Nonstandard symmetric gradient Sobolev spaces

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- D.Breit & A.C. Symmetric gradient Sobolev spaces endowed with rearrangement-invariant norms, preprint

Consider vector-valued functions

$$\mathbf{u} : \Omega \to \mathbb{R}^n,$$

where Ω is an open set in $\mathbb{R}^n$, $n \geq 2$.
Then the (weak) gradient

$$\nabla \mathbf{u} : \Omega \to \mathbb{R}^{n\times n}.$$

Typically, the analysis of systems of PDE's where the unknown is a function $\mathbf{u}$ as above entails the use of the classical Sobolev spaces

$$W_0^{1,p}(\Omega) = W_0^1 L^p(\Omega) = \{\mathbf{u} \in L^p(\Omega) : |\nabla \mathbf{u}| \in L^p(\Omega), \text{“}\mathbf{u} = 0\text{” on } \partial\Omega\},$$

for $p \in [1, \infty]$.

However, some mathematical models of physical systems described by a function $\mathbf{u}$ are not governed by the its full gradient $\nabla\mathbf{u}$, but just by its symmetric part

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \tfrac{1}{2}\big(\nabla\mathbf{u} + \nabla\mathbf{u}^T\big),$$

where $\nabla\mathbf{u}^T =$ transpose of $\nabla\mathbf{u}$.

For instance:

- Generalized Newtonain fluids
- Plasticity
- Nonlinear elasticity

A theory has been developed for the symmetric gradient Sobolev spaces

$$E^1_0L^p(\Omega) = \{\mathbf{u} \in L^p(\Omega) : |\boldsymbol{\varepsilon}(\mathbf{u})| \in L^p(\Omega), \text{``}\mathbf{u} = 0\text{'' on } \partial\Omega\},$$

for $p \in [1, \infty]$. Clearly,

$$W^1_0L^p(\Omega) \to E^1_0L^p(\Omega) \quad \forall\, p \in [1, \infty].$$

Embeddings for $E^1_0 L^p(\Omega)$. If $1 < p < \infty$, then

$$W^1_0 L^p(\Omega) = E^1_0 L^p(\Omega).$$

This is a consequence of Korn's inequality: if $1 < p < \infty$, then

$$\|\nabla \mathbf{u}\|_{L^p(\Omega)} \leq c \|\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^p(\Omega)} \quad \forall\ \mathbf{u} \in E^1_0 L^p(\Omega).$$

This inequality fails if $p = 1$ or $p = \infty$.
Hence, if $1 < p < \infty$, Sobolev embeddings for $E^1_0 L^p(\Omega)$ are exactly the same as those for $W^1_0 L^p(\Omega)$.
In particular, if $|\Omega| < \infty$,

$$E^1_0 L^p(\Omega) \to \begin{cases} L^{p^*}(\Omega) & 1 < p < n \\ \exp L^{n'}(\Omega) & p = n \\ L^\infty(\Omega) \text{ and } C^{1-\frac{n}{p}}(\Omega) & p > n. \end{cases} \tag{1}$$

Alternate proof of (1) by representation formulas for $\mathbf{u} \in E^1_0 L^p(\Omega)$ by integral operators with kernels having Riesz type singularities.

Although

$$W_0^1 L^1(\Omega) \neq E_0^1 L^1(\Omega),$$

still

$$E_0^1 L^1(\Omega) \to L^{n'}(\Omega). \qquad (2)$$

The proof of embedding (2) is in the spirit of that by Gagliardo and Nirenberg for $W_0^1 L^1(\Omega)$.

Neither of the approaches available for $p > 1$ applies to $p = 1$, and the approach for $p = 1$ does not carry over to $p > 1$.

Some of the mathematical models for physical systems mentioned above are driven by nonlinearities of non-power type.
They require the use of symmetric gradient Orlicz-Sobolev spaces $E^1_0L^A(\Omega)$ associated with Young functions $A : [0,\infty) \to [0,\infty]$ (i.e. convex and $A(0) = 0$). They are defined as

$$E^1_0L^A(\Omega) = \{\mathbf{u} \in L^A(\Omega) : |\boldsymbol{\varepsilon}(\mathbf{u})| \in L^A(\Omega), \text{“}\mathbf{u} = 0\text{” on } \partial\Omega\},$$

where the Orlicz space

$$L^A(\Omega) = \Big\{\mathbf{u} : \ \exists\ \lambda > 0 \text{ s.t. } \int_\Omega A\Big(\frac{|\mathbf{u}(x)|}{\lambda}\Big)\, dx < \infty\Big\}.$$

For instance:

• Prandt-Eyring model for Non-Newtonian fluids: $A(t) \approx t\log(1+t)$ near infinity,

• Fluids in liquid body armors: $A(t) \approx e^t - 1$ near infinity.

Pb. Sobolev embeddings for $E^1_0L^A(\Omega)$.

Question risen by M.Bulíček and J.Malek (2009).

Parallel problem solved for $W^1_0L^A(\Omega)$ in [C., Indiana Univ. Math. J. (1996)].

None of the methods described for $E^1_0L^p(\Omega)$, namely when $A(t) = t^p$, applies to arbitrary $A(t)$.

In particular, Korn's inequality

$$\|\nabla \mathbf{u}\|_{L^A(\Omega)} \leq c\|\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^A(\Omega)} \quad \forall\ \mathbf{u} \in E^1_0L^A(\Omega)$$

holds if and only if $A \in \Delta_2 \cap \nabla_2$ [Fuchs, Irish Math. Sci. Bull. (2010)], [Breit-Diening, J. Math. Fluid. Mech. (2012)]. Earlier special cases e.g. in [Acerbi-Mingione, ARMA (2002)].

A version of Korn's inequality in $E^1_0L^A(\Omega)$ holds for every A, with a possibly slightly different Young function B on the left-hand side.

One has that

$$\|\nabla \mathbf{u}\|_{L^B(\Omega)} \leq c\|\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^A(\Omega)} \quad \forall\ \mathbf{u} \in E^1_0L^A(\Omega)$$

if and only if

$$t\int_0^t \frac{B(s)}{s^2}\,ds \leq A(ct) \quad \text{and} \quad t\int_0^t \frac{\widetilde{A}(s)}{s^2}\,ds \leq \widetilde{B}(ct) \qquad \text{for large } t$$

[C., JFA (2014)], [Breit-C.-Diening, SIAM J. Math. Anal. (2017)].

Does not enable to get optimal Sobolev embeddings, for instance when $A(t) = t$.

In [Breit-C., preprint], optimal embeddings are established in the more general class of symmetric gradient Sobolev spaces $E^1_0X(\Omega)$, defined as

$$E^1_0X(\Omega) = \{|\mathbf{u}| \in X(\Omega) : |\boldsymbol{\varepsilon}(\mathbf{u})| \in X(\Omega), \text{“}\mathbf{u} = 0\text{” on } \partial\Omega\},$$

and built upon any rearrangement-invariant space $X(\Omega)$.
Namely, $X(\Omega)$ is a Banach function space such that

$$\|\mathbf{u}\|_{X(\Omega)} = \|\mathbf{v}\|_{X(\Omega)} \qquad \text{if } |\mathbf{u}|^* = |\mathbf{v}|^*,$$

where $|\mathbf{u}|^*$ denotes the decreasing rearrangement of $|\mathbf{u}|$.
In a sense, the norm $\|\mathbf{u}\|_{X(\Omega)}$ only depends on global integrability properties of $\mathbf{u}$ over Ω.
Examples : Lebesgue spaces, Orlicz spaces, Lorentz spaces.

General features of symmetric gradient Sobolev embeddings for $E^1_0X(\Omega)$.

Emebddings into r.i. spaces

Assume that $Y(\Omega)$ is a rearrangement-invariant space. Then

$$W^1_0X(\Omega) \to Y(\Omega) \quad \Leftrightarrow \quad E^1_0X(\Omega) \to Y(\Omega).$$

Symmetric gradient Sobolev embeddings into *rearrangement-invariant* target spaces are *always equivalent* to the corresponding embeddings for the full gradient built upon the same domain spaces.

Emebddings into spaces of continuous functions

Assume that $C^\sigma(\Omega)$ is a space of functions with modulus of continuity $\sigma(\,\cdot\,)$. Then

$$W^1_0X(\Omega) \to C^\sigma(\Omega) \quad \not\Rightarrow \quad E^1_0X(\Omega) \to C^\sigma(\Omega).$$

Symmetric gradient Sobolev embeddings into target spaces of *uniformly continuous* functions *may be weaker* than the corresponding embeddings for the full gradient built upon the same domain spaces.

Optimal Orlicz target space for $E^1_0 L^A(\Omega)$.

Assume that $|\Omega| < \infty$.
Given a Young function A, define the function $H : [0,\infty) \to [0,\infty)$ as

$$H(t) = \left(\int_0^t \left(\frac{s}{A(s)} \right)^{\frac{1}{n-1}} ds \right)^{\frac{1}{n'}} \quad \text{for } t > 0,$$

and the Young function A_n by

$$A_n(t) = A(H^{-1}(t)) \quad \text{for } t > 0.$$

Theorem: optimal Orlicz target for $E^1_0L^A(\Omega)$

The embedding $E^1_0L^A(\Omega) \to L^{A_n}(\Omega)$ holds, and

$$\|\mathbf{u}\|_{L^{A_n}(\Omega)} \leq c\|\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^A(\Omega)} \qquad (3)$$

for every $\mathbf{u} \in E^1_0L^A(\Omega)$. Moreover, the target space in inequality (3) is optimal among all Orlicz spaces.

In particular,

$$L^{A_n}(\Omega) = L^\infty(\Omega)$$

if and only if $A(t)$ grows so fast near infinity that

$$\int^\infty \left(\frac{s}{A(s)}\right)^{\frac{1}{n-1}} ds < \infty. \qquad (4)$$

When $A(t) = t^p$, condition (4) $\Leftrightarrow$ $p > n$.

- The same space $L^{A_n}(\Omega)$ is also optimal for $W^1_0L^A(\Omega) \to L^{A_n}(\Omega)$.

Ex. Assume that either $p = 1$ and $\alpha \geq 0$, or $p > 1$ and $\alpha \in \mathbb{R}$. Then

$$E^1_0 L^p(\log L)^\alpha(\Omega) \to \begin{cases} L^{\frac{np}{n-p}}(\log L)^{\frac{n\alpha}{n-p}}(\Omega) & \text{if } 1 \leq p < n \\ \exp L^{\frac{n}{n-1-\alpha}}(\Omega) & \text{if } p = n \text{ and } \alpha < n-1 \\ \exp\exp L^{\frac{n}{n-1}}(\Omega) & \text{if } p = n \text{ and } \alpha = n-1 \\ L^\infty(\Omega) & \text{if either } p = n \text{ and } \alpha > n-1, \\ & \quad \text{or } p > n\,, \end{cases}$$

all the target spaces being optimal in the class of Orlicz spaces.

If $1 \leq p < n$, then the Sobolev embedding

$$W_0^{1,p}(\Omega) \to L^{p^*}(\Omega)$$

admits the improvement

$$W_0^{1,p}(\Omega) \to L^{p^*,p}(\Omega),$$

where $L^{p^*,p}(\Omega)$ is the Lorentz space equipped with the norm

$$\|u\|_{L^{p^*,p}(\Omega)} = \|s^{-\frac{1}{n}} u^*(s)\|_{L^p(0,|\Omega|)}.$$

Moreover, $L^{p^*,p}(\Omega)$ is optimal among all r.i. target spaces.
A parallel improvement holds also in the borderline case when $p = n$.

By Korn's inequality, if $1 < p < n$, one also has

$$E^1_0 L^p(\Omega) \to L^{p^*,p}(\Omega).$$

The result holds for $p = 1$ as well:

$$E^1_0 L^1(\Omega) \to L^{n',1}(\Omega).$$

This is a very recent result of [Spector-Van Schaftingen, Atti Accad. Lincei (2019)]. Previously announced (not published) by [Tartar].

Proof in the spirit of Gagliardo-Nirenberg and Fournier.

Optimal rearrangement-invariant target space for $E^1_0L^A(\Omega)$.

Let $a : [0,\infty) \to [0,\infty)$ be such that

$$A(t) = \int_0^t a(\tau)\,\mathrm{d}\tau \quad \text{for } t \geq 0.$$

Define the Young function $\widehat{A}$ as

$$\widehat{A}(t) = \int_0^t \widehat{a}(\tau)\,\mathrm{d}\tau \quad \text{for } t \geq 0,$$

where where $\widehat{a} : [0,\infty) \to [0,\infty)$ obeys

$$\widehat{a}^{-1}(t) = \left(\int_{a^{-1}(t)}^{\infty} \left(\int_0^s \left(\frac{1}{a(r)} \right)^{\frac{1}{n-1}} dr \right)^{-n} \frac{ds}{a(s)^{\frac{n}{n-1}}} \right)^{\frac{1}{1-n}} \quad \text{for } \; t > 0.$$

One has $\widehat{A}(t) \lesssim A(t)$, and $\widehat{A}(t) \approx A(t)$ if $A(t) \ll t^n$.

Let $L(\widehat{A}, n)(\Omega)$ be the Orlicz-Lorentz space equipped with the norm

$$\|u\|_{L(\widehat{A},n)(\Omega)} = \left\|s^{-\frac{1}{n}} u^*(s)\right\|_{L^{\widehat{A}}(0,|\Omega|)}.$$

Theorem: optimal r.i. target for $E^1_0 L^A(\Omega)$

The embedding

$$E^1_0 L^A(\Omega) \to L(\widehat{A}, n)(\Omega) \tag{5}$$

holds, and

$$\|\mathbf{u}\|_{L(\widehat{A},n)(\Omega)} \leq c \|\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^A(\Omega)}$$

for every $\mathbf{u} \in E^1_0 L^A(\Omega)$. The target space $L(\widehat{A}, n)(\Omega)$ in embedding (5) is optimal among all rearrangement-invariant spaces.

- The same space $L(\widehat{A}, n)(\Omega)$ is also optimal for

$$W^1_0 L^A(\Omega) \to L(\widehat{A}, n)(\Omega)$$

[C., Rev. Mat. Iberoamericana (2004].)

A key step in the proofs is a reduction principle for symmetric gradient Sobolev spaces with arbitrary r.i. norms.

Theorem: reduction principle

Let $X(\Omega)$ and $Y(\Omega)$ be r.i. spaces. The following facts are equivalent:
(i) The embedding $E^1_0X(\Omega) \to Y(\Omega)$ holds, namely $\exists\ c_1$ s.t.

$$\|\mathbf{u}\|_{Y(\Omega)} \le c_1 \|\boldsymbol{\varepsilon}(\mathbf{u})\|_{X(\Omega)} \qquad \forall\ \mathbf{u} \in E^1_0X(\Omega);$$

(ii) The embedding $W^1_0X(\Omega) \to Y(\Omega)$ holds, namely $\exists\ c_2$ s.t.

$$\|\mathbf{u}\|_{Y(\Omega)} \le c_2 \|\nabla \mathbf{u}\|_{X(\Omega)} \quad \forall\ \mathbf{u} \in W^1_0X(\Omega);$$

(iii) $\exists\ c_3$ s.t.

$$\left\| \int_s^{|\Omega|} f(r) r^{-1+\frac{1}{n}} dr \right\|_{Y(0,|\Omega|)} \le c_3 \|f\|_{X(0,|\Omega|)} \qquad \forall \quad 0 \le f \searrow .$$

This reduction principle enables one to characterize the optimal r.i. target space for $E^1_0X(\Omega)$.
Let $X_1(\Omega)$ be the r.i. space whose norm obeys

$$\|u\|_{X_1'(\Omega)} = \|s^{\frac{1}{n}} u^{**}(s)\|_{X'(0,|\Omega|)},$$

where $u^{**}(s) = \frac{1}{s}\int\limits_0^s u^*(r)\,dr$.

Theorem: optimal r.i. target for $E^1_0X(\Omega)$

The embedding $E^1_0X(\Omega) \to X_1(\Omega)$ holds, i.e.

$$\|\mathbf{u}\|_{X_1(\Omega)} \le c\|\boldsymbol{\varepsilon}(\mathbf{u})\|_{X(\Omega)} \qquad \forall\ \mathbf{u} \in E^1_0X(\Omega). \tag{6}$$

Moreover, $X_1(\Omega)$ is the optimal (smallest possible) rearrangement-invariant target space in (6).

The space $X_1(\Omega)$ was shown to be optimal for $W^1_0X(\Omega) \to X_1(\Omega)$ in [Edmunds-Kerman-Pick, JFA (2000)].

Approach to the reduction principle.

The proof for $W^1_0X(\Omega) \to Y(\Omega)$ is a consequence of the Pólya-Szegö inequality

$$\|\nabla u\|_{X(\Omega)} \geq \|\nabla u^\bigstar\|_{X(\Omega^\bigstar)}$$

for (scalar-valued) $u \in W^1_0X(\Omega)$. Where

$$u^\bigstar = \text{symmetric rearrangement of } u.$$

Does not apply to the symmeric gradient $\varepsilon(\mathbf{u})$.
We rely on an alternate approach, based on K-functionals.
Exploited in [Kerman-Pick, Forum Math. (2006)] for higher-order Sobolev embeddings.
The K-functional of a couple of normed spaces (Z_0, Z_1) is defined for $\zeta \in Z_0 + Z_1$ and $t > 0$ as

$$K(\zeta, t, Z_0, Z_1) = \inf_{\substack{\zeta = \zeta_0 + \zeta_1 \\ \zeta_0 \in Z_0,\, \zeta_1 \in Z_1}} \big(\|\zeta_0\|_{Z_0} + t\|\zeta_1\|_{Z_1}\big).$$

The K-functional for the couple $(W^{1,1}(\mathbb{R}^n), W^{1,\infty}(\mathbb{R}^n))$ is a fundamental result in interpolation theory [De Vore - Scherer, Ann. Math. (1979)]:

$$K(\mathbf{u}, t, W^{1,1}(\mathbb{R}^n), W^{1,\infty}(\mathbb{R}^n)) \approx \int_0^t \mathbf{u}^*(s) + (\nabla \mathbf{u})^*(s)\, ds \quad \text{for } t > 0.$$

A crucial step in our work is the K-functional for $(E^1L^1(\mathbb{R}^n), E^1L^\infty(\mathbb{R}^n))$.

Theorem: K-functional

One has that

$$K(\mathbf{u}, t, E^1L^1(\mathbb{R}^n), E^1L^\infty(\mathbb{R}^n)) \approx \int_0^t \mathbf{u}^*(s) + \boldsymbol{\varepsilon}(\mathbf{u})^*(s)\, ds \quad \text{for } t > 0,$$

and

$$K(\mathbf{u}, t, E^1_0L^1(\mathbb{R}^n), E^1_0L^\infty(\mathbb{R}^n)) \approx \int_0^t \boldsymbol{\varepsilon}(\mathbf{u})^*(s) ds \quad \text{for } t > 0.$$

The proof of the K-functional makes use of Lipschitz truncation results for symmetric gradient Sobolev spaces related to those of employed in regularity theory for PDSs and in the Calculus of Variations, and, in particular in [Breit-Diening-Fuchs, JDE (2012)].

With the K-functional at disposal, the outline of the proof of the reduction principle is as follows:

- Endpoint embeddings:

$$E^1_0 L^1(\mathbb{R}^n) \to L^{n',1}(\mathbb{R}^n),$$

$$E^1_0 L^{n,1}(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n).$$

- K functionals for the endpoint spaces (by reiteration theorem) :

$$K(t, \mathbf{u}; L^{n',1}(\Omega), L^\infty(\Omega)) \approx \int_0^{t^{n'}} s^{-\frac{1}{n}} \mathbf{u}^*(s)ds \quad \text{for } t > 0.$$

$$K(\mathbf{u}, t, E_0^1 L^1(\mathbb{R}^n), E_0^1 L^{n,1}(\mathbb{R}^n)) \approx \int_0^{t^{n'}} \boldsymbol{\varepsilon}(\mathbf{u})^*(s)ds + t \int_{t^{n'}}^\infty \boldsymbol{\varepsilon}(\mathbf{u})^*(s) s^{-\frac{1}{n'}} ds$$

- Basic K-functional inequality:

$$K(t, \mathbf{u}; L^{n',1}(\Omega), L^\infty(\Omega)) \le cK(\mathbf{u}, t/c, E_0^1 L^1(\mathbb{R}^n), E_0^1 L^{n,1}(\mathbb{R}^n)) \quad \text{for } t > 0.$$

Optimal target space of continuous functions for $E^1_0 L^A(\Omega)$.

Consider now embeddings of the form

$$E^1_0 L^A(\Omega) \to C^\sigma(\Omega),$$

where $\sigma : [0,\infty) \to [0,\infty)$ is an increasing function s.t. $\sigma(0) = 0$.
Here, we assume that Ω is bounded.

Theorem: embeddings of $E^1_0 L^A(\Omega)$ into $C^\sigma(\Omega)$

Let A be a Young function.
(i) The embedding $E^1_0 L^A(\Omega) \to C^0(\Omega)$ holds, namely

$$\|\mathbf{u}\|_{C^0(\Omega)} \leq c\|\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^A(\Omega)} \qquad \forall\ \mathbf{u} \in E^1_0 L^A(\Omega)$$

if and only if

$$\int^\infty \left(\frac{t}{A(t)}\right)^{\frac{1}{n-1}} dt < \infty \,. \tag{7}$$

(ii) Assume that A fulfills (7). Define the Young functions

$$\xi_A(t) = t^{n'} \int_t^\infty \frac{\widetilde{A}(\tau)}{\tau^{1+n'}}\,d\tau \quad \text{and} \quad \eta_A(t) = t\int_0^t \frac{\widetilde{A}(\tau)}{\tau^2}\,d\tau \quad \text{for } t > 0.$$

Let

$$\sigma_A(r) = \frac{r^{1-n}}{\xi_A^{-1}(r^{-n})} + \frac{r^{1-n}}{\eta_A^{-1}(r^{-n})} \quad \text{for } r > 0.$$

Then $E_0^1 L^A(\Omega) \to C^{\sigma_A}(\Omega)$ and $\exists\ c$ s.t.

$$\|\mathbf{u}\|_{C^{\sigma_A}(\Omega)} \le c\|\boldsymbol{\varepsilon}(\mathbf{u})\|_{L^A(\Omega)} \qquad \forall\ \mathbf{u} \in E_0^1 L^A(\Omega).$$

Moreover, the space $C^{\sigma_A}(\Omega)$ is optimal.

• The theorem follows as a special case of a result on optimal embeddings $E_0^1 X(\Omega) \to C^\sigma(\Omega)$, for general r.i. spaces $X(\Omega)$.
Different approach, via representation formulas (no K-functionals).

Comparison with embeddings for $W^1_0 L^A(\Omega)$.

One has that

$$E^1_0 L^A(\Omega) \to C^0(\Omega)$$

if and only if

$$W^1_0 L^A(\Omega) \to C^0(\Omega)$$

if and only if

$$\int^{\infty} \left(\frac{t}{A(t)} \right)^{\frac{1}{n-1}} dt < \infty \,.$$

However, the optimal space $C^{\varrho_A}(\Omega)$ in

$$W_0^1 L^A(\Omega) \to C^{\varrho_A}(\Omega)$$

is given by

$$\varrho_A(r) = \frac{r^{1-n}}{\xi_A^{-1}(r^{-n})} \quad \text{for } r > 0$$

[C., Ann. Scuola Norm. Sup. Pisa [1996]. For certain Young functions A, one has that

$$C^{\varrho_A}(\Omega) \subsetneq C^{\sigma_A}(\Omega).$$

Therefore, the optimal space for $W_0^1 L^A(\Omega)$ can be smaller than that for $E_0^1 L^A(\Omega)$.

Tipically, this is the case when $L^A(\Omega)$ is "close" to $L^\infty(\Omega)$.

Ex. 1. One has that

$$W^1_0 L^\infty(\Omega) \to C^{0,1}(\Omega) = \mathrm{Lip}(\Omega),$$

whereas

$$E^1_0 L^\infty(\Omega) \to C^{\sigma_\infty}(\Omega)$$

with

$$\sigma(r) \approx r \log(1/r) \quad \text{as } r \to 0^+.$$

Ex. 2. Given $\beta > 0$, one has that

$$W^1_0 \exp L^\beta(\Omega) \to C^{\varrho_\beta}(\Omega),$$

where

$$\varrho_\beta(r) \approx r\big(\log(1/r)\big)^{\frac{1}{\beta}} \quad \text{as } r \to 0^+.$$

On the other hand,

$$E^1_0 \exp L^\beta(\Omega) \to C^{\sigma_\beta}(\Omega),$$

where

$$\sigma_\beta(r) \approx r\big(\log(1/r)\big)^{1+\frac{1}{\beta}} \quad \text{as } r \to 0^+.$$

Versions of the results presented hold also for the spaces

$$E^1 X(\mathbb{R}^n).$$

In this case the point is that $|\mathbb{R}^n| = \infty$.

Moreover, versions of the results hold also in

$$E^1 X(\Omega).$$

Since functions need not vanish on $\partial\Omega$, some regularity of Ω is needed.

We allow for minimal regularity assumptions on Ω.

Our results about $E^1 X(\Omega)$ hold for (ε, δ)-domains Ω.

An open set $\Omega \subset \mathbb{R}^n$ is called an (ε, δ)-domain if $\exists\ \varepsilon, \delta > 0$ such that $\forall$ $x, y \in \Omega$, with $|x-y| < \delta$, there exists a rectifiable curve γ connecting x and y, with length $\ell(\gamma)$, satisfying

$$\ell(\gamma) \leq \frac{1}{\varepsilon}|x-y|, \tag{8}$$

$$\operatorname{dist}(z, \partial\Omega) \geq \varepsilon \frac{|x-z||y-z|}{|x-y|} \quad \forall z \in \gamma. \tag{9}$$

The notion of (ε, δ)-domain was used in [Jones, Acta Math. (1981)], where it is shown that any domain of this kind is an extension domain for the Sobolev space $W^{m,p}(\Omega)$ for every $m \in \mathbb{N}$ and $p \in [1, \infty]$.

The class of (ε, δ)-domains is known to be, in a sense, the largest one supporting the extension property for Sobolev spaces.

The proof of our embeddings for the space $E^1X(\Omega)$ requires additional results of possible independent interest.

We need a global approximation result on (ε, δ)-domains.

Theorem: Smooth approximation up to the boundary on (ε, δ)-domains

Let Ω be an (ε, δ)-domain and let $\mathbf{u} \in E^1L^1(\Omega)$. Then there exists a sequence $\{\mathbf{u}_j\} \subset C^\infty(\overline{\Omega})$ such that $\mathbf{u}_j \to \mathbf{u}$ in $E^1L^1(\Omega)$.

A version of Jones extension theorem for Sobolev functions on (ε, δ)-domain is also needed.
An extension operator for $E^1L^1(\Omega)$ is in [Gmeineder-Raita, JFA (2019)].
We need an extension operator on more general domains and enjoying additional properties.

Theorem: Extension operator on (ε, δ)-domains

Let Ω be an (ε, δ)-domain. Then there $\exists$ a linear extension operator

$$\mathscr{E}_\Omega : L^1_{\rm loc}(\Omega) \to L^1_{\rm loc}(\mathbb{R}^n)$$

s.t.:

$$\mathscr{E}_\Omega \mathbf{u} = \mathbf{u} \quad \text{in } \Omega \qquad \forall\ \mathbf{u} \in L^1_{\rm loc}(\Omega),$$

$$\mathscr{E}_\Omega : L^1(\Omega) \to L^1(\mathbb{R}^n), \quad \mathscr{E}_\Omega : L^\infty(\Omega) \to L^\infty(\mathbb{R}^n),$$

$$\mathscr{E}_\Omega : E^1L^1(\Omega) \to E^1L^1(\mathbb{R}^n), \quad \mathscr{E}_\Omega : E^1L^\infty(\Omega) \to E^1L^\infty(\mathbb{R}^n).$$

Moreover, $\exists$ linear bounded operators $\mathscr{L}^i$, $i = 1, 2$, such that

$$\mathscr{L}^i : L^1(\Omega) \to L^1(\mathbb{R}^n), \quad \mathscr{L}^i : L^\infty(\Omega) \to L^\infty(\mathbb{R}^n) \quad \text{for } i = 1, 2,$$

and

$$\boldsymbol{\varepsilon}(\mathscr{E}_\Omega \mathbf{u}) = \mathscr{L}^1 \boldsymbol{\varepsilon}(\mathbf{u}) + \mathscr{L}^2 \mathbf{u}$$

for $\mathbf{u} \in E^1L^1(\Omega) + E^1L^\infty(\Omega)$.