \, dy \le C,$$

where

$$\widetilde{\mu} = \begin{cases} \mu & \text{if } 2 \le p < q \le s, \\ (p-1)\mu & \text{others} \end{cases}$$

and C depends on $R_0, \delta, [A(x,F)]_{\frac{p}{p-1},\mu;\widetilde{\Omega}}$, data.

Thank you for your attention!