Well-posedness of variational formulation of "modular" PDEs

Anastasia Molchanova

University of Vienna

joint research with

A. Menovschikov (Hradec Králové) & L. Scarpa (Milan)

31.05.2021

TECHNISCHE UNIVERSITÄT WIEN

universität wien

FWF Der Wissenschaftsfonds.

REWIRE Reinforcing Women in Research

Function spaces

Lesbegue space $\quad L^p \quad := \left\{ f \in \mathcal{M} : \int |f|^p \, d\mathrm{x} < \infty \right\}$

Function spaces

Lesbegue space	L^p	$:= \left\{ f \in \mathcal{M} : \int \lvert f \rvert^p \, d\mathrm{x} < \infty \right\}$
Orlicz space	L^Φ	$:= \left\{ f \in \mathcal{M} : \int \Phi(\lvert f \rvert) \, d\mathrm{x} < \infty \right\}$

Function spaces

Lesbegue space $L^p := \left\{ f \in \mathcal{M} : \int |f|^p \, d\mathrm{x} < \infty \right\}$

Orlicz space $L^\Phi := \left\{ f \in \mathcal{M} : \int \Phi(|f|) \, d\mathrm{x} < \infty \right\}$

$L \log L$ or $\exp L$

Function spaces

Lesbegue space $L^p := \left\{ f \in \mathcal{M} : \int |f|^p \, d\mathrm{x} < \infty \right\}$

Orlicz space $L^\Phi := \left\{ f \in \mathcal{M} : \int \Phi(|f|) \, d\mathrm{x} < \infty \right\}$

$L \log L$ or $\exp L$

Musielak–Orlicz space $L^{\Phi(\cdot)} := \left\{ f \in \mathcal{M} : \int \Phi(\mathrm{x}, |f|) \, d\mathrm{x} < \infty \right\}$

Function spaces

Lesbegue space $L^p := \left\{ f \in \mathcal{M} : \int |f|^p \, d\mathrm{x} < \infty \right\}$

Orlicz space $L^\Phi := \left\{ f \in \mathcal{M} : \int \Phi(|f|) \, d\mathrm{x} < \infty \right\}$

$L \log L$ or $\exp L$

Musielak–Orlicz space $L^{\Phi(\cdot)} := \left\{ f \in \mathcal{M} : \int \Phi(\mathrm{x}, |f|) \, d\mathrm{x} < \infty \right\}$

Lebesgue spaces
with variable exponent

Modular spaces

Modular spaces

$\mathcal{X}$ — a real Banach space; $\mathcal{X}^*$ — its dual;

Convex (semi-)modular φ on $\mathcal{X}$:

- $\varphi\colon \mathcal{X} \to [0, \infty]$ is a convex functional;
- $(\varphi(0) = 0)$ $\varphi(x) = 0$ iff $x = 0$;
- if $\varphi(\alpha x) = 0$ for all $\alpha > 0$, then $x = 0$;
- $\varphi(-x) = \varphi(x)$ for all $x \in \mathcal{X}$.

Modular spaces

$\mathcal{X}$ — a real Banach space; $\mathcal{X}^*$ — its dual;

Convex (semi-)modular φ on $\mathcal{X}$:

- $\varphi\colon \mathcal{X} \to [0, \infty]$ is a convex functional;
- $(\varphi(0) = 0)$ $\varphi(x) = 0$ iff $x = 0$;
- if $\varphi(\alpha x) = 0$ for all $\alpha > 0$, then $x = 0$;
- $\varphi(-x) = \varphi(x)$ for all $x \in \mathcal{X}$.

Modular spaces $E_\varphi \subset L_\varphi \subset \mathcal{X}$:

$$L_\varphi := \{x \in \mathcal{X} : \exists\, \alpha > 0 : \quad \varphi(\alpha x) < +\infty\},$$
$$E_\varphi := \{x \in \mathcal{X} : \forall\, \alpha > 0 : \quad \varphi(\alpha x) < +\infty\}.$$

Modular spaces

$\mathcal{X}$ — a real Banach space; $\mathcal{X}^*$ — its dual;

Convex (semi-)modular φ on $\mathcal{X}$:

- $\varphi\colon \mathcal{X} \to [0, \infty]$ is a convex functional;
- $(\varphi(0) = 0)$ $\varphi(x) = 0$ iff $x = 0$;
- if $\varphi(\alpha x) = 0$ for all $\alpha > 0$, then $x = 0$;
- $\varphi(-x) = \varphi(x)$ for all $x \in \mathcal{X}$.

Modular spaces $E_\varphi \subset L_\varphi \subset \mathcal{X}$:

$$L_\varphi := \{x \in \mathcal{X} :\ \exists\, \alpha > 0 : \quad \varphi(\alpha x) < +\infty\},$$
$$E_\varphi := \{x \in \mathcal{X} :\ \forall\, \alpha > 0 : \quad \varphi(\alpha x) < +\infty\}.$$

Luxemburg norm

$$\|x\|_\varphi := \inf \{\lambda > 0 : \quad \varphi(x/\lambda) \leq 1\}.$$

Modular spaces

$\mathcal{X}$ — a real Banach space; $\mathcal{X}^*$ — its dual;

Convex (semi-)modular φ on $\mathcal{X}$:

- $\varphi \colon \mathcal{X} \to [0, \infty]$ is a convex functional;
- $(\varphi(0) = 0)$ $\varphi(x) = 0$ iff $x = 0$;
- if $\varphi(\alpha x) = 0$ for all $\alpha > 0$, then $x = 0$;
- $\varphi(-x) = \varphi(x)$ for all $x \in \mathcal{X}$.

Modular spaces $E_\varphi \subset L_\varphi \subset \mathcal{X}$:

$$L_\varphi := \{x \in \mathcal{X} : \exists\, \alpha > 0 : \quad \varphi(\alpha x) < +\infty\},$$
$$E_\varphi := \{x \in \mathcal{X} : \forall\, \alpha > 0 : \quad \varphi(\alpha x) < +\infty\}.$$

Luxemburg norm

$$\|x\|_\varphi := \inf\left\{\lambda > 0 : \quad \varphi(x/\lambda) \le 1\right\}.$$

$(x_n)_n \subset L_\varphi$ modular-converges to $x \in L_\varphi$ iff

$$\exists\, \alpha > 0 : \ \varphi(\alpha(x_n - x)) \to 0 \text{ as } n \to \infty.$$

More about modular spaces

- If there exists a strictly increasing function $\rho\colon [0, +\infty) \to [0, +\infty)$ with $\rho(0) = 0$ such that

$$\varphi(x) \geq \rho(\|x\|_{\mathcal{X}}) \qquad \forall\, x \in \mathcal{X}, \tag{🐟}$$

then $(E_\varphi, \|\cdot\|_\varphi)$ and $(L_\varphi, \|\cdot\|_\varphi)$ are Banach spaces and

$$E_\varphi \hookrightarrow L_\varphi \hookrightarrow \mathcal{X}.$$

More about modular spaces

- If there exists a strictly increasing function $\rho\colon [0,+\infty) \to [0,+\infty)$ with $\rho(0) = 0$ such that

$$\varphi(x) \geq \rho(\|x\|_{\mathcal{X}}) \qquad \forall\, x \in \mathcal{X}, \qquad (🐟)$$

  then $(E_\varphi, \|\cdot\|_\varphi)$ and $(L_\varphi, \|\cdot\|_\varphi)$ are Banach spaces and

$$E_\varphi \hookrightarrow L_\varphi \hookrightarrow \mathcal{X}.$$

$$(\varrho(z) \approx z^s, \quad s > 0)$$

More about modular spaces

- If there exists a strictly increasing function $\rho\colon [0,+\infty) \to [0,+\infty)$ with $\rho(0) = 0$ such that

$$\varphi(x) \geq \rho(\|x\|_{\mathcal{X}}) \qquad \forall\, x \in \mathcal{X},$$

(🐟)

  then $(E_\varphi, \|\cdot\|_\varphi)$ and $(L_\varphi, \|\cdot\|_\varphi)$ are Banach spaces and

$$E_\varphi \hookrightarrow L_\varphi \hookrightarrow \mathcal{X}.$$

- If φ is a lower semicontinuous convex semi-modular on $\mathcal{X}$ and $L_\varphi \hookrightarrow \mathcal{X}$ densely, then the convex conjugate $\varphi^*\colon \mathcal{X}^* \to [0,+\infty]$, defined by

$$\varphi^*(y) := \sup_{x\in\mathcal{X}} \{\langle y, x\rangle_{\mathcal{X}^*,\mathcal{X}} - \varphi(x)\}, \qquad y \in \mathcal{X},$$

  is a lower semicontinuous convex semi-modular on $\mathcal{X}^*$.

Restriction

Let φ be a lsc convex semi-modular on $\mathcal{X}$ satisfying the growth condition 🐟. Then

Restriction

Let φ be a lsc convex semi-modular on $\mathcal{X}$ satisfying the growth condition . Then

- the restriction $\bar{\varphi} := \varphi_{|E_\varphi}$ is a lsc convex semi-modular on E_φ;

Restriction

Let φ be a lsc convex semi-modular on $\mathcal{X}$ satisfying the growth condition . Then

- the restriction $\bar{\varphi} := \varphi_{|E_\varphi}$ is a lsc convex semi-modular on E_φ;
- the convex conjugate $\bar{\varphi}^* \colon E_\varphi^* \to [0, +\infty]$,

$$\bar{\varphi}^*(y) := \sup_{x \in E_\varphi} \{\langle y, x \rangle_{E_\varphi^*, E_\varphi} - \varphi(x)\}, \qquad y \in E_\varphi^*,$$

  is a lsc convex semi-modular on E_φ^*;

Restriction

Let φ be a lsc convex semi-modular on $\mathcal{X}$ satisfying the growth condition . Then

- the restriction $\bar{\varphi} := \varphi_{|E_\varphi}$ is a lsc convex semi-modular on E_φ;
- the convex conjugate $\bar{\varphi}^* \colon E_\varphi^* \to [0, +\infty]$,

$$\bar{\varphi}^*(y) := \sup_{x \in E_\varphi} \{\langle y, x \rangle_{E_\varphi^*, E_\varphi} - \varphi(x)\}, \qquad y \in E_\varphi^*,$$

  is a lsc convex semi-modular on E_φ^*;
- the spaces

$$L_{\bar{\varphi}^*} := \{y \in E_\varphi^* : \ \exists\, \alpha > 0 : \quad \bar{\varphi}^*(\alpha y) < +\infty\},$$
$$E_{\bar{\varphi}^*} := \{y \in E_\varphi^* : \ \forall\, \alpha > 0 : \quad \bar{\varphi}^*(\alpha y) < +\infty\}$$

  are Banach spaces with a norm $\|y\|_{\bar{\varphi}^*} := \inf\{\lambda > 0 : \ \bar{\varphi}^*(y/\lambda) \leq 1\}$;

Restriction

Let φ be a lsc convex semi-modular on $\mathcal{X}$ satisfying the growth condition . Then

- the restriction $\bar{\varphi} := \varphi_{|E_\varphi}$ is a lsc convex semi-modular on E_φ;
- the convex conjugate $\bar{\varphi}^* \colon E_\varphi^* \to [0, +\infty]$,

$$\bar{\varphi}^*(y) := \sup_{x \in E_\varphi} \{\langle y, x \rangle_{E_\varphi^*, E_\varphi} - \varphi(x)\}, \qquad y \in E_\varphi^*,$$

  is a lsc convex semi-modular on E_φ^*;
- the spaces

$$L_{\bar{\varphi}^*} := \{y \in E_\varphi^* : \ \exists\, \alpha > 0 : \quad \bar{\varphi}^*(\alpha y) < +\infty\},$$
$$E_{\bar{\varphi}^*} := \{y \in E_\varphi^* : \ \forall\, \alpha > 0 : \quad \bar{\varphi}^*(\alpha y) < +\infty\}$$

  are Banach spaces with a norm $\|y\|_{\bar{\varphi}^*} := \inf\{\lambda > 0 : \ \bar{\varphi}^*(y/\lambda) \leq 1\}$;
- $E_{\bar{\varphi}^*} \hookrightarrow L_{\bar{\varphi}^*} = E_\varphi^*$;

Restriction

Let φ be a lsc convex semi-modular on $\mathcal{X}$ satisfying the growth condition . Then

- the restriction $\bar{\varphi} := \varphi_{|E_\varphi}$ is a lsc convex semi-modular on E_φ;
- the convex conjugate $\bar{\varphi}^*\colon E_\varphi^* \to [0, +\infty]$,

$$\bar{\varphi}^*(y) := \sup_{x \in E_\varphi} \{\langle y, x\rangle_{E_\varphi^*, E_\varphi} - \varphi(x)\}, \qquad y \in E_\varphi^*,$$

  is a lsc convex semi-modular on E_φ^*;
- the spaces

$$L_{\bar{\varphi}^*} := \{y \in E_\varphi^* :\ \exists\, \alpha > 0 : \quad \bar{\varphi}^*(\alpha y) < +\infty\},$$
$$E_{\bar{\varphi}^*} := \{y \in E_\varphi^* :\ \forall\, \alpha > 0 : \quad \bar{\varphi}^*(\alpha y) < +\infty\}$$

  are Banach spaces with a norm $\|y\|_{\bar{\varphi}^*} := \inf\{\lambda > 0 :\ \bar{\varphi}^*(y/\lambda) \le 1\}$;
- $E_{\bar{\varphi}^*} \hookrightarrow L_{\bar{\varphi}^*} = E_\varphi^*$;
- if $\varrho(z) \approx z^s$ with $s > 1$ and $E_\varphi \hookrightarrow \mathcal{X}$ densely, then $\mathcal{X}^* \hookrightarrow E_{\bar{\varphi}^*}$.

Variational setting for PDEs

Reflexive separable Banach spaces

V — real reflexive separable Banach space; $\qquad V \hookrightarrow H \hookrightarrow V^*$ densely;

$\varphi\colon V \to (0,\infty]$ — proper lsc convex function;
$D(x) := \{x \in V : \varphi(x) < +\infty\}$ — effective domain;
$\partial\varphi(x) := \{y \in V^* : \varphi(z) - \varphi(x) \geq \langle y, x - z\rangle \ \forall x \in D(\varphi)\}$.

Cauchy problem:

$$\partial_t u(t) + \partial\varphi(u(t)) \ni f(t) \text{ in } V^*,\ 0 < t < T, \qquad u(0) = u_0.$$

Reflexive separable Banach spaces

V — real reflexive separable Banach space; $V \hookrightarrow H \hookrightarrow V^*$ densely;

$\varphi \colon V \to (0, \infty]$ — proper lsc convex function;
$D(x) := \{x \in V : \varphi(x) < +\infty\}$ — effective domain;
$\partial\varphi(x) := \{y \in V^* : \varphi(z) - \varphi(x) \geq \langle y, x - z \rangle \ \forall x \in D(\varphi)\}$.

Cauchy problem:

$$\partial_t u(t) + \partial\varphi(u(t)) \ni f(t) \text{ in } V^*,\ 0 < t < T, \qquad u(0) = u_0.$$

Under certain growth conditions, $u_0 \in D(\varphi)$ and $f \in L^{p'}(0, T; V^*)$, $\exists!\ (u, \xi)$:

$$u \in L^p(0, T; V) \cap C([0, T]; H) \cap W^{1,p'}(0, T; V^*),$$
$$\xi \in L^{p'}(0, T; V^*), \qquad \varphi(u) \in L^1(0, T),$$

such that $\xi \in \partial\varphi(u)$ and

$$\partial_t u + \xi = f \quad \text{in } V^*, \quad 0 < t < T, \qquad u(0) = u_0.$$

~~Reflexive~~ separable Banach spaces

V — real (non)reflexive (non)separable Banach space;

?

Assumptions

H0 H is a real separable Hilbert space,
$\varphi\colon H \to [0, \infty]$ is a lsc convex semi-modular on H,
there exist constants $c > 0$ and $s > 1$ s.t.

$$\varphi(x) \geq c \, \|x\|_H^s \qquad \forall\, x \in H.$$

Assumptions

H0 H is a real separable Hilbert space,
$\varphi\colon H \to [0,\infty]$ is a lsc convex semi-modular on H,
there exist constants $c > 0$ and $s > 1$ s.t.

$$\varphi(x) \geq c\,\|x\|_H^s \qquad \forall\, x \in H.$$

H1 E_φ is dense in H,
there exists a separable reflexive Banach space $V_0 \hookrightarrow E_\varphi$ continuously and densely, s.t. φ is bounded on bounded subsets of V_0.

Assumptions

H0 H is a real separable Hilbert space,
$\varphi\colon H \to [0,\infty]$ is a lsc convex semi-modular on H,
there exist constants $c > 0$ and $s > 1$ s.t.

$$\varphi(x) \geq c\,\|x\|_H^s \qquad \forall\, x \in H.$$

H1 E_φ is dense in H,
there exists a separable reflexive Banach space $V_0 \hookrightarrow E_\varphi$ continuously and densely, s.t. φ is bounded on bounded subsets of V_0.

H2 either one of the following conditions holds:

- H2i $E_\varphi \hookrightarrow L_\varphi$ densely, or
- H2ii $H \hookrightarrow L_{\bar{\varphi}^*}$ densely.

New duality

$$V \hookrightarrow H \hookrightarrow V^*$$

New duality

$$V_0 \hookrightarrow E_\varphi \hookrightarrow L_\varphi \hookrightarrow H \hookrightarrow E_{\bar{\varphi}^*} \hookrightarrow L_{\bar{\varphi}^*} = E_\varphi^* \hookrightarrow V_0^*$$

New duality

$$V_0 \hookrightarrow E_\varphi \hookrightarrow L_\varphi \hookrightarrow H \hookrightarrow E_{\bar{\varphi}^*} \hookrightarrow L_{\bar{\varphi}^*} = E_\varphi^* \hookrightarrow V_0^*$$

$$\not\parallel$$

$$(L_\varphi)^*$$

New duality

$$L_{\varphi} \hookrightarrow H \hookrightarrow \qquad L_{\bar{\varphi}^*}$$

$$\not\parallel$$

$$(L_{\varphi})^*$$

New duality

$$L_\varphi \hookrightarrow H \hookrightarrow L_{\bar{\varphi}^*}$$

Assume **H0–H2**.
Then there exists a unique continuous bilinear form

$$[\cdot,\cdot]\colon L_{\bar{\varphi}^*} \times L_\varphi \to \mathbb{R},$$

s.t. $[y,\cdot]\colon L_\varphi \to \mathbb{R}$ and $[\cdot,x]\colon L_{\bar{\varphi}^*} \to \mathbb{R}$ are linear and continuous, and

$$[y,x] = (y,x)_{H,H} \qquad \forall\, x \in L_\varphi,\ \forall\, y \in H,$$
$$[y,x] = \langle y,x\rangle_{E_\varphi^*,E_\varphi} \qquad \forall\, x \in E_\varphi,\ \forall\, y \in L_{\bar{\varphi}^*}.$$

New duality

$$L_\varphi \hookrightarrow H \hookrightarrow L_{\bar\varphi^*}$$

Assume **H0–H2**.
Then there exists a unique continuous bilinear form

$$[\cdot,\cdot]\colon L_{\bar\varphi^*} \times L_\varphi \to \mathbb{R},$$

s.t. $[y,\cdot]\colon L_\varphi \to \mathbb{R}$ and $[\cdot,x]\colon L_{\bar\varphi^*} \to \mathbb{R}$ are linear and continuous, and

$$\begin{aligned} &[y,x] = (y,x)_{H,H} && \forall\, x \in L_\varphi,\ \forall\, y \in H,\\ &[y,x] = \langle y,x\rangle_{E^*_\varphi,E_\varphi} && \forall\, x \in E_\varphi,\ \forall\, y \in L_{\bar\varphi^*}. \end{aligned}$$

And the generalized Hölder inequality holds:

$$[y,x] \le 2\,\|y\|_{\bar\varphi^*}\,\|x\|_\varphi \qquad \forall\,(x,y) \in L_\varphi \times L_{\bar\varphi^*}.$$

Bochner spaces

$u\colon [0,T] \to \mathcal{X}$ is strongly mesurable iff there exists a sequence of simple functions v_n s.t. $\|v - v_n\| \to 0$;

Bochner spaces

$u\colon [0,T] \to \mathcal{X}$ is strongly mesurable iff there exists a sequence of simple functions v_n s.t. $\|v - v_n\| \to 0$;

$u\colon [0,T] \to \mathcal{X}$ is weakly measurable iff $\forall y \in \mathcal{X}^*$, $\langle v, y\rangle$ is measurable.

Bochner spaces

$u\colon [0,T] \to \mathcal{X}$ is strongly mesurable iff there exists a sequence of simple functions v_n s.t. $\|v - v_n\| \to 0$;

$u\colon [0,T] \to \mathcal{X}$ is weakly measurable iff $\forall y \in \mathcal{X}^*$, $\langle v, y \rangle$ is measurable.

Pettis' measurability theorem.

Let $\mathcal{X}$ be separable. Then $u\colon [0,T] \to \mathcal{X}$ is strongly measurable iff u is weakly measurable.

Bochner spaces

$$
\begin{aligned}
L^1_w(0,T;L_\varphi) &:= \Big\{ v\colon [0,T] \to L_\varphi : \quad [y,v] \in L^1(0,T) \quad \forall\, y \in L_{\bar{\varphi}^*} \Big\}\,,\\
L^1_w(0,T;L_{\bar{\varphi}^*}) &:= \Big\{ v\colon [0,T] \to L_{\bar{\varphi}^*} : \quad [v,x] \in L^1(0,T) \quad \forall\, x \in L_\varphi \Big\}\,,\\
L^1(0,T;L_\varphi) &:= \Big\{ v\colon [0,T] \to L_\varphi \text{ str.meas.} : \quad \|v\|_\varphi \in L^1(0,T) \Big\}\,,\\
L^1(0,T;L_{\bar{\varphi}^*}) &:= \Big\{ v\colon [0,T] \to L_{\bar{\varphi}^*} \text{ str.meas.} : \quad \|v\|_{\bar{\varphi}^*} \in L^1(0,T) \Big\}\,,
\end{aligned}
$$

Bochner spaces

$$
\begin{aligned}
L^1_w(0,T;L_\varphi) &:= \Big\{ v\colon [0,T] \to L_\varphi : \quad [y,v] \in L^1(0,T) \quad \forall\, y \in L_{\bar\varphi^*} \Big\}\,,\\
L^1_w(0,T;L_{\bar\varphi^*}) &:= \Big\{ v\colon [0,T] \to L_{\bar\varphi^*} : \quad [v,x] \in L^1(0,T) \quad \forall\, x \in L_\varphi \Big\}\,,\\
L^1(0,T;L_\varphi) &:= \Big\{ v\colon [0,T] \to L_\varphi \text{ str.meas.} : \quad \|v\|_\varphi \in L^1(0,T) \Big\}\,,\\
L^1(0,T;L_{\bar\varphi^*}) &:= \Big\{ v\colon [0,T] \to L_{\bar\varphi^*} \text{ str.meas.} : \quad \|v\|_{\bar\varphi^*} \in L^1(0,T) \Big\}\,,
\end{aligned}
$$

$$
\begin{aligned}
W^{1,1}_w(0,T;L_{\bar\varphi^*}) := \Big\{ & v\colon [0,T] \to L_{\bar\varphi^*} : \exists v' \in L^1_w(0,T;L_{\bar\varphi^*}) : \\
& [v(t),x] = [v(0),x] + \int_0^t [v'(s),x]\,ds \quad \forall\, x \in L_\varphi \Big\}\,.
\end{aligned}
$$

Chain rule

Assume **H0–H2**, and let $u \in W^{1,1}_w(0,T;L_{\bar\varphi^*}) \cap L^1_w(0,T;L_\varphi)$:

$$\partial_t u = u_1' + u_2', \qquad \text{with} \qquad u_1' \in L^1_w(0,T;L_{\bar\varphi^*}), \quad u_2' \in L^1(0,T;H).$$

Chain rule

Assume **H0–H2**, and let $u \in W^{1,1}_w(0,T;L_{\bar{\varphi}^*}) \cap L^1_w(0,T;L_\varphi)$:

$$\partial_t u = u_1' + u_2', \qquad \text{with} \qquad u_1' \in L^1_w(0,T;L_{\bar{\varphi}^*}), \quad u_2' \in L^1(0,T;H).$$

If $\exists \alpha > 0$: $\varphi(\alpha u)$, $\varphi^*(\alpha u_1') \in L^1(0,T)$, then

Chain rule

Assume **H0–H2**, and let $u \in W^{1,1}_w(0,T;L_{\bar{\varphi}^*}) \cap L^1_w(0,T;L_\varphi)$:

$$\partial_t u = u_1' + u_2', \qquad \text{with} \qquad u_1' \in L^1_w(0,T;L_{\bar{\varphi}^*}), \quad u_2' \in L^1(0,T;H).$$

If $\exists \alpha > 0$: $\varphi(\alpha u)$, $\varphi^*(\alpha u_1') \in L^1(0,T)$, then

- $u \in C^0([0,T];H)$,

Chain rule

Assume **H0–H2**, and let $u \in W^{1,1}_w(0,T;L_{\bar\varphi^*}) \cap L^1_w(0,T;L_\varphi)$:

$$\partial_t u = u_1' + u_2', \qquad \text{with} \qquad u_1' \in L^1_w(0,T;L_{\bar\varphi^*}), \quad u_2' \in L^1(0,T;H).$$

If $\exists \alpha > 0$: $\varphi(\alpha u)$, $\varphi^*(\alpha u_1') \in L^1(0,T)$, then

- $u \in C^0([0,T];H)$,
- the function $t \mapsto \|u(t)\|_H^2$, $t \in [0,T]$, is absolutely continuous,

Chain rule

Assume **H0–H2**, and let $u \in W^{1,1}_w(0,T;L_{\bar\varphi^*}) \cap L^1_w(0,T;L_\varphi)$:

$$\partial_t u = u_1' + u_2', \qquad \text{with} \qquad u_1' \in L^1_w(0,T;L_{\bar\varphi^*}), \quad u_2' \in L^1(0,T;H).$$

If $\exists \alpha > 0$: $\varphi(\alpha u)$, $\varphi^*(\alpha u_1') \in L^1(0,T)$, then

- $u \in C^0([0,T];H)$,
- the function $t \mapsto \|u(t)\|_H^2$, $t \in [0,T]$, is absolutely continuous,
- $[\partial_t u, u] = \dfrac{d}{dt}\dfrac{1}{2}\|u\|_H^2$ a.e. in $(0,T)$.

Existence

Assume **H0–H2** and let

$$u_0 \in H, \qquad f \in L^1(0,T;H).$$

Existence

Assume **H0–H2** and let

$$u_0 \in H, \qquad f \in L^1(0,T;H).$$

Then $\exists!\,(u,\xi)$:

$$u \in W^{1,1}_w(0,T;L_{\bar\varphi^*}) \cap C^0([0,T];H) \cap L^1_w(0,T;L_\varphi),$$
$$\xi \in L^1_w(0,T;L_{\bar\varphi^*}),$$
$$\varphi(u),\ \bar\varphi^*(\xi) \in L^1(0,T),$$

Existence

Assume **H0–H2** and let

$$u_0 \in H, \qquad f \in L^1(0,T;H).$$

Then $\exists!\,(u,\xi)$:

$$\begin{aligned}
&u \in W^{1,1}_w(0,T;L_{\bar\varphi^*}) \cap C^0([0,T];H) \cap L^1_w(0,T;L_\varphi),\\
&\xi \in L^1_w(0,T;L_{\bar\varphi^*}),\\
&\varphi(u),\ \bar\varphi^*(\xi) \in L^1(0,T),
\end{aligned}$$

such that

$$\partial_t u + \xi = f \quad \text{in } L_{\bar\varphi^*} \quad \text{a.e. in } (0,T), \qquad u(0)=u_0,$$ ()

Existence

Assume **H0–H2** and let

$$u_0 \in H, \qquad f \in L^1(0,T;H).$$

Then $\exists!\,(u,\xi)$:

$$u \in W^{1,1}_w(0,T;L_{\bar\varphi^*}) \cap C^0([0,T];H) \cap L^1_w(0,T;L_\varphi),$$
$$\xi \in L^1_w(0,T;L_{\bar\varphi^*}),$$
$$\varphi(u),\ \bar\varphi^*(\xi) \in L^1(0,T),$$

such that

$$\partial_t u + \xi = f \quad \text{in } L_{\bar\varphi^*} \quad \text{a.e. in } (0,T), \qquad u(0) = u_0,$$ ()

and

$$\varphi(u) + [\xi, x-u] \le \varphi(x) \quad \forall\, x \in E_\varphi, \quad \text{a.e. in } (0,T).$$ ()

Existence

Assume **H0–H2** and let

$$u_0 \in H, \qquad f \in L^1(0,T;H).$$

Then $\exists!\,(u,\xi)$:

$$u \in W^{1,1}_w(0,T;L_{\bar\varphi^*}) \cap C^0([0,T];H) \cap L^1_w(0,T;L_\varphi),$$
$$\xi \in L^1_w(0,T;L_{\bar\varphi^*}),$$
$$\varphi(u),\ \bar\varphi^*(\xi) \in L^1(0,T),$$

such that

$$\partial_t u + \xi = f \quad \text{in } L_{\bar\varphi^*} \quad \text{a.e. in } (0,T), \qquad u(0) = u_0,$$ ()

and

$$\varphi(u) + [\xi, x-u] \le \varphi(x) \quad \forall\, x \in E_\varphi, \quad \text{a.e. in } (0,T).$$ ()

Moreover, $\forall\, t \in [0,T]$

$$\frac{1}{2}\,\|u(t)\|_H^2 + \int_0^t [\xi(s), u(s)]\, ds = \frac{1}{2}\,\|u_0\|_H^2 + \int_0^t (f(s), u(s))\, ds.$$

Continuous dependence on the data

Assume **H0–H2**, and let (u_0^1, f_1) and (u_0^2, f_2) be reasonable data.

Then, for any respective solutions (u_1, ξ_1) and (u_2, ξ_2) to  and  :

$$\|u_1 - u_2\|^2_{C^0([0,T];H)} + \|[\xi_1 - \xi_2, u_1 - u_2]\|_{L^1(0,T)} \leq 2\left(\left\|u_0^1 - u_0^2\right\|^2_H + \|f_1 - f_2\|^2_{L^1(0,T;H)}\right).$$

Some examples

- Separable reflexive Banach space V:

$$\begin{cases} \partial_t u(t,x) + \partial\varphi(u(t,x)) \ni f(t,x) & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

Some examples

- Separable reflexive Banach space V:

$$\begin{cases} \partial_t u(t,x) + \partial\varphi(u(t,x)) \ni f(t,x) & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

$$V = V_0 = E_\varphi = L_\varphi, \qquad V^* = V_0^* = E_{\bar{\varphi}^*} = L_{\bar{\varphi}^*} = E_\varphi^* = L_\varphi^*.$$

Some examples

- Musielak–Orlicz space L^{Φ}:

$$\begin{cases} \partial_t u(t,x) + \partial\Phi(x, u(t,x)) \ni f(t,x) & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

Some examples

- Musielak–Orlicz space L^{Φ}:

$$\begin{cases} \partial_t u(t,x) + \partial\Phi(x, u(t,x)) \ni f(t,x) & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases}$$

Φ satisfies Δ_2 or ∇_2, or both.

Some examples

- Dynamic boundary conditions:

$$\begin{cases} \partial_t u - \Delta u + \partial M(\cdot, u) \ni f & \text{in } (0,T) \times \Omega, \\ u = u_{\partial\Omega} & \text{in } (0,T) \times \partial\Omega, \\ \partial_t u_{\partial\Omega} + \partial_{\mathbf{n}} u + \partial M_{\partial\Omega}(\cdot, u_{\partial\Omega}) \ni f_{\partial\Omega} & \text{in } (0,T) \times \partial\Omega, \\ (u, u_{\partial\Omega})(0) = (u_0, u_{0,\partial\Omega}) & \text{in } \Omega \times \partial\Omega, \end{cases}$$

Some examples

- Dynamic boundary conditions:

$$\begin{cases} \partial_t u - \Delta u + \partial M(\cdot, u) \ni f & \text{in } (0,T)\times\Omega, \\ u = u_{\partial\Omega} & \text{in } (0,T)\times\partial\Omega, \\ \partial_t u_{\partial\Omega} + \partial_{\mathbf{n}} u + \partial M_{\partial\Omega}(\cdot, u_{\partial\Omega}) \ni f_{\partial\Omega} & \text{in } (0,T)\times\partial\Omega, \\ (u, u_{\partial\Omega})(0) = (u_0, u_{0,\partial\Omega}) & \text{in } \Omega\times\partial\Omega, \end{cases}$$

$$L_\varphi = \Big\{ \mathbf{v} \in H^1(\Omega)\times H^{1/2}(\partial\Omega) : \\ v_{|\partial\Omega} = v_{\partial\Omega},\ v \in L^M(\Omega),\ v_{\partial\Omega} \in L^{M_{\partial\Omega}}(\partial\Omega) \Big\},$$

$$E_\varphi = \Big\{ \mathbf{v} \in H^1(\Omega)\times H^{1/2}(\partial\Omega) : \\ v_{|\partial\Omega} = v_{\partial\Omega},\ v \in E^M(\Omega),\ v_{\partial\Omega} \in E^{M_{\partial\Omega}}(\partial\Omega) \Big\}.$$

Details

A. Menovschikov, A.M., L. Scarpa, *An extended variational theory for nonlinear evolution equations via modular spaces*,
ArXiv: https://arxiv.org/abs/2012.05518

THE END