Regularity results for degenerate elliptic functionals with non standard growth

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Università degli Studi di Napoli "Federico II"

"Monday Nonstandard Seminar"- February 15, 2021

Analysis of the regularity properties of local minimizers of integral functionals of the form

$$\mathcal{F}(u,\Omega) := \int_\Omega F(x,Du)\,dx \qquad\qquad \Omega \subset \mathbb{R}^n$$

- $\xi \to F(x,\xi)$ is degenerate convex or
- $x \to F(x,\xi)$ is degenerate elliptic

- $\xi \to F(x,\xi)$ satisfy non standard growth conditions

- $x \to F(x,\xi)$ belongs to a Sobolev class

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BIBLIOGRAPHY

[CGGP] **G.Cupini, F. Giannetti, R.Giova & APdN.** *Regularity for minimizers of integrals with non standard growth conditions and discontinuous coefficients.* J. Differential Equations (2018)

[CMMP] **G.Cupini, P. Marcellini, E. Mascolo & APdN.** *Lipschitz regularity for degenerate elliptic integrals with (p, q)-growth.* Arxiv Preprint (2021)

Let us consider

$$\mathcal{F}(v,\Omega) = \int_\Omega F(x, Dv) \qquad \text{(F)}$$

- Ω bounded open subset of $\mathbb{R}^n$ n>2
- $v : \Omega \to \mathbb{R}^N$ $N \geq 2$
- (p, q)- growth. There exist exponents $2 \leq p \leq q$ and positive constants $\ell, L > 0$ s.t.

$$\text{(F0)} \qquad \ell|\xi|^p \leq F(x,\xi) \leq L(1 + |\xi|^2)^{\frac{q}{2}}$$

a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN}$.

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a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN}$.

Assumptions on $F(x,\xi)$ at ∞

There exists $\overline{\mathrm{R}} > 0$ such that

• Radial structure: there exists $\widetilde{F} : \Omega \times [\bar{R}, +\infty)$ s.t.

$$\text{(F1)} \qquad F(x,\xi) = \widetilde{F}(x, |\xi|),$$

a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN} \setminus B_{\bar{R}}(0)$.

• p-uniform convexity at ∞: $F(\cdot,\xi) \in C^2(\mathbb{R}^{nN} \setminus B_{\bar{R}}(0))$ and there exists $\nu > 0$ s.t.

$$\text{(F2)} \qquad \langle D_{\xi\xi}F(x,\xi)\lambda, \lambda\rangle \geq \nu(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}|\lambda|^2$$

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- Growth of $D_{\xi\xi}F$: There exists a constant $L_1 > 0$ s.t.

$$\text{(F3)} \qquad |D_{\xi\xi}F(x,\xi)| \leq L_1(1+|\xi|^2)^{\frac{q-2}{2}}$$

a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN} \setminus B_{\bar{R}}(0)$.

- Sobolev type regularity with respect to the x variable: The partial map

$$x \to D_\xi F(x,\xi)$$

is weakly differentiable for every $\xi \in \mathbb{R}^{nN} \setminus B_{\bar{R}}(0)$ and there exists $K(x) \in L^{\sigma}_{\mathrm{loc}}(\Omega), \ \sigma > 1$, s.t.

$$\text{(F4)} \qquad |D_x D_\xi F(x,\xi)| \leq |K(x)|(1+|\xi|^2)^{\frac{q-1}{2}}$$

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W.l.o.g. from now on we shall suppose $\bar{R} = 1$.

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Model functional

- $$\int_\Omega \left((|Du|-1)_+^p + a(x)(|Du|-1)_+^q\right)\, dx, \quad q > p > 1$$

$a(x) \in W^{1,\sigma}_{\rm loc}(\Omega)$ $0 \le a(x) \le 1$ ($\sigma \le n$ $\Rightarrow$ a possibly discontinous)

$$K := D_x a \in L^\sigma_{\rm loc}(\Omega)$$

Its Euler- Lagrange equation can be interpreted as the flow in a *origin-destination network*, in *traffic problems* that take into account the behavior of the flow in every position x of the path and with different constrained capacities.

Brasco, Carlier & Santabrogio (2010) - Brasco & Carlier (2014) - Bousquet & Brasco (2016) - Colombo & Figalli (2014)

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The Gap

p and q too far apart

$\Downarrow$

counterexample to the regularity

- P. Marcellini (1986)
- Hong (1992)

p and q sufficiently close

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minimizers are regular

- P. Marcellini (1991)

Sufficient Conditions on the gap

$$\frac{q}{p} = c(n) \to 1 \quad \text{as} \quad n \to \infty$$

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Uniform convexity out side a ball

Functionals with standard growth

- Chipot & Evans (1986), $p = 2$
- Giaquinta & Modica (1986), $p \geq 2$
- Fonseca, Fusco & Marcellini (2002) existence through reg.
- Leone, APdN & Verde (2007), $1 < p < 2$
- Foss, APdN & Verde (2008), (2011)

Integrands with non standard growth

- Cupini, Guidorzi & Mascolo (2003)
- Cupini & Migliorini (2003)
- Celada, Cupini & Guidorzi (2007)
- Diening, Stroffolini & Verde (2011) (φ-growth)

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Sobolev Coefficients

$W^{1,n}$ assumption for the coefficients + uniform p-convexity in the whole $\mathbb{R}^{nN}$+ standard growth conditions (i.e. p=q)

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Regularity of local minimizers

Beltrami equations, $p = n = 2$

- Clop, Faraco, Mateu, Orobitg & Zhong '09

Systems and integral functionals

- APdN (2011) $2 \leq p < n$
- Giannetti & APdN (2015) variable exponents
- Cruz Uribe, Moen & Rodney (2014) Dirichlet problems
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- Giannetti, APdN & Scheven (2016-2019) parabolic setting
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Regularity if

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uniform convexity only at ∞
but
smooth dependence on the x

$W^{1,n}$ coefficients
but
uniform convexity
in the whole $\mathbb{R}^{nN}$ and
$p = q$

Sobolev Coefficients & uniform convexity only at ∞

- **Eleuteri, Marcellini & Mascolo** AMPA (2015)

Uniform convexity only at ∞ + Sobolev Coefficients

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Lipschitz continuity of the local minimizers

- (p, q)- growth condition on F
- $K(x) \in L^{\sigma}(\Omega)$, with $\sigma > n$

Sharp bound on the gap $\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{\sigma}$

- **Esposito, Leonetti & Mingione** (JDE 2004)
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Non standard growth conditions

Question. May we weaken the assumption on K and still obtain some regularity of the local minimizers?

Sharp bound for the local boundedness of the minimizers

$$q \leq \frac{np}{n-p} = p^*$$

- Boccardo, Marcellini & Sbordone B.U.M.I. (1990)

...

- G. Cupini, P. Marcellini & E.Mascolo Discr. Cont. Dyn. Syst.(2009)

Limit case $q = p^*$

- G. Cupini, P. Marcellini & E.Mascolo J. Optim. Theory Appl. (2015) - Nonlinear. Anal. (2016)

Theorem 3. ([CGGP])

Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a local minimizer of $\mathcal{F}(u, \Omega)$, under the assumptions (F0)–(F4) with $\sigma \geq p + 2$. If

$$1 < p \leq q < \min\left\{p + 1 - \frac{p+2}{\sigma}, p^*\right\}$$

Then

- $u \in L^{\infty}_{\mathrm{loc}}$
- $Du \in L^{p+2}_{\mathrm{loc}}$
- $|Du|^{\frac{p}{2}-1} Du \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{Nn} \setminus B_2(0))$

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Comments on the gap: $q < \min\left\{p+1-\frac{p+2}{\sigma}, p^*\right\}$

- The bound $q < p^*$ is used only to deal with locally bounded minimizers.
- The bound on q depends on the assumptions on K.
- The bound improves if $\sigma \to \infty$. If $x \to f(x,\xi) \in Lip(\Omega)$ we find $q \le \min\{p+1, p^*\}$ (which is the usual one when dealing with bounded minimizers
  - Carozza, Kristensen & APdN- Ann. Inst. H. Poincaré (2011))
- The result holds true also if $\sigma < n$ provided

$$1 < p < \sigma - 2 < n - 2$$

- $\sigma = p + 2 \Rightarrow q = p$ we extend a previous result to the case of functionals uniformly convex only at ∞.
  - Giova & APdN - Adv. Calc. Var. (2017)

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The proof is achieved through

- A priori estimate
- Approximation argument

1. Tools for the a priori estimate

- Local boundedness of the minimizers

The radial structure assumption and the bound $p \le q < p^*$ imply that

$$||u||_{L^\infty(B_\rho)} \le c\left(\int_{B_{2\rho}} F(x, Du)\,dx\right)^{\vartheta},$$

some $\vartheta = \vartheta(n, p, q)$.

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2. Tools for the a priori estimate

• A new interpolation inequality

If $v \in C^2(\Omega)$ and $\eta \in C_0^\infty(\Omega)$, we have

$$\int \eta^2 (|Dv| - 1)_+^2 |Dv|^p \, dx$$

$$\leq c||v||_{L^\infty}^2 \int (\eta^2 + |D\eta|^2)|Dv|^p \, dx$$

$$+c||v||_{L^\infty}^2 \int \eta^2 \frac{(|Dv| - 1)_+^2}{(1 + (|Dv| - 1)_+)^2} |Dv|^{p-2}|D^2 v|^2 \, dx$$

for a positive constant $c = c(p)$.

- Carozza, Kristensen & APdN- Ann. Institut H. Poincaré (2011)

- Sobolev embedding Theorem imply

$$\frac{(|Du|-1)_+^2}{(1+(|Du|-1)_+)^2}|Du|^{p-2}|D^2u|^2 \in L^1_{\mathrm{loc}}(\Omega)$$

$$\Downarrow$$

$$Du \in L^{\frac{pn}{n-2}}_{\mathrm{loc}}(\Omega)$$

- Interpolation Inequality

$$\frac{(|Du|-1)_+^2}{(1+(|Du|-1)_+)^2}|Du|^{p-2}|D^2u|^2 \in L^1_{\mathrm{loc}} + u \in L^\infty_{\mathrm{loc}}$$

$$\Downarrow$$

$$Du \in L^{p+2}_{\mathrm{loc}}(\Omega)$$

Remark We take advantage from the assumption $u \in L^\infty$ if

$$p+2 \geq \frac{pn}{n-2} \iff 1 < p \leq n-2$$

The approximation

Approximation through integrands

- smooth with respect to the x- variable.
- p-uniformly convex with respect to the gradient variable in the whole $\mathbb{R}^{nN}$.
- satisfying standard growth conditions

- Main difficulty:
The functional is NOT strictly convex $\Longrightarrow$ we may loose uniqueness for the minimizers!

- We add a penalization term to the approximating functionals
More precisely

$$\mathcal{F}_{\varepsilon,j,h}(v, B_r) = \int_{B_r} f_{j,h}(x, Dv) + \int_{B_r} \arctan |u_\varepsilon - v|^2$$

with $f_{j,h}$ strictly convex, with standard growth and smooth with respect to x.

- Celada, Cupini & Guidorzi (2007)

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We prove that

- $v_{\varepsilon,j,h}$ minimizers of the problem

$$\min\left\{\mathcal{F}_{\varepsilon,j,h}(v, B_r) : \ v = u \ su \ B_r\right\}$$

converge (strongly) to u in $L^p(B_r \cap \{|Du| > 2\})$.

- The higher differentiability and the higher integrability true for $v_{\varepsilon,j,h}$ are preserved in passing to the limit and so they hold true also for u.

A new point of view

Consider the model functional

$$\int_\Omega |Du|^p + a(x)(1+|Du|)^q \qquad 0 \le a(x) \le M$$

It clearly satisfies non standard growth, but it also satisfies the following standard degenerate

$$a(x)(1+|Du|)^q \le |Du|^p + a(x)(1+|Du|)^q \le M(1+|Du|^q)$$

The uniform convexity is lost on the set $\{x \in \Omega : a(x) = 0\}$.

- Cupini, Marcellini & Mascolo Nonlinear Analysis (2018).

Low regularity results

- Trudinger (1971) local boundedness result under a suitable summability assumption for the degeneracy
- Fabes, Kenig & Serapioni (1982) Hölder continuity under a degeneracy which is an A_2 weight
- Iwaniec & Sbordone (2001) (2003) Higher integrability of the gradient with exponentially integrable degeneracy

What about higher regularity?

- Balci, Diening, Giova & APdN (2020)

Calderon Zygmund estimate for weighetd p-Laplacian of the form

$$\mathrm{div}(|M(x)\nabla u|^{p-2}M(x)\nabla u) = \mathrm{div}(|M(x)F|^{p-2}M(x)F)$$

under a BMO assumption on $\log M$

- Cupini, Marcellini, Mascolo & APdN (2020)

$$a \in W^{1,r} \qquad \text{and} \qquad a^{-1} \in L^s$$

$$\Downarrow$$

Lipschitz continuity of the minimizers

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$$\mathrm{div}(|M(x)\nabla u|^{p-2}M(x)\nabla u) = \mathrm{div}(|M(x)F|^{p-2}M(x)F)$$

under a BMO assumption on $\log M$

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$$a \in W^{1,r} \qquad \text{and} \qquad a^{-1} \in L^s$$

$$\Downarrow$$

Lipschitz continuity of the minimizers

What about higher regularity?

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Lipschitz continuity of the minimizers

Assumptions on $F(x, \xi)$

• Radial structure: (and) there exists $\widetilde{F} : \Omega \times [0, +\infty)$ s.t.

$$F(x, \xi) = \widetilde{F}(x, |\xi|), \qquad \text{(F1)}$$

a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN}$.

• degenerate ellipticity : $F(\cdot, \xi) \in C^2(\mathbb{R}^{nN})$ and there exist $2 \leq p < q$, a non negative measurable function $a(x)$ and a constant $L > 0$ s.t.

$$a(x)(1 + |\xi|^2)^{\frac{p-2}{2}} |\lambda|^2 \leq \langle D_{\xi\xi} F(x, \xi)\lambda, \lambda \rangle \leq L(1 + |\xi|^2)^{\frac{q-2}{2}} \qquad \text{(F2)}$$

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a.e. $x\in\Omega$ and for every $\xi\in\mathbb{R}^{nN}$.

• Sobolev type regularity with respect to the x variable: The partial map

$$x \to D_\xi F(x, \xi)$$

is weakly differentiable for every $\xi \in \mathbb{R}^{nN}$ and there exists a non negative measurable function $K(x)$, s.t.

$$|D_x D_\xi F(x, \xi)| \le K(x)(1 + |\xi|^2)^{\frac{q-1}{2}} \tag{F3}$$

a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN}$.

Theorem 2. ([CMMP])

Let F satisfy assumptions (F1)–(F3) with

$$\frac{1}{a} \in L^s_{loc}(\Omega) \qquad and \qquad K \in L^r_{loc}(\Omega)$$

with $r, s > n$ such that

$$\frac{q}{p} < \frac{s}{s+1}\left(1 + \frac{1}{n} - \frac{1}{r}\right)$$

Then for a local minimizer $u \in W^{1,1}_{loc}(\Omega)$ of $F(u)$ we have

$$Du \in L^\infty_{loc}(\Omega)$$

and

$$a(x)(1 + |Du|^2)^{\frac{p-2}{4}} |D^2 u| \in L^2_{loc}(\Omega)$$

Some comments

- $r = s = +\infty$ the bound reduces to

$$\frac{q}{p} < 1 + \frac{1}{n}$$

  - Cupini, Guidorzi Mascolo (2003)
  - Esposito, Leonetti Mingione (2003)
  - Marcellini (2020)

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  - Eleuteri, Marcellini, Mascolo (2015)

An example

The minimizer of the one dimensional integral

$$F(u) = \int_{-1}^{1} a(x)|u'(x)|^p \, dx$$

solves the Euler's first variation

$$\int_{-1}^{1} a(x)|u'(x)|^{p-2}u'(x)\varphi'(x) \, dx = 0 \qquad \forall \varphi \in C_0^1(-1,1)$$

and so

$$|u'(x)|^{p-1} = \frac{c}{a(x)}$$

Therefore if a vanishes u' can not be bounded and viceversa.

$a(x) = |x|^\alpha$ satisfies our assumption if and only if

$$\alpha < \frac{1}{s} \qquad \wedge \quad \alpha > 1 - \frac{1}{r} \quad \Longrightarrow \quad \frac{1}{r} + \frac{1}{s} > 1$$

which is complementary to our bound that for $p = q$ and $n = 1$ reduces to

$$\frac{1}{r} + \frac{1}{s} < 1$$

The Proof

- A priori estimate
- Approximation argument

• Weighted Sobolev inequality

If $a^{-1} \in L^s(\Omega)$ and $u \in W_0^{1,\frac{ps}{s+1}}(\Omega)$ then

$$||u||_{L^{\sigma^*}(\Omega)} \leq c||a^{-1}||_{L^s(\Omega)}^{\frac{1}{p}} \left(\int_\Omega a|Du|^p \, dx \right)^{\frac{1}{p}},$$

with $\sigma = \frac{ps}{s+1}$.

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Sketch of the proof of the a priori estimate

Fix $\rho < r < s < t < \lambda r < R$ and a cut-off function $\eta \in C_0^\infty(B_t)$ tra B_s e B_t. We test the second variation of the Euler Lagrange system

$$\int_\Omega D_{\xi\xi}F(x, Du)D^2uD\varphi + D_xD_\xi F(x, Du)D\varphi = 0,$$

with $\varphi = \eta^2 Du(1 + |Du|)^\gamma$, some $\gamma > 0$.

$$\nu \int_{B_t} \eta^2 a(x)(1 + |Du|)^{p-2+\gamma}|D^2u|^2 \leq$$

$$\overset{(F3)}{\leq} C(p+\gamma)^4 \left\{ \int_{B_t} \frac{K^2(x)}{a(x)}(1 + |Du|)^{2q-p+\gamma} + \textit{lower order terms} \right\}$$

The use of the weighted Sobolev inequality in the l.h.s yields

$$\nu \int_{B_t} \eta^2 a(x)(1+|Du|)^{p-2+\gamma}|D^2u|^2 \gtrsim \left(\int_{B_s} (1+|Du|)^{(p+\gamma)\frac{2_s^*}{2}} \right)^{\frac{2}{2_s^*}}$$

where

$$2_s^* = \left(\frac{2s}{s+1} \right)^*$$

The assumptions

$$K \in L^r \qquad \frac{1}{a} \in L^s$$

allow us to use Hölder's inequality in the r.h.s to get

$$\int_{B_R} \frac{K^2(x)}{a(x)} (1+|Du|)^{2q-p+\gamma}$$

$$\leq \left(\int_{B_t} \frac{1}{a^s} \right)^{\frac{1}{s}} \left(\int_{B_t} K^r \right)^{\frac{2}{r}} \left(\int_{B_t} (1+|Du|)^{(2q-p+\gamma)\frac{rs}{rs-2s-r}} \right)^{\frac{rs-2s-r}{rs}}$$

Joining the estimates we obtain

$$\left(\int_{B_s}(1+|Du|)^{(p+\gamma)\frac{2_s^*}{2}}\right)^{\frac{2}{2_s^*}}$$

$$\lesssim c(p+\gamma)^4\left(\int_{B_t}(1+|Du|)^{(2q-p+\gamma)\frac{rs}{rs-2s-r}}\right)^{\frac{rs-2s-r}{rs}}$$

$$\lesssim c(p+\gamma)^4||Du||_{L^\infty}^{2(q-p)}\left(\int_{B_t}(1+|Du|)^{(p+\gamma)\frac{rs}{rs-2s-r}}\right)^{\frac{rs-2s-r}{rs}}$$

i.e. setting $m=\frac{rs}{rs-2s-r}$

$$\left(\int_{B_s}[(1+|Du|)^{(p+\gamma)m}]^{\frac{2_s^*}{2m}}\right)^{\frac{2m}{2_s^*}}\lesssim(p+\gamma)^{4m}||Du||_{L^\infty}^{2(q-p)m}\int_{B_t}(1+|Du|)^{(p+\gamma)m}$$

The bound

$$\frac{q}{p} < \frac{s}{s+1}\left(1 + \frac{1}{n} - \frac{1}{r}\right) \qquad (*)$$

allows the use of the Moser iteration argument to obtain the boundedness of Du. We can write (*) in the equivalent form

$$\frac{1}{ps} + \frac{1}{qr} + \frac{1}{p} - \frac{1}{q} < \frac{1}{qn}$$

Approximation

We apply the a priori estimate to $v_h \in u + W_0^{1,\frac{ps}{s+1}}(B_R)$ minimizers of the functionals

$$\mathcal{F}_h(Dv) = \int_{B_R} f_h(x, Dv)$$

with

$$f_h(x,\xi) = f(x,\xi) + \frac{1}{h}(1+|\xi|^2)^{\frac{ps}{2(s+1)}}$$

and let $h \to \infty$

• Higher differentiability of the approximating minimizers

If $v \in W^{1,\infty}_{loc}(\Omega)$ is a local minimizer of $H(v) = \int_\Omega h(x, |Dv|)\, dx$ where

$$\lambda(1+t^2)^{\frac{p-2}{2}} \leq h_{tt}(x,t) \leq L(1+t^2)^{\frac{q-2}{2}}$$

$$|h_{tx}(x,t)| \leq K(x)(1+t^2)^{\frac{q-1}{2}}$$

with $K \in L^r_{loc}(\Omega)$ and

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}$$

then

$$v \in W^{2,2}_{loc}(\Omega) \qquad \textit{and} \qquad (1+|Dv|^2)^{\frac{p-2}{4}}|D^2 v| \in L^2_{loc}(\Omega)$$


Arigato
Mahalo
Grazie
Spasiba
THANK YOU
Thoinks Moite
Gracias
Merci
Danke
Shukran