Regularity results for Minimizers of Discontinuous Quasiconvex Integrals with General Growth

C.S.Goodrich (UNSW, Sydney, Australia),
Giovanni Scilla (Federico II, Napoli, ITALY),
Bianca Stroffolini (Federico II, Napoli, ITALY)

Monday's Nonstandard Seminar
Mimuw, University of Warsaw
March 8, 2021

Outline

The Problem

Regularity results for local minimizers of

$$F(u): = \int_\Omega f(x,u,Du)\,dx$$

where $f:\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\to\mathbb{R}$ satisfies

- $\mathrm{P}\to f(\cdot,\cdot,\mathrm{P})\in C^1(\mathbb{R}^{N\times n})\cap C^2(\mathbb{R}^{N\times n}\backslash\{0\})$
- coercivity: $\nu\varphi(|\mathrm{P}|)\le f(x,\mathrm{u},\mathrm{P})-f(x,\mathrm{u},0)$;
- general growth: $|Df(x,\mathrm{u},\mathrm{P})|\le L\varphi'(|\mathrm{P}|)\,,\quad |D^2f(x,\mathrm{u},\mathrm{P})|\le L\varphi''(|\mathrm{P}|)$;
- degenerate quasiconvex

$$\int_B f(x,\mathrm{u},\mathrm{P}+D\eta(y))-f(x,\mathrm{u},\mathrm{P})\,\mathrm{d}y\ge\nu\int_B\varphi''(\mu+|\mathrm{P}|+|D\eta(y)|)\,|D\eta(y)|^2\,\mathrm{d}y$$

- VMO condition in x uniformly in (u,P);

$$|f(x,\mathrm{u},\mathrm{P})-(f(\cdot,\mathrm{u},\mathrm{P}))_{x_0,r}|\le v_{x_0}(x,r)\varphi(|\mathrm{P}|)\,,\qquad\text{for all } x\in B_r(x_0)$$

where $x_0\in\Omega$, $r\in(0,1]$ and $\mathrm{P}\in\mathbb{R}^{N\times n}$ and $v_{x_0}:\mathbb{R}^n\times[0,1]\to[0,2L]$ are bounded functions such that

$$\lim_{\varrho\to0}\mathcal{V}(\varrho)=0\,,\ \text{ where } \mathcal{V}(\varrho):=\sup_{x_0\in\Omega}\sup_{0<r\le\varrho}\fint_{B_r(x_0)}v_{x_0}(x,r)\,\mathrm{d}x\,,$$

and

$$(f(\cdot,\mathrm{u},\mathrm{P}))_{x_0,r}:=\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}f(x,\mathrm{u},\mathrm{P})\,\mathrm{d}x\,;$$

- f is uniformly continuous with respect to the u variable;

$$|f(x,\mathrm{u},\mathrm{P}) - f(x,\mathrm{u}_0,\mathrm{P})| \le L\omega(|\mathrm{u}-\mathrm{u}_0|)\varphi(|\mathrm{P}|)\,,$$

  where $\omega : [0,\infty) \to [0,1]$ is a nondecreasing, concave modulus of continuity; i.e., $\lim_{t\downarrow 0}\omega(t) = \omega(0) = 0$.

- the second derivatives $D^2 f$ are Hölder continuous away from 0; with some exponent $\beta_0 \in (0,1)$ such that uniformly in (x,u) and for $0 < |\mathrm{P}| \le \frac{1}{2}|\mathrm{Q}|$

$$|D^2 f(x,\mathrm{u},\mathrm{P}) - D^2 f(x,\mathrm{u},\mathrm{P}+\mathrm{Q})| \le c_0\,\varphi''(|\mathrm{Q}|)\,|\mathrm{Q}|^{-\beta_0}|\mathrm{P}|^{\beta_0}\,;$$

- the function $\mathrm{P} \to Df(x,\mathrm{u},\mathrm{P})$ behaves asymptotically at 0 as the φ-Laplacian.

$$\lim_{t\to 0}\frac{Df(x,\mathrm{u},t\mathrm{P})}{\varphi'(t)} = \mathrm{P}\,,$$

  uniformly in $\{\mathrm{P} \in \mathbb{R}^{N\times n} :\ |\mathrm{P}| = 1\}$ and uniformly for all $x \in \Omega$ and $\mathrm{u} \in \mathbb{R}^N$.

φ N-function

- $\varphi(0) = 0$
- φ' right continuous, non-decreasing
- $\varphi'(0) = 0$, $\varphi'(t) > 0$ for $t > 0$, and $\lim_{t\to\infty} \varphi'(t) = \infty$.

Orlicz-Sobolev

- L^φ : $f \in L^\varphi$ iff there exists $K > 0$ such that $\int \varphi(\frac{|f|}{K})\, dx < \infty$
- $W^{1,\varphi}$: $f \in W^{1,\varphi}$ iff $f, Df \in L^\varphi$.

The φ-Laplacian

$$F(u) = \int_\Omega \varphi(|Du|)dx$$

Euler-Lagrange system:

$$-\operatorname{div}\left(\frac{\varphi'(|Du|)}{|Du|}Du\right) = 0$$

Known results

- Marcellini '89-'96 general growth
- Lieberman : scalar case '91 ; vectorial case '93
- Mingione-Siepe '99
- Esposito-Mingione '00 nearly linear growth
- Fuchs-Mingione '00
- Marcellini-Papi '06
- Bildhauer Fuchs
- Diening, S. et al.
- $\cdots$
- Beck, Mingione;
- De Filippis, Mingione ; De Marco, Marcellini.

Marcellini's, Marcellini-Papi's approach: Euler system

$$u \in W^{1,\infty}, A \in C^1 \Longrightarrow u \in C^{1,\alpha}$$

without excess decay estimate!

Excess functional:

$$\Phi(x_0, r) = \fint_{B_r} |V(Du) - V(Du)_{x_o,r}|^2 \, dx$$

where $V(z) = |z|^{\frac{p-2}{2}} z$.

Question

What are suitable assumptions on φ that guarantee everywhere $C^{1,\alpha}$-regularity for local minimizers?

Uhlenbeck-type results (DSV)

- $\varphi \in C^1([0,\infty)) \bigcap C^2((0,\infty))$
- H1. $\varphi'(t) \sim t\varphi''(t)$ uniformly in $t > 0$
- H2. Hölder continuity for φ"

$$|\varphi''(s+t) - \varphi''(t)| \leq c\,\varphi''(t) \left(\frac{|s|}{t}\right)^{\beta} \quad \beta > 0$$

for all $t > 0$ and $s \in \mathbb{R}$ with $|s| < \frac{1}{2}t$.

Excess decay for φ-Laplacian system (Diening, B.S., Verde)

Let $u \in W^{1,\varphi}_{loc}(\Omega, \mathbb{R}^n)$ local minimizer for

$$\int_\Omega \varphi(|Du|)dx$$

with φ like before and $V(P) = \sqrt{\varphi'(|P|)|P|}\frac{P}{|P|}$

$$\Downarrow$$

"excess decay estimate"

$$\fint_{B_\rho} |V(Du) - (V(Du))_\rho|^2 \leq c(\frac{\rho}{R})^\alpha \fint_{B_R} |V(Du) - (V(Du))_R|^2 \, \forall \rho < R$$

$$\Downarrow$$

Du locally Hölder continuous.

A useful regularization is

Shifted function

$$\varphi_a'(t) := \frac{\varphi'(a+t)}{a+t}t$$

Almost φ-harmonic maps

The φ-harmonic approximation Lemma (Diening, B.S., Verde)

For every $\varepsilon > 0$ and $\theta \in (0,1)$, $\exists \delta = \delta(\varepsilon, \theta, \varphi) > 0$ s.t. if $u \in W^{1,\varphi}(B, \mathbb{R}^N)$ is *almost φ-harmonic* i.e. $\forall \xi \in C_0^\infty(B, \mathbb{R}^N)$

$$\left| \fint_B \frac{\varphi'(|\nabla u|)}{|\nabla u|} \langle \nabla u, \nabla \xi \rangle \, dx \right| \leq \delta \left(\fint_B \varphi(|\nabla u|) \, dx + \varphi(\|\nabla \xi\|)_\infty \right),$$

then the unique φ-harmonic map h with $h = u$ on ∂B satisfies

$$\left(\fint_B |V(\nabla u) - V(\nabla h)|^{2\theta} \, dx \right)^{\frac{1}{\theta}} < \varepsilon \fint_B \varphi(|\nabla u|) \, dx.$$

Generalization of the p-harmonic approximation (Duzaar, Mingione)

Modified version of Celada, Ok.

Consider a bilinear form on *Hom*$(\mathbb{R}^n, \mathbb{R}^N)$ which is (strongly) elliptic in the sense of Legendre-Hadamard, i.e. if for all $a \in \mathbb{R}^N$, $b \in \mathbb{R}^n$ it holds

$$\mathcal{A}_{ij}^{\alpha\beta} a^i b_\alpha a^j b_\beta \geq \kappa_A |a|^2 |b|^2$$

for some $\kappa_A > 0$.
The function u is *almost $\mathcal{A}$-harmonic*, iff:

$$\left| \fint_B \mathcal{A} \nabla u \cdot \nabla \xi \, dx \right| \leq \delta \left(\fint_{\widetilde{B}} |\nabla u| \, dx \right) \|\nabla \xi\|_{L^\infty(B)}$$

for all $\xi \in C_0^\infty(B, \mathbb{R}^N)$.

$\mathcal{A}$-harmonic approximation in Orlicz spaces (Diening, Lengeler, S., Verde)

Let φ be an N-function and let $s > 1$. Then for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \varphi, s) > 0$ such that the following holds: let $u \in W^{1,\varphi}(\widetilde{B})$ be *almost $\mathcal{A}$-harmonic* on B. Then denoting by h the unique $\mathcal{A}$-harmonic comparison map and by $w = h - u \in W_0^{1,\varphi}(B)$, it holds

$$\begin{aligned}
&\fint_B \varphi\left(\frac{|w|}{r_B}\right) dx + \fint_B \varphi(|\nabla w|)\, dx \\
&\le \varepsilon\left(\left(\fint_B (\varphi(|\nabla u|))^s\, dx\right)^{\frac{1}{s}} + \fint_{\widetilde{B}} \varphi(|\nabla u|)\, dx\right).
\end{aligned}$$

Historical background

What kind of regularity we can hope for?

The continuous coefficients case

- Scalar case: Cupini-Fusco-Petti

$$F(x,u,z) = \nu(\mu^2 + |z|^2)^{\frac{p}{2}} + f(x,u,z)$$

  with f convex in z and modulus of continuity in $(x,u) \Longrightarrow C^{0,\alpha}_{loc}, \forall\alpha \in (0,1)$;

- Vectorial case: Duzaar and Gastel systems (Dini) and linear growth$\Longrightarrow C^{1,\alpha}_{loc}$
- Foss and Mingione continuous in x
  F is C^2 w.r. to z, p-growth, p uniform strict quasiconvexity, F continuous w. r . to (x,u), F_{zz} continuous
  $\Longrightarrow$ Partial Hölder continuity.

Hybrid excess functional : “renormalized oscillation of the gradient ” and the oscillation of the function.

The VMO setting I

Equations in non-divergence form:

- Chiarenza, Frasca, Longo;
- Di Fazio, Ragusa;
- Di Fazio, Palagachev, Ragusa;
- Di Fazio, Zamboni.

Quadratic functionals/systems

- Danecek, Viszus.

The VMO setting II

Nonlinear setting:

- Bögelein, Duzaar, Habermann, Scheven;
- Bögelein.

Our result

Theorem

Let $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ be local minimizer of the functional, then there exists an open subset $\Omega_0 \subset \Omega$ such that

$$u \in C^{0,\alpha}_{\mathrm{loc}}\left(\Omega_0, \mathbb{R}^N\right) \qquad \text{and} \qquad |\Omega \setminus \Omega_0| = 0$$

for every $\alpha \in (0,1)$. Moreover, $\Omega \setminus \Omega_0 \subset \Sigma_1 \cup \Sigma_2$ where

$$\Sigma_1 := \left\{ x_0 \in \Omega : \liminf_{\varrho \searrow 0} \fint_{B_\varrho(x_0)} |V_{|(Du)_{x_0,\varrho}|}(Du - (Du)_{x_0,\varrho})|^2 \, \mathrm{d}x > 0 \right\},$$

$$\Sigma_2 := \left\{ x_0 \in \Omega : \limsup_{\varrho \searrow 0} |(Du)_{x_0,\varrho}| = +\infty \right\}.$$

Two relevant quantities

Proof by Bögelein or Duzaar et al. in the p-setting, uses homogeneity of the function . In particular, an analog of the Campanato excess

$$\Psi_\alpha(x_0,\varrho) := \varrho^{-\alpha p} \fint_{B_\varrho(x_0)} |u - (u)_{x_0,\varrho}|^p \,\mathrm{d}x$$

plays a key role in the iteration process. Clearly this could not be easily handled in the Orlicz setting.
Our strategy is to find carefully the two quantities who play the role both in the non-degenerate and in the degenerate cases. The first leading quantity is the excess functional "renormalized ":

$$\Phi(x_0,\varrho) := \fint_{B_\varrho(x_0)} \varphi_{|(Du)_{x_0,\varrho}|}(|Du - (Du)_{x_0,\varrho}|) \,\mathrm{d}x$$

The second one is a "Morrey-type" excess

$$\Theta(x_0,\varrho) := \varrho\varphi^{-1}\left(\fint_{B_\varrho(x_0)} \varphi(|Du|) \,\mathrm{d}x\right)$$

Sketch of the proof

We distinguish two regimes:

- nondegenerate:

$$\Phi(x_0, \varrho) \leq \varepsilon\varphi(|(D\mathrm{u})_{x_0,\varrho}|)\,,$$

  - Ekeland principle;
  - $\mathcal{A}$-harmonic approximation ;
  - decay of the renormalized excess.
- degenerate

$$\Phi(x_0, \varrho) \geq \kappa\varphi(|(D\mathrm{u})_{x_0,\varrho}|)$$

  - φ-harmonic approximation;
  - decay of the Morrey excess.
- Final iteration.

Ekeland principle

Lemma

Let (X, d) be a complete metric space, and assume that $F : X \to [0, \infty]$ be not identically ∞ and lower semicontinuous with respect to the metric topology on X. If for some $u \in X$ and some $\kappa > 0$, there holds

$$F(u) \leq \inf_X F + \kappa\,,$$

then there exists $v \in X$ with the properties

$$d(u, v) \leq 1 \quad \text{and} \quad F(v) \leq F(w) + \kappa d(v, w) \quad \forall w \in X\,.$$

Our metric setting

Consider the "frozen" density:

$$g(\mathrm{P}) \equiv g_{x_0,\varrho}(\mathrm{P}) := \Big(f(\cdot,(\mathrm{u})_{x_0,\varrho},\mathrm{P})\Big)_{x_0,\varrho} \qquad \text{for all } \mathrm{P} \in \mathbb{R}^{N\times n},$$

and

$$K(x_0,\varrho) := \tilde{H}(x_0,\varrho)\Psi(x_0,\varrho)$$

$$\tilde{H}(x_0,\varrho) := \frac{1}{1+(2L)^{1-\frac{1}{s}}}\left([\omega(\Theta(x_0,\varrho))]^{1-\frac{1}{s}} + [\mathcal{V}(\varrho)]^{1-\frac{1}{s}}\right),$$

$$\Theta(x_0,\varrho) := \varrho\varphi^{-1}\left(\fint_{B_\varrho(x_0)} \varphi(|D\mathrm{u}|)\,\mathrm{d}x\right)$$

As for the complete metric space (X,d), we consider

$$X := \left\{\mathrm{w} \in \mathrm{u} + W_0^{1,1}(B_{\varrho/2}(x_0)) : \fint_{B_\varrho/2(x_0)} \varphi(|D\mathrm{w}|)\,\mathrm{d}x \le \fint_{B_\varrho/2(x_0)} \varphi(|D\mathrm{u}|)\,\mathrm{d}x\right\}$$

with the metric

$$d(\mathrm{w}_1,\mathrm{w}_2) := \frac{1}{c_*\varphi^{-1}(K(\varrho))}\fint_{B_{\varrho/2}(x_0)} |D\mathrm{w}_1 - D\mathrm{w}_2|\,\mathrm{d}x, \qquad \text{for } \mathrm{w}_1,\mathrm{w}_2 \in \mathrm{u} + W_0^{1,1}(B_{\varrho/2(x_0)},\mathbb{R}^N),$$

and note that the functional is lower semicontinuous in the metric topology:

$$\mathcal{G}[\mathrm{w}] := \fint_{B_{\varrho/2}(x_0)} g(D\mathrm{w})\,\mathrm{d}x \quad \text{in } \mathrm{u} + W_0^{1,1}(B_{\varrho/2}(x_0),\mathbb{R}^N),$$

The comparison map

We would get a comparison map $\mathrm{v} \in \mathrm{u} + W_0^{1,1}(B_{\varrho/2}(x_0), \mathbb{R}^N)$ by proving the following lemma.

Lemma

Assume that $\mathrm{u} \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ *is a minimizer of the functional , Then there exists a minimizer* $\mathrm{v} \in \mathrm{u} + W_0^{1,1}(B_{\varrho/2}(x_0), \mathbb{R}^N)$ *of the functional*

$$\widetilde{\mathcal{G}}[\mathrm{w}] := \fint_{B_{\varrho/2}(x_0)} g(D\mathrm{w})\,\mathrm{d}x + \frac{K(x_0,\varrho)}{\varphi^{-1}(K(x_0,\varrho))} \fint_{B_{\varrho/2}(x_0)} |D\mathrm{v} - D\mathrm{w}|\,\mathrm{d}x\,,$$

that satisfies

$$\fint_{B_{\varrho/2}(x_0)} |D\mathrm{v} - D\mathrm{u}|\,\mathrm{d}x \leq c_* \varphi^{-1}(K(x_0,\varrho)) \tag{1}$$

for some constant $c_* = c_*(n, N, \Delta_2(\varphi), \nu, L)$*. Moreover,* v *fulfills the following Euler-Lagrange variational inequality:*

$$\left| \fint_{B_{\varrho/2}(x_0)} \langle Dg(D\mathrm{v}) | D\eta \rangle\,\mathrm{d}x \right| \leq \frac{K(x_0,\varrho)}{\varphi^{-1}(K(x_0,\varrho))} \fint_{B_{\varrho/2}(x_0)} |D\eta|\,\mathrm{d}x \tag{2}$$

for every $\eta \in C_0^\infty(B_{\varrho/2}(x_0), \mathbb{R}^N)$.

$\mathcal{A}$-harmonic approximation

We set

$$\mathcal{A} := \frac{D^2 g((Du)_{x_0,\varrho})}{\varphi''(|(Du)_{x_0,\varrho}|)} \equiv \frac{\left(D^2 f(\cdot, (u)_{x_0,\varrho}, (Du)_{x_0,\varrho})\right)_{x_0,\varrho}}{\varphi''(|(Du)_{x_0,\varrho}|)} . \tag{3}$$

We point out that $\mathcal{A}$ defined above is a bilinear form on $\mathbb{R}^{N\times n}$, satisfying Legendre-Hadamard conditions.

Lemma

Let $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ *be a minimizer of the functional and assume that for a ball* $B_\varrho(x_0) \subseteq \Omega$ *the non-degeneracy assumption*

$$\Phi(x_0, \varrho) \leq \varphi(|(Du)_{x_0,\varrho}|) \quad \text{and} \quad \varrho \leq 1 ,$$

is satisfied. Then, u *is approximately* $\mathcal{A}$*-harmonic on the ball* $B_{\varrho/2}(x_0)$*, in the sense that there exists* $\beta_1 = \beta_1(n, N, \mu_1, \mu_2, \nu, L, \beta_0) \in (0, \frac{1}{2})$ *such that*

$$\begin{aligned} &\left| \fint_{B_{\varrho/2}(x_0)} \langle \mathcal{A}(Du - (Du)_{x_0,\varrho}) | D\eta \rangle \, \mathrm{d}x \right| \\ &\leq c |(Du)_{x_0,\varrho}| \|D\eta\|_\infty \left\{ [H(x_0,\varrho)]^{\beta_1} + \frac{\Phi(x_0,\varrho)}{\varphi(|(Du)_{x_0,\varrho}|)} + \left(\frac{\Phi(x_0,\varrho)}{\varphi(|(Du)_{x_0,\varrho}|)} \right)^{\frac{1+\beta_0}{2}} \right\} \end{aligned}$$

holds for every $\eta \in C_c^\infty(B_{\varrho/2}(x_0), \mathbb{R}^N)$*.*

Decay estimates in the nondegenerate case

Lemma

For every $\varepsilon \in (0,1)$ *there exist* $\delta_1, \delta_2 \in (0,1]$, *where* $\delta_i = \delta_i(n, N, \mu_1, \mu_2, \beta_0, \nu, L, \varepsilon)$, $i = 1, 2$, *with the following property: if*

$$\frac{\Phi(x_0, \varrho)}{\varphi(|(Du)_{x_0,\varrho}|)} \leq \delta_1$$
$$[H(x_0, \varrho)]^{\beta_1} \leq \delta_2$$

then the excess improvement estimate

$$\Phi(x_0, \vartheta\varrho) \leq c_{\mathrm{dec}} \vartheta^2 \left[1 + \frac{\varepsilon}{\vartheta^{n+2}}\right] \Phi_*(x_0, \varrho)$$

holds for every $\vartheta \in (0,1)$ *for some constant* $c_{\mathrm{dec}} = c_{\mathrm{dec}}(n, N, \mu_1, \mu_2, \nu, L, c_1) > 0$, *where* Φ_* *is defined as*

$$\Phi_*(x_0, \varrho) := \Phi(x_0, \varrho) + \varphi(|(Du)_{x_0,\varrho}|)[H(x_0, \varrho)]^{\beta_1} ,$$

Iteration

Lemma

If the conditions

$$\frac{\Phi(x_0,\varrho)}{\varphi(|(Du)_{x_0,\varrho}|)} \le \varepsilon_* \quad \text{and} \quad \Theta(x_0,\varrho) \le \delta_* \,.$$

hold on $B_\varrho(x_0) \subseteq \Omega$ *for* $\varrho \in (0,\varrho_*]$, *then*

$$\frac{\Phi(x_0,\vartheta^m\varrho)}{\varphi(|(Du)_{x_0,\vartheta^m\varrho}|)} \le \varepsilon_* \quad \text{and} \quad \Theta(x_0,\vartheta^m\varrho) \le \delta_*$$

for every $m = 0,1,\ldots.$. *As a consequence, for any* $\alpha \in (0,1)$ *the following Morrey-type estimate holds:*

$$\Theta(y,r) \le c\delta_* \left(\frac{r}{\varrho}\right)^{\alpha}$$

for all $y \in B_{\varrho/2}(x_0)$ *and* $r \in (0,\varrho/2]$.

The φ-harmonic approximation

Lemma

Let $u \in W^{1,\varphi}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ *be a local minimizer of the functional , then there exists* $\beta_2 = \beta_2(n, N, \mu_1, \mu_2, c_0, L) > 0$ *such that, for every* $\delta > 0$ *and for* $\sigma = \sigma(\delta) > 0$ *, the inequality*

$$\left| \fint_{B_{\varrho/2}(x_0)} \left\langle \frac{\varphi'(|Du|)}{|Du|} Du \middle| D\eta \right\rangle \mathrm{d}x \right|$$

$$\leq c \left(\delta + [\tilde{H}(x_0, \varrho)]^{\beta_2} + \frac{\varphi^{-1}(\Psi(x_0, \varrho))}{\sigma} \right) \left(\fint_{B_\varrho(x_0)} \varphi(|Du|)\,\mathrm{d}x + \varphi(\|D\eta\|_\infty) \right)$$

holds for every $\eta \in C^\infty_c(B_{\varrho/2}(x_0), \mathbb{R}^N)$ *for some constant* $c = c(n, N, \mu_1, \mu_2, c_0, \nu, L) > 0$.

Decay improvement in the degenerate case

Lemma

Let $\gamma_0 > 0$ be the exponent of the excess decay for φ-harmonic maps. Then, for every $0 < \gamma < \gamma_0$ and every $\kappa, \mu \in (0,1)$ there exist $\varepsilon_\#, \tau \in (0,1)$ and $\varrho_\# \in (0,1]$ depending on $n, N, \mu_1, \mu_2, c_0, \beta_0, L, \nu, \gamma, \gamma_0, \mu$ and κ ($\varepsilon_\#$ also depends on $\sigma(\delta)$, and $\varrho_\#$ also depends on ω and $\mathcal{V}$) with the following property: if

$$\kappa\varphi(|(Du)_{x_0,\varrho}|) \leq \Phi(x_0,\varrho) \leq \varepsilon_\#$$

for $B_\varrho(x_0) \subseteq \Omega$ with $\varrho \in (0, \varrho_\#]$, then

$$\Phi(x_0,\tau\varrho) \leq \tau^{2\gamma}\Phi(x_0,\varrho) \quad \text{and} \quad \Theta(x_0,\tau\varrho) < \mu\,.$$

Final iteration

Lemma

There exist constants $\varepsilon_\#$, δ_* and $\tilde{\varrho}$ such that the conditions

$$\Phi(x_0, \varrho) < \varepsilon_\# \quad \text{and} \quad \Theta(x_0, \varrho) < \delta_* \,,$$

for $B_\varrho(x_0) \subseteq \Omega$ with $\varrho \in (0, \tilde{\varrho}]$ imply

$$u \in C^{0,\alpha}(\overline{B_{\varrho/2}(x_0)}) \,.$$

We introduce the set of integers

$$\mathbb{S} := \left\{ k \in \mathbb{N}_0 : \ \kappa \varphi(|(Du)_\varrho|) \le \Phi(\tau^k \varrho) \right\} \,,$$

and we distinguish between the cases $\mathbb{S} = \mathbb{N}_0$ and $\mathbb{S} \neq \mathbb{N}_0$.
The case $\mathbb{S} = \mathbb{N}_0$. We prove by induction that the bounds

$$\Phi(\tau^k \varrho) < \varepsilon_\# \quad \text{and} \quad \Theta(\tau^k \varrho) < \delta_* \tag{4}$$

hold for every $k \in \mathbb{N}_0$. Also the Morrey-type estimate

$$\Theta(y, r) \le c \delta_* \left(\frac{r}{\varrho} \right)^\alpha$$

holds for every $\alpha \in (0, 1)$, for all $y \in B_{\varrho/2}(x_0)$ and $r \in (0, \varrho/2]$.

The case $\mathbb{S} \neq \mathbb{N}_0$. In this case, there exists $k_0 := \min \mathbb{N} \backslash \mathbb{S}$. Since $k \in \mathbb{S}$ for any integer $k < k_0$ we can iterate as in the case $\mathbb{S} = \mathbb{N}_0$ for $k = 0, 1, \ldots, k_0 - 1$ to infer that (4) holds for any $k \leq k_0$. By the definition of $\mathbb{S}$ we have

$$\Phi(\tau^{k_0} \varrho) < \kappa \varphi(|(Du)_\varrho|) ,$$

which together with the second inequality in (4) with $k = k_0$ ensures that

$$\frac{\Phi(\vartheta^m \tau^{k_0} \varrho)}{\varphi(|(Du)_{\vartheta^m \tau^{k_0} \varrho}|)} \leq \kappa \quad \text{and} \quad \Theta(\vartheta^m \tau^{k_0} \varrho) \leq \delta_* \tag{5}$$

for every $m \in \mathbb{N}_0$.

Now, we consider an arbitrary radius $r \in (0, \varrho]$. If $r \in (\tau^{k_0} \varrho/2, \varrho]$ we find $0 \leq k \leq k_0$ such that $\tau^{k+1} \leq r \leq \theta^k$ and then we can argue as in the case $\mathbb{S} = \mathbb{N}_0$. In the case $r \in (0, \tau^{k_0} \varrho/2]$, instead, we find $m \in \mathbb{N}_0$ such that $\vartheta^{m+1} \tau^{k_0} \varrho < r \leq \vartheta^m \tau^{k_0} \varrho$ and we are in the non degenerate situation (iteration at smaller scale)

$$r^{1-\alpha} \varphi^{-1}(E(B_r(y))) \leq c(\vartheta^m \tau^{k_0} \varrho)^{1-\alpha} \varphi^{-1}(E(B_{\vartheta^m \tau^{k_0} \varrho})) \leq \frac{c\delta_*}{(\vartheta^m \tau^{k_0} \varrho)^\alpha}$$

for every $y \in B_{\vartheta^m \tau^{k_0} \varrho/2} \subseteq B_{\varrho/2}$ whence

$$\Theta(y, r) \leq c \frac{\delta_*}{(\vartheta^m \tau^{k_0})^\alpha} \left(\frac{r}{\varrho} \right)^\alpha .$$

At this point, we can argue as in the case $\mathbb{S} = \mathbb{N}_0$ thus concluding the proof.

Summary

- nondegenerate;
- degenerate;
- Final iteration: switching radius.

THANK YOU FOR YOUR ATTENTION