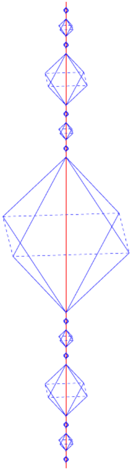
Fakultät für Mathematik



Regularity VS Lavrentiev gap: borderline case of double-phase potential

Anna Kh.Balci

Lavrentiev Gap

What is the *natural* space to minimize

$$\mathcal{F}(w) = \int_\Omega \Phi(x, \nabla w)\, dx?$$

1. All $w \in W^{1,\Phi(\cdot)}$ with finite energy? (1915 Tonelli's Existence Theorem)
2. Smooth functions $w \in H^{1,\Phi(\cdot)}$?

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Lavrentiev's example 1926

$$\inf_{\text{all } w} \mathcal{F}(w) < \inf_{\text{smooth } w} \mathcal{F}(w)$$

M. A. Lavrentiev

[Mania]: $f(x, w, \nabla w) = (x - w^3)^2 (w')^6$,

$w(0) = 0$ and $w(1) = 1$,

$w_{\min}(x) = x^{\frac{1}{3}}$, $\mathcal{F}(w_{\min}) = 0$.

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Density of Smooth Functions

Let
$$W^{1,p}(\Omega) := \{w : \|w\|_{1,p} := \|w\|_p + \|\nabla w\|_p < \infty\},$$
$$H^{1,p}(\Omega) := \text{closure of } C^1(\Omega) \text{ in } W^{1,p}.$$

Meyers and Serrin, 1964, "$H = W$"

$$W^{1,p}(\Omega) = H^{1,p}(\Omega) \qquad \text{for all domains and } p \in [1, \infty).$$

Local result due to Friedrichs.

Main tool:

Friedrichs mollifies: $w * \varphi_\varepsilon \to w$ in $W^{1,p}$.

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Sobolev-Orlicz Spaces: $H \neq W$

We assume Δ_2 and ∇_2 conditions and

$$|t|^{p_-} \lesssim \Phi(x,t) \lesssim |t|^{p_+},$$

where $1 < p_- \leq p_+ < \infty$, $c_0 \geq 0$, $c_1, c_2 > 0$.

$$\|f\|_{L^{\Phi(\cdot)}(\Omega)} := \inf\left\{\gamma > 0 \,:\, \int_\Omega \Phi(x, |f(x)/\gamma|)\,dx \leq 1\right\},$$
$$L^{\Phi(\cdot)}(\Omega) := \left\{f \in L^0(\Omega) \,:\, \|f\|_{L^{\Phi(\cdot)}(\Omega)} < \infty\right\}.$$

We define the generalized Orlicz-Sobolev spaces $W^{1,\Phi(\cdot)}$ and $H^{1,\Phi(\cdot)}$

$$W^{1,\Phi(\cdot)}(\Omega) := \{w \in W^{1,1}(\Omega) \,:\, \nabla w \in L^{\Phi(\cdot)}(\Omega)\},$$
$$\|w\|_{W^{1,\Phi(\cdot)}(\Omega)} := \|w\|_{L^1(\Omega)} + \|\nabla w\|_{L^{\Phi(\cdot)}(\Omega)},$$
$$H^{1,\Phi(\cdot)}(\Omega) := \left(\text{closure of } C^\infty(\Omega) \cap W^{1,\Phi(\cdot)}(\Omega)\right).$$

Regular case: $H = W$

An integrand $\Phi(x,t)$ is regular in the domain Ω if for all $u \in W_0^{1,\Phi(\cdot)}(\Omega)$ there exists a smooth sequence $u_\varepsilon \in C_0^\infty(\Omega)$ such that

1. $u_\varepsilon \to u$ in $W_0^{1,1}(\Omega)$;
2. $\lim_{\varepsilon\to 0} \int_\Omega \Phi(x,\nabla u_\varepsilon)\,dx = \int_\Omega \Phi(x,\nabla u)\,dx := \mathcal{F}(u)$.

This is equivalent to $H = W$:

$$\begin{aligned} \text{Sheffeé's theorem:} \quad & \Phi(x,\nabla u_\varepsilon) \to \Phi(x,\nabla u) \text{ in } L^1(\Omega), \\ \text{Convexity}+\Delta_2: \quad & \Phi(x,|\nabla u_\varepsilon - \nabla u|) \lesssim \Phi(x,|\nabla u_\varepsilon|) + \Phi(x,|\nabla u|). \end{aligned}$$

Thus $\Phi(x,|\nabla u_\varepsilon - \nabla u|)$ is uniformly inegrable, goes to zero.

The Sufficient Condition for Regularity

Lemma (Zhikov 1995)

Assume, that there exist Carathéodory functions $\Phi_\varepsilon(x,t)$*:*

1. *Non-standard growth conditions:* $|t|^{p^-} \lesssim \Phi(x,t) \lesssim |t|^{p^+}$;
2. $\Phi_\varepsilon(x,0) = 0$;
3. $\Phi(x,t) \lesssim \Phi_\varepsilon(x,t) + 1$ *for* $x \in \bar{\Omega},\ t \lesssim \varepsilon^{\frac{-d}{p^-}}$;
4. $\Phi_\varepsilon(x,t) \lesssim \Phi(y,t) + 1$ *for* $|x-y| \lesssim \varepsilon,\ t \in \mathbb{R}_+$.

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The other form in Harjulehto and Hästö 2019: **ADec-condition**

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Fine Properties

Zhikov-Fan, $\Phi(x,t) := t^{p(x)}$, 1995

If $p(x)$ is log-Hölder continuous, i.e.

$$|p(x) - p(y)| \leq \frac{c}{\log\left(e + \frac{1}{|x-y|}\right)}, \quad \text{then } \Phi(x,t) \text{ is regular.}$$

$$\Phi_\varepsilon(x,y) := |t|^{p_\varepsilon(x)},\ p_\varepsilon(x) = \min\{p(y), |y-x| \leq 2k\varepsilon\}.$$

Barabanov-Zhikov 1995, $\Phi(x,t) := t^p + a(x)t^q$

If $a(x)$ is Lipschitz continuous and

$$q < \frac{d+1}{d}p, \quad \text{then } \Phi(x,t) \text{ is regular.}$$

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Applications

If $H = W$ there are many questions to study for different models:

Regularity of solution:	Acerbi-Mingione, Fonseca,Maly, Baroni,Colombo...
Boudedness of integral operators:	Diening, Cruz-Uribe, Fiorenza, Samko...
Calderon-Zygmund estimates:	Mingione, Diening, Byun, Hästö...
Obstacles:	Byun, Oh, Ok, ...
Manifolds:	De Fillipis, Mingione...
General Muselak-Orlicz spaces:	Chlebicka, Zatorska-Goldstein, Lee ...
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First 2D example: Lavrentiev Gap, $H \neq W$

V. V. Zhikov
1940-2017

Zhikov's Example 1986, $H \neq W$

$$\mathcal{F}(w) = \int_\Omega \frac{|\nabla w|^{p(x)}}{p(x)}\,dx \to \min,$$

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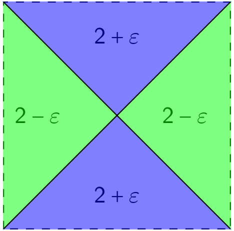

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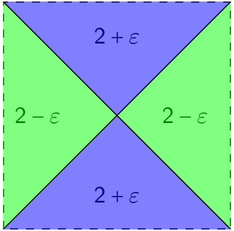

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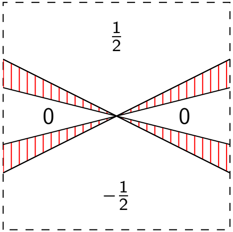

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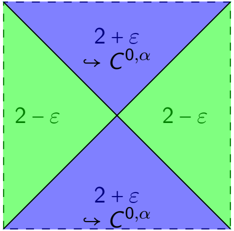

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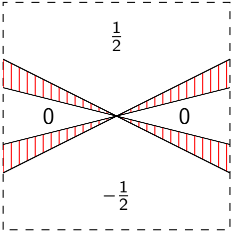

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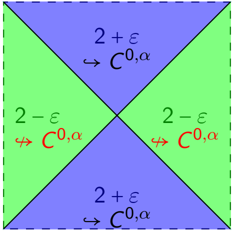

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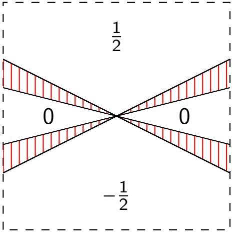

Double Phase Potential

Let $\mathcal{F}(w) := \int_\Omega |\nabla w|^p + a(x)|\nabla w|^q \, dx$ with $1 < p < q$ and $a \geq 0$.

Marcellini '80's; Esposito-Leonetti-Mingione '04:

Gap if $p < d < d + \alpha < q$ – lack of the higher regularity Dimensional threshold ?

General procedure for Lavrentiev gap examples **Balci, Diening, Surnachev 2020**

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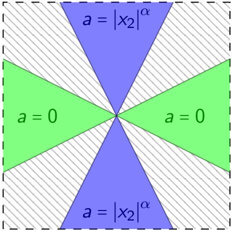

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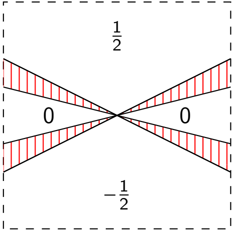

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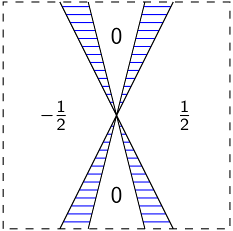

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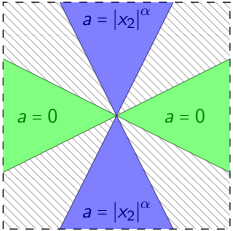

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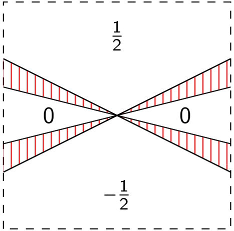

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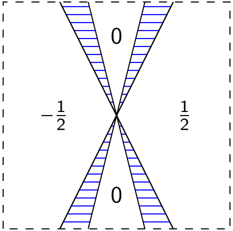

Gap if $p < d < d + \alpha < q$ – lack of the higher regularity Dimensional threshold ? NO!

General procedure for Lavrentiev gap examples **Balci, Diening, Surnachev 2020**

General algorithm: Balci, Diening, Surnachev 2020

Using fractal geometry we construct function u and b such that

Theorem Balci, Diening, Surnachev, CalVar and PDE's 2020

For all $p_0 \in (1, \infty)$ we have $d = 2$

1. There exist u, v with $\nabla u \in L^{\Phi(\cdot)}$ and $\nabla^\perp v \in L^{\Phi^*(\cdot)}$.
2. $\langle \nabla^\perp v, \nabla u \rangle = 0$ for C_0^∞ but $\langle \nabla^\perp v, \nabla u \rangle \neq 0$ for $W^{1,\Phi(\cdot)}$.
3. $W^{1,\Phi(\cdot)} \neq H^{1,\Phi(\cdot)}$ and $W_0^{1,\Phi(\cdot)} \neq H_0^{1,\Phi(\cdot)}$.
4. Lavrentiev gap.

The Dimension Conjecture: d is not important for the special cases.
$d \geq 2$ $\nabla^\perp v$ replaced by $b = \operatorname{div} A$

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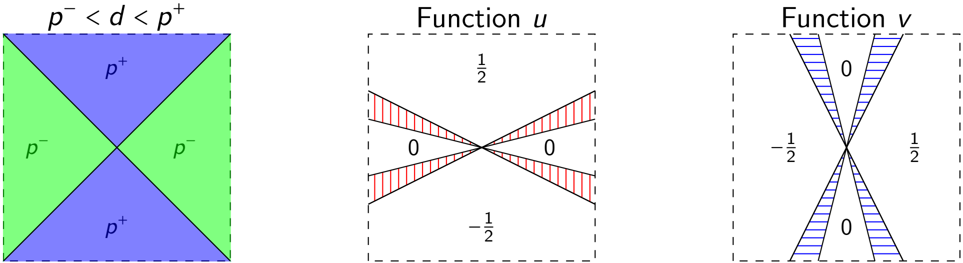
Basic building block



Contact set S of u has dimension zero. $W^{1,p^+} \hookrightarrow C^0$ controls u on S, but not W^{1,p^-}.

Idea: Increase $\mathfrak{D} := \dim(S)$ to reduce $p^\pm$.

For example: $S := \frac{1}{3}$-Cantor-set.

$\mathfrak{D} = \log(2)/\log(3) \approx 0.631, \quad 2 = 3^{\mathfrak{D}}$.

Basic building block

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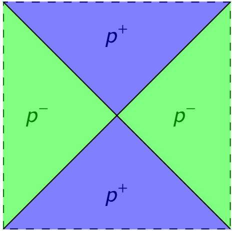
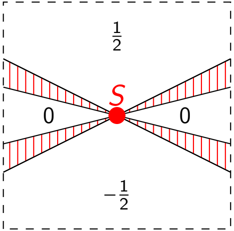


Function u



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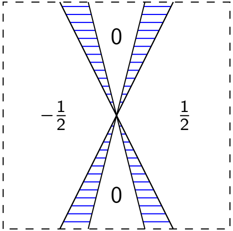

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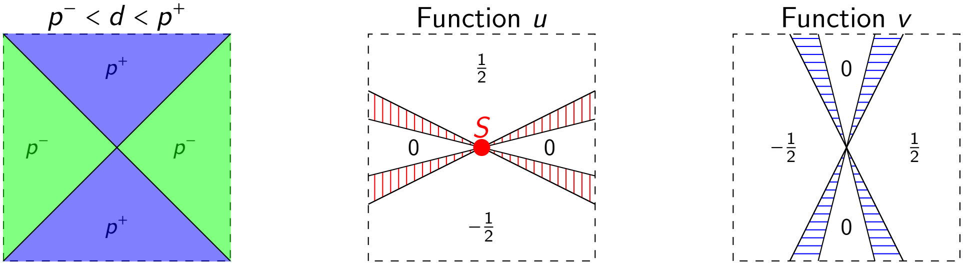

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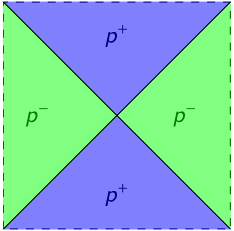

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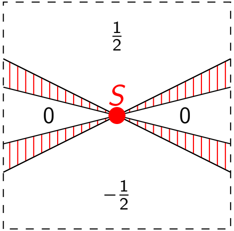

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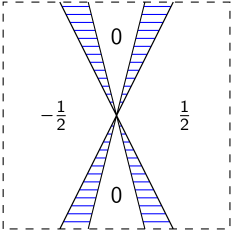

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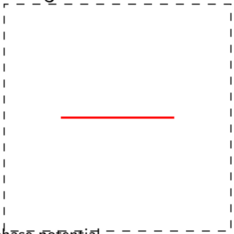
$\frac{1}{3}$-Cantor set



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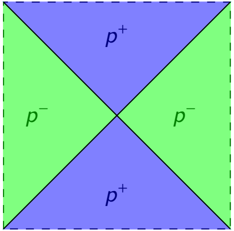

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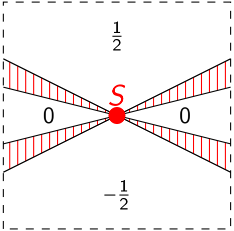

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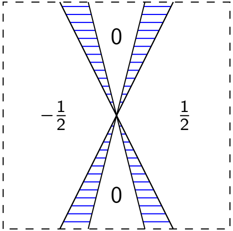

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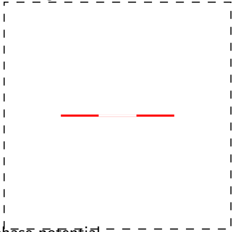

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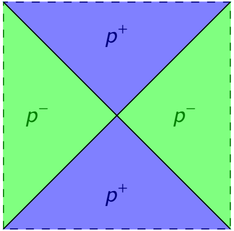

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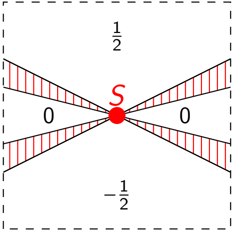

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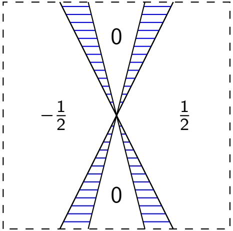

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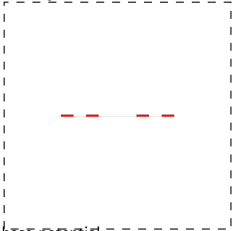

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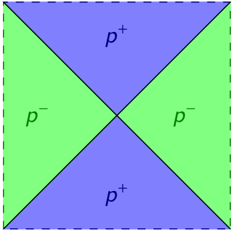

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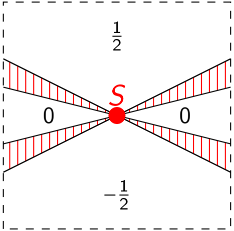
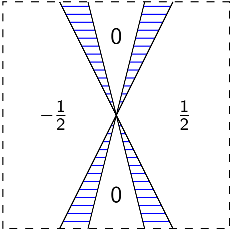


Function v



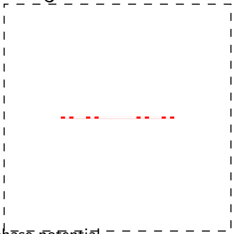
Contact set S of u has dimension zero. $W^{1,p^+} \hookrightarrow C^0$ controls u on S, but not W^{1,p^-}.

Idea: Increase $\mathfrak{D} := \dim(S)$ to reduce $p^\pm$.

For example: $S := \frac{1}{3}$-Cantor-set.

$\mathfrak{D} = \log(2)/\log(3) \approx 0.631, \quad 2 = 3^{\mathfrak{D}}$.

$\frac{1}{3}$-Cantor set



Cantor Necklace $p \approx 2 - \mathfrak{D}$, $2 = 3^{\mathfrak{D}}$

$$p(x) := \begin{cases} p^+ & \text{green}, \\ p^- & \text{blue}. \end{cases}$$

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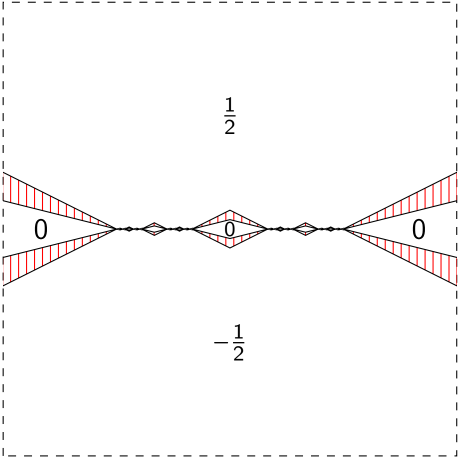

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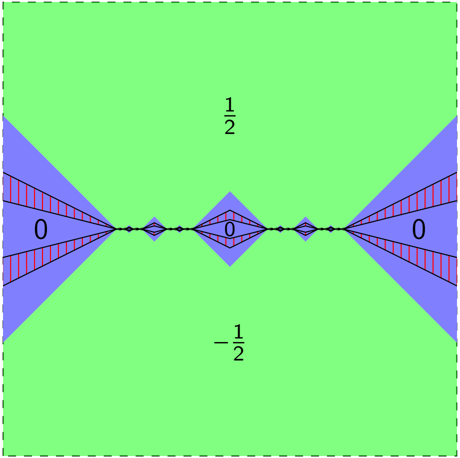
Function u and exponent p



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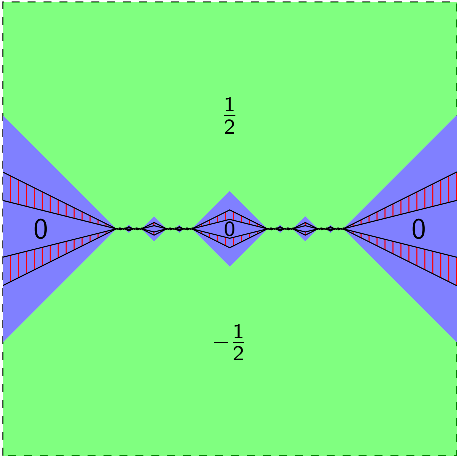

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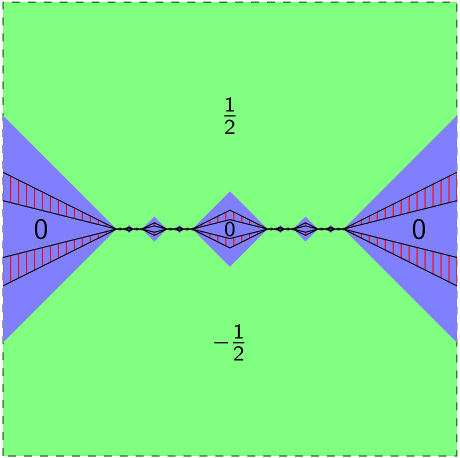

Integrability of ∇u

$$\int_\Omega |\nabla u|^{p(x)}\, dx$$

$$\approx \sum_{\text{scale } j} \sum_{\text{block } \mathfrak{B}} \int_{\mathfrak{B}} |\nabla u|^{p^-}\, dx$$

$$\approx \sum_{j \geq 0} 2^j \cdot (3^{-j})^2 (3^j)^{p^-}$$

$$= \sum_{j \geq 0} 3^{(\mathfrak{D} - 2 + p^-)j} < \infty$$

if $p^- < 2 - \mathfrak{D}$.

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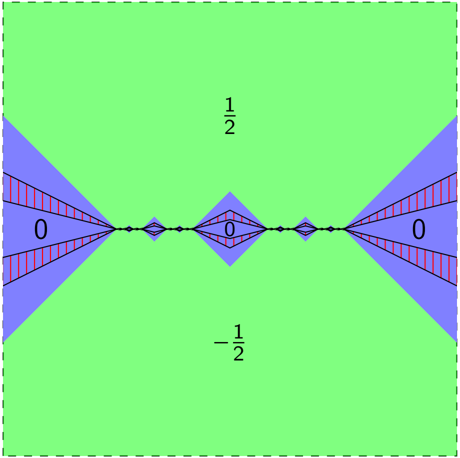

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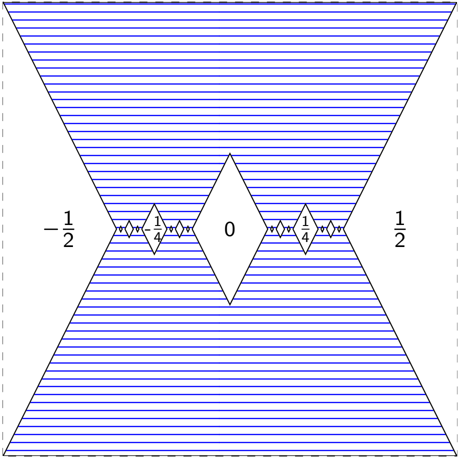

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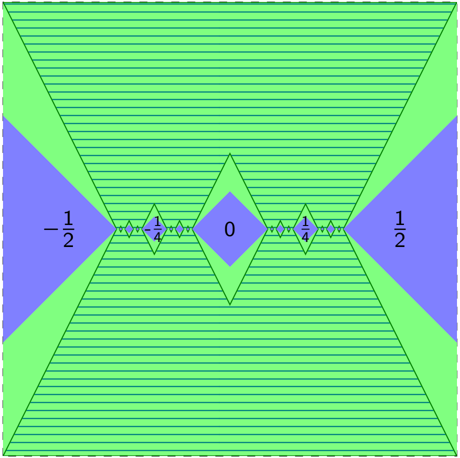

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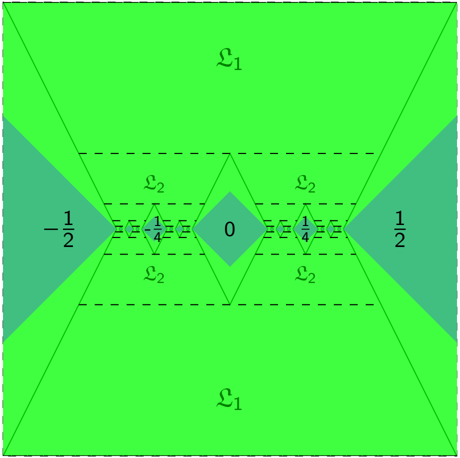

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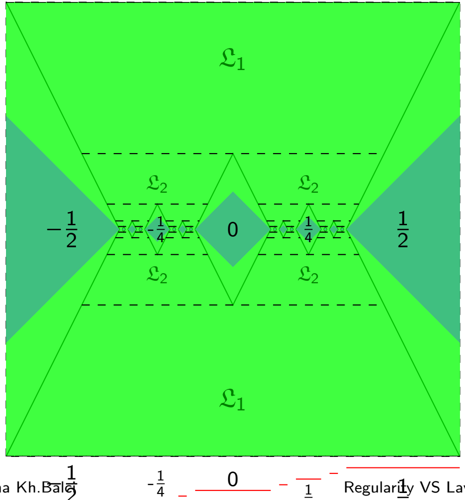

Integrability of $\nabla^\perp v \in L^{p'(\cdot)}$

$$p^+ > 2 - \mathfrak{D}$$

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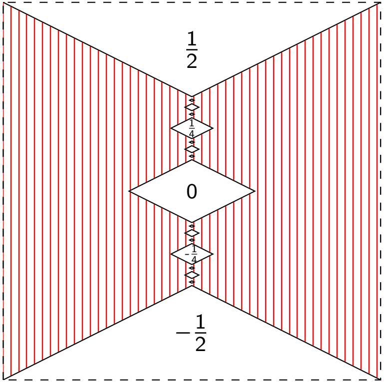
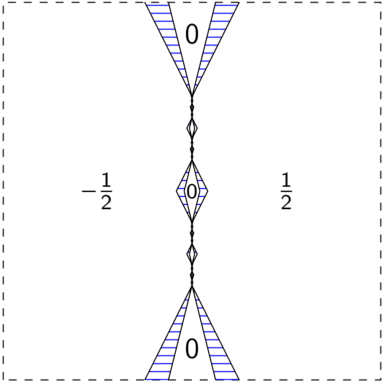
Case $p \approx (2-\mathfrak{D})' > 2$

Function u



Function v



In 2D roles of u and v just change.

Application 2: Double Phase Potential

Let $\mathcal{F}(w) := \int_\Omega |\nabla w|^p + a(x)|\nabla w|^q \, dx$ with $1 < p < q$ and $a \geq 0$.

Resent Positive results:

Colombo and Mingione '15:

if w is a bounded minimizer of $\mathcal{F}$ and $q \leq p + \alpha$, then w is automatically in $W^{1,q}(\Omega)$.

Baroni, Colombo and Mingione '18:

if w is a minimizer of $\mathcal{F}$, $h \in C^{0,\gamma}(\overline{\Omega})$ and $q \leq p + \frac{\alpha}{1-\gamma}$, then w is automatically in $W^{1,q}(\Omega)$.

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Balci, Diening and Surnachev, CalcVar, 2020

Gap for $q > p + \alpha \max\{1, \frac{p-1}{d-1}\}$.

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Motivation: Baroni, Colombo, Mingione

Borderline case of double-phase: $\Phi(x,t) = |t|^p + a(x)|t|^p \log(e+|t|)$

Baroni, Colombo, Mingione (2015)

Let $a(\cdot)$ be log-Hölder continuous. Then $u \in C^{0,\beta}_{\text{loc}}(\Omega)$ for some $\beta \in (0,1)$.
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Example of Lavrentiev gap? **Balci, Surnachev 2020**

$$\Phi(x,t) := |\nabla t|^p \log^{-\beta}(e+|\nabla t|) + a(x)|\nabla t|^p \log^{\alpha}(e+|\nabla t|)\,dx$$

the case $\beta = 0$, $\alpha = 1$ corresponds to **[BCM15]**.

More general $\Phi(x,t) := \varphi(t) + a(x)\psi(t)$.

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Borderline case: Sufficient Condition for Regularity

With the help of general Lemma of Zhikov we get

Balci, Surnachev 2020

Let $\Phi(x,t) := |\nabla t|^p \log^{-\beta}(e + |\nabla t|) + a(x)|\nabla t|^p \log^{\alpha}(e + |\nabla t|)\, dx.$

Assume that the weight $a(x)$ is non-negative, bounded and has the modulus of continuity

$$\omega(r) \leq \frac{k_0}{\log^{\alpha+\beta}(r^{-1})}, \quad \text{if } r \leq \frac{1}{4}.$$

Then the integrand Φ is regular: $C_0^\infty(\Omega)$ is dense in $W_0^{1,\Phi(\cdot)}(\Omega)$.

Checkerboard setup: full description

$$\mathcal{F}(w) := \int_\Omega \frac{1}{2}|\nabla w|^2 \log^{-\beta}(e+|\nabla w|) + a(x)\frac{1}{2}|\nabla w|^2 \log^{\alpha}(e+|\nabla w|)\,dx$$

The weight $a(x)$ as defined as

$$a(x) = \begin{cases} 1, & \text{if } |x_1| < |x_2| \\ 0, & \text{if } |x_1| \geq |x_2|. \end{cases}$$

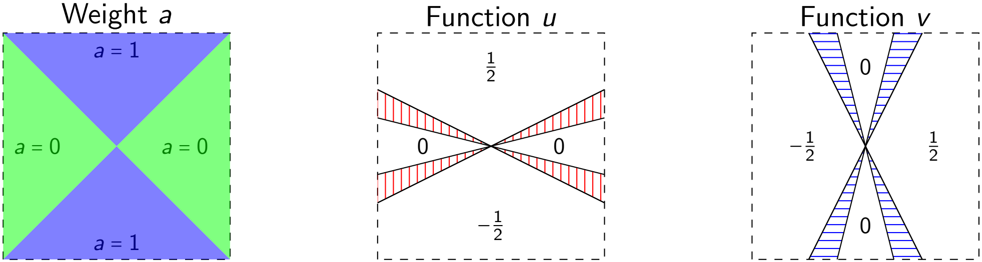

Figure: One saddle point

Borderline Case: Density

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Theorem (Balci, Surnachev ArXiv 2020)

$H_0^{1,\Phi(\cdot)} = W_0^{1,\Phi(\cdot)}$ *if* $\min(\alpha,\beta) \le 1$.

$H_0^{1,\Phi(\cdot)} \neq W_0^{1,\Phi(\cdot)}$ *if* $\alpha > 1$ *and* $\beta > 1$.

This is the case of one saddle point.

1. $\alpha, \beta > 1$ – example of Lavrentiev gap.
2. $\alpha \le 1$ – the saddle point is removable.
3. $\alpha > 1$, $\beta \in [0,1]$ – use the estimates for the modulus of continuity of $u \in W^{1,\Phi(\cdot)}$, $a = 1$.

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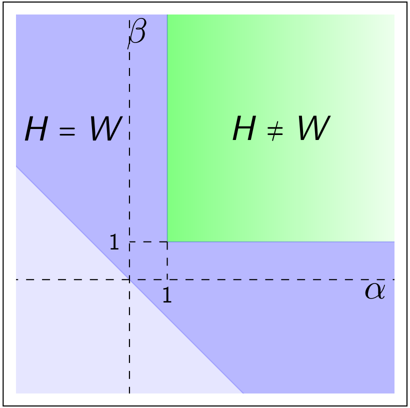

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Proof Sketch: Case $\alpha > 1$, $\beta > 1$

$$\mathcal{F}(w) = \int_\Omega \Phi(x, w)\,dx$$
$$= \int_\Omega \frac{1}{2}|\nabla w|^2 \log^{-\beta}(e + |\nabla w|) + a(x)\frac{1}{2}|\nabla w|^2 \log^{\alpha}(e + |\nabla w|)\,dx.$$

Lavrentiev gap: $$\int_\Omega \Phi(x, |\nabla u|)\,dx \lesssim \int_{Green} |\nabla u|^2 \log^{-\beta}(e + |\nabla u|)\,dx$$
$$\lesssim \int_0^2 \frac{dt}{t \log^{\beta}(e+t)} < \infty, \text{ provided } \beta > 1.$$

$b = \nabla^\perp v$: $$\int_\Omega \Phi^*(x, |b|)\,dx \lesssim \int_{Blue} |b|^2 \log^{-\alpha}(e + |b|)\,dx$$
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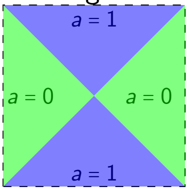
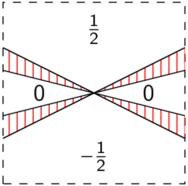
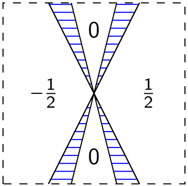
Weight a



Function u



Function v



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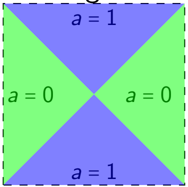

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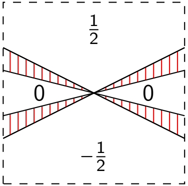

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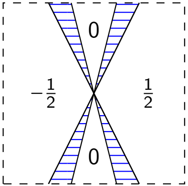

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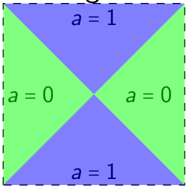

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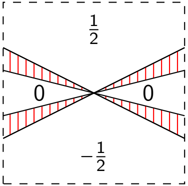

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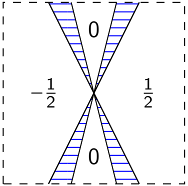

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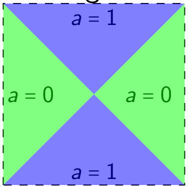

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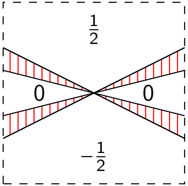

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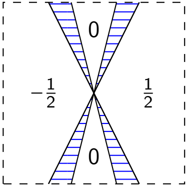

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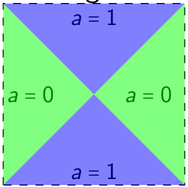

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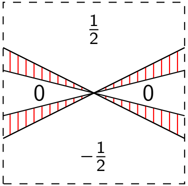

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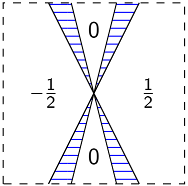

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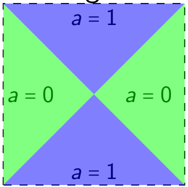
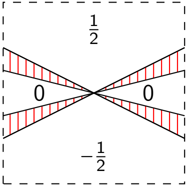
Then $\mathcal{F}(t\tilde{u}) + \mathcal{S}(t\tilde{u}) < 0$ for some $t > 0$.

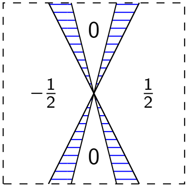
Weight a



Function u



Function v



Proof Sketch: Case $\alpha \leq 1$

$$\mathcal{F}(w) = \int_\Omega \Phi(x, w)\, dx$$
$$= \int_\Omega \frac{1}{2}|\nabla w|^2 \log^{-\beta}(e + |\nabla w|) + a(x)\frac{1}{2}|\nabla w|^2 \log^{\alpha}(e + |\nabla w|)\, dx.$$

If $\alpha \leq 1$ then $H = W$.

The saddle point is removable by cut-off.

C_0^∞ approximating sequence: $u_\varepsilon = u\eta_\varepsilon$.

$$\eta_\varepsilon(r) = \begin{cases} 1, & r \geq \varepsilon, \\ \frac{\log(1/\varepsilon) - \log\log(1/r)}{\log(1/\varepsilon) - \log\log(1/\varepsilon)}, & e^{-1/\varepsilon} < r < \varepsilon, \\ 0, & r \leq e^{-1/\varepsilon}. \end{cases}$$

$\int_\Omega \Phi(x, |\nabla \eta_\varepsilon|)\, dx \to 0$ as $\varepsilon \to 0$.

Sufficient to show that

$$\int_\Omega |\nabla \eta_\varepsilon|^2 \log(e + |\nabla \eta_\varepsilon|)\, dx \to 0.$$

$$\int_\Omega \Phi(x, |\nabla(u - u_\varepsilon)|)\, dx \to 0.$$

Weight a

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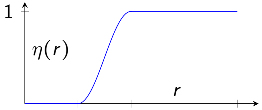
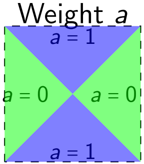




Proof sketch:Case $\alpha > 1$, $\beta \le 1$

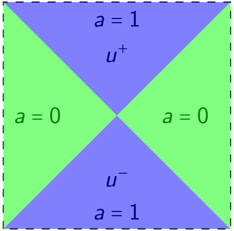
$$\mathcal{F}(w) := \int_\Omega \frac{1}{2}|\nabla w|^2 \log^{-\beta}(e + |\nabla w|) + a(x)\frac{1}{2}|\nabla w|^2 \log^{\alpha}(e + |\nabla w|)\,dx$$
$$= \int_\Omega \Phi(x, w)\,dx = \int_\Omega \varphi(t) + a(x)\psi(t)\,dx.$$

Lemma (On continuity)

If $\alpha > 1$ and $u \in W^{1,\Phi(\cdot)}(\Omega)$ then it is continuous in Blue with modulus of continuity

$$\omega(t) \lesssim \|\nabla u\|_{L^{\psi(\cdot)}(Blue)} \log^{\frac{1-\alpha}{2}}(1/t), \quad t < 1/e.$$

Weight a



Limit values from below and above u^-, u^+.
If $\alpha > 1$, $\beta \le 1$ and $u \in W^{1,\Phi(\cdot)}(\Omega)$, then $u^+ = u^-$. If $u^+ = u^-$, then $u \in H^{1,\Phi(\cdot)}(\Omega)$.

UNIVERSITÄT BIELEFELD

Fakultät für Mathematik

Borderline case: Summary

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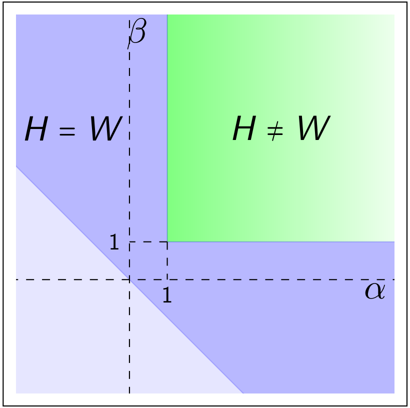
Theorem (Balci, Surnachev ArXiv 2020)

$H_0^{1,\Phi(\cdot)} = W_0^{1,\Phi(\cdot)}$ *if* $\min(\alpha,\beta) \leq 1$.

$H_0^{1,\Phi(\cdot)} \neq W_0^{1,\Phi(\cdot)}$ *if* $\alpha > 1$ *and* $\beta > 1$.



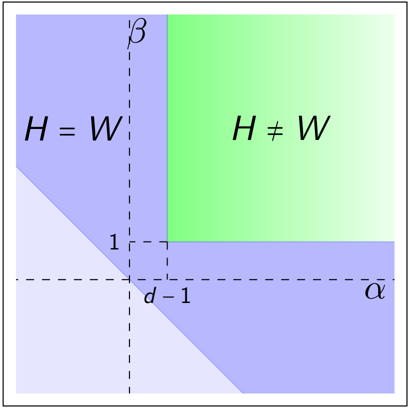
Borderline case: Summary

$$\mathcal{F}(w) := \int_\Omega \frac{1}{d}|\nabla w|^d \log^{-\beta}(e+|\nabla w|) + a(x)\frac{1}{d}|\nabla w|^d \log^{\alpha}(e+|\nabla w|)\,dx$$

Theorem (Balci, Surnachev ArXiv 2020)

$H_0^{1,\Phi(\cdot)} \neq W_0^{1,\Phi(\cdot)}$ *if* $\alpha > d-1$ *and* $\beta > 1$.
otherwise $H_0^{1,\Phi(\cdot)} = W_0^{1,\Phi(\cdot)}$.

Also works for any d.
New density results+examples of Lavrentiev gap.
Surprising hidden regularity even for bad weights.



H-Minimizer VS W-Minimizer

Different notions of $\Phi(\cdot)$-harmonic functions:

$$\mathcal{E}_1 := \inf \mathcal{G}\big(W_0^{1,\Phi(\cdot)}(\Omega)\big) \qquad \inf \mathcal{G}\big(H_0^{1,\Phi(\cdot)}(\Omega)\big) := \mathcal{E}_2$$

$$W_g^{1,\Phi(\cdot)}(\Omega) := g + W_0^{1,\Phi(\cdot)}(\Omega) \qquad H_g^{1,\Phi(\cdot)}(\Omega) := g + H_0^{1,\Phi(\cdot)}(\Omega)$$

$$h_W(g) = \arg\min \mathcal{F}\big(W_g^{1,\Phi(\cdot)}(\Omega)\big) \qquad h_H(g) = \arg\min \mathcal{F}\big(H_g^{1,\Phi(\cdot)}(\Omega)\big)$$

If there is Lavrentiev gap, the $h_W \neq h_H$!
Idea: use tu as a boundary value, then for sufficiently large t $h_W \neq h_H$.

Balci, Surnachev 2020

Let $\alpha, \beta > 1$. Any H-minimizer h_H is continuous in Ω. Any W-minimizer h_W that is not equal to h_H is discontinuous at the origin.

Can we calculate h_W numerically?

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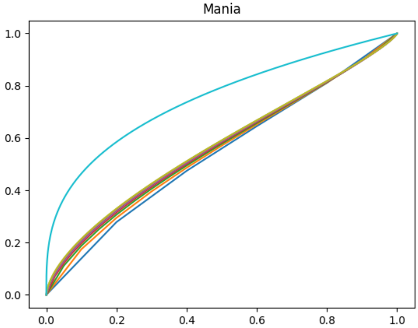
Numerics for Problems with Lavrentiev Gap



$$\mathcal{F}(w) := \int_0^1 (x - w(x)^3)^2 (w'(x))^6 \, \mathrm{d}x$$

for $w(0) = 0$ and $w(1) = 1$

$$w_{\min}(x) = x^{\frac{1}{3}}$$

$$0 = \inf_{\text{all } w} \mathcal{F}(w) < \inf_{\text{smooth } w} \mathcal{F}(w)$$

Problem: Standard FEM fails to converge to correct solution.

[Ball, Knowes; Carstensen, Ortner] Partial results for DG-methods.
Idea is to use Crouzeix-Raviart FEM for functionals with x-dependence.

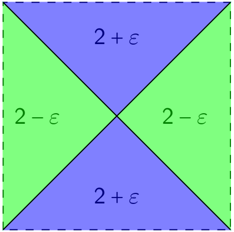
Numerics for Lavrentiev gap

$$-\operatorname{div}(|\nabla h|^{p(\cdot)-2}|\nabla h|) = 0 \quad \text{in } \Omega,$$
$$h = tu \quad \text{on } \partial\Omega.$$

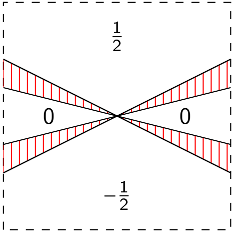
J. Storn

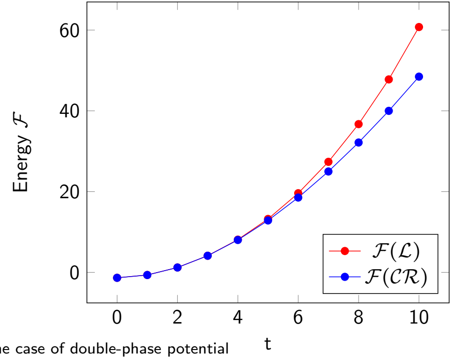
for log-Hölder $p(\cdot)$ standard FEM:
Breit, Diening, Schwarzacher 2015.

Exponent p



Function $u \notin H$





Summary and further research

1. General procedure using fractals: variable exponent, double-phase, weighted p-energy.
2. We have the full description for the model

$$\Phi(x,t) := |\nabla t|^p \log^{-\beta}(e + |\nabla t|) + a(x)|\nabla t|^p \log^{\alpha}(e + |\nabla t|)\, dx \quad \text{if } p = d.$$

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3. New numerical results for special cases.

What about general p? Need **thin and ultra-thin Cantor sets**.

We study Lavrentiev gap for partial spaces of differential forms.
Tomorrow: 14:15 - **Swarnendu Sil** Nonlinear Stein theorem for differential forms via ZOOM-Conference ID 926 5310 0938 Password: 1928

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