Unique continuation for sublinear parabolic equations

Agnid Banerjee
TIFR CAM, Bangalore, India

(Based on joint works with Vedansh Arya and Ramesh Manna)

March 22, 2021

Preliminaries

Let Ω be a connected, open subset of $\mathbb{R}^n$ and u be a real valued function defined on Ω.

Definition

We say that a function u vanishes to infinite order at some $x_0 \in \Omega$ if for given $k > 0$, there exists $C_k > 0$ such that

$$|u(x)| \leq C_k |x - x_0|^k \text{ for all } x \text{ near } x_0.$$

- If u is smooth then above definition is equivalent to $D^\alpha u(x_0) = 0$ for all α.

Basics Contd.

Strong Unique Continuation Property

We say that a function $u \not\equiv 0$ satisfies strong unique continuation property (sucp) if it cannot vanish to infinite order at any point $x_0 \in \Omega$.

Basics Contd.

Strong Unique Continuation Property

We say that a function $u \not\equiv 0$ satisfies strong unique continuation property (sucp) if it cannot vanish to infinite order at any point $x_0 \in \Omega$.

Example: Any real analytic function.

Basics Contd.

Strong Unique Continuation Property

We say that a function $u \not\equiv 0$ satisfies strong unique continuation property (sucp) if it cannot vanish to infinite order at any point $x_0 \in \Omega$.

Example: Any real analytic function. Weak unique continuation property (wucp) is defined to be one when a function cannot vanish in any open subset of a domain.

Basics Contd.

Strong Unique Continuation Property

We say that a function $u \not\equiv 0$ satisfies strong unique continuation property (sucp) if it cannot vanish to infinite order at any point $x_0 \in \Omega$.

Example: Any real analytic function. Weak unique continuation property (wucp) is defined to be one when a function cannot vanish in any open subset of a domain.

Basics Contd.

Strong Unique Continuation Property

We say that a function $u \not\equiv 0$ satisfies strong unique continuation property (sucp) if it cannot vanish to infinite order at any point $x_0 \in \Omega$.

Example: Any real analytic function. Weak unique continuation property (wucp) is defined to be one when a function cannot vanish in any open subset of a domain.

Definition

An operator L is said to have the strong/weak unique continuation property if any non-trivial solution satisfies the strong/weak unique continuation property.

Basics Contd.

Strong Unique Continuation Property

We say that a function $u \not\equiv 0$ satisfies strong unique continuation property (sucp) if it cannot vanish to infinite order at any point $x_0 \in \Omega$.

Example: Any real analytic function. Weak unique continuation property (wucp) is defined to be one when a function cannot vanish in any open subset of a domain.

Definition

An operator L is said to have the strong/weak unique continuation property if any non-trivial solution satisfies the strong/weak unique continuation property.

Example $L = \Delta$ in which case, the sucp follows from real analyticity of the solution.

Background

Background

- For operators of the form $L = \text{div}(A(x)\nabla.) + b(x).\nabla + c(x)$, with A Lipschitz and $b, c \in L^\infty$, sucp was established in the early 1960's by Aronszajn-Krzywicki-Szarki[AKS] using Carleman estimates.

Background

- For operators of the form $L = \text{div}(A(x)\nabla.) + b(x).\nabla + c(x)$, with A Lipschitz and $b, c \in L^\infty$, sucp was established in the early 1960's by Aronszajn-Krzywicki-Szarki[AKS] using Carleman estimates.
- In 1979, F. Almgren discovered a remarkable monotonicity formula in his study of regularity of mass minimizing currents.

Background

- For operators of the form $L = \text{div}(A(x)\nabla .) + b(x).\nabla + c(x)$, with A Lipschitz and $b, c \in L^\infty$, sucp was established in the early 1960's by Aronszajn-Krzywicki-Szarki[AKS] using Carleman estimates.
- In 1979, F. Almgren discovered a remarkable monotonicity formula in his study of regularity of mass minimizing currents.

If $\Delta u = 0$ in B_1. Then the so called Almgren frequency

$$N(u,r) = \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2} \tag{1.1}$$

is monotone increasing as a function of r.

Bounded frequency $\implies$ sucp

Bounded frequency $\implies$ sucp

One consequence of the monotonicity of the frequency(infact, only boundedness suffices!) is the following doubling property:

$$\int_{B_{2r}} u^2 \leq C(n, ||u|_{L^2(B_1)}) \int_{B_r} u^2 \tag{1.2}$$

Bounded frequency $\implies$ sucp

One consequence of the monotonicity of the frequency(infact, only boundedness suffices!) is the following doubling property:

$$\int_{B_{2r}} u^2 \leq C(n, ||u|_{L^2(B_1)}) \int_{B_r} u^2 \tag{1.2}$$

It is well known that doubling $\implies$ sucp.

Contd.

Contd.

In 1986, Garofalo and Lin proved that solutions to elliptic equations treated by [AKS] (i.e. Lipschitz principal part and bounded lower order terms) satisfy a delicate generalization of the Almgren's monotonicity formula.

Contd.

In 1986, Garofalo and Lin proved that solutions to elliptic equations treated by [AKS] (i.e. Lipschitz principal part and bounded lower order terms) satisfy a delicate generalization of the Almgren's monotonicity formula.

Such a monotonicity formula was then used to show that solutions as well as their gradients are in some A_p class of Muckenhoupt and in particular satisfy the doubling inequality which implies sucp.

Contd.

In 1986, Garofalo and Lin proved that solutions to elliptic equations treated by [AKS] (i.e. Lipschitz principal part and bounded lower order terms) satisfy a delicate generalization of the Almgren's monotonicity formula.

Such a monotonicity formula was then used to show that solutions as well as their gradients are in some A_p class of Muckenhoupt and in particular satisfy the doubling inequality which implies sucp.

Remark Sucp fails when the principal part $A \in C^{0,\alpha}$ for any $\alpha < 1$ and the counterexamples are due to Plis and Miller.

Parabolic case

Parabolic case

A parabolic version of Almgren's monotonicity formula was discovered by C. Poon in 1996. More precisely, Poon showed that if u is a bounded solution to

$$\Delta u - u_t = b.\nabla u + cu \tag{1.3}$$

in say $\mathbb{R}^n \times (t_1, t_0)$, then the following quantity

Parabolic case

A parabolic version of Almgren's monotonicity formula was discovered by C. Poon in 1996. More precisely, Poon showed that if u is a bounded solution to

$$\Delta u - u_t = b.\nabla u + cu \tag{1.3}$$

in say $\mathbb{R}^n \times (t_1, t_0)$, then the following quantity

$$N(r) = \frac{r^2 \int_{t=t_0-r^2} |\nabla u(x,t)|^2 G_{x_0,t_0}(x,t)dx}{\int_{t=t_0-r^2} u(x,t)^2 G_{x_0,t_0}(x,t)dx} \tag{1.4}$$

is bounded where G_{x_0,t_0} is the backward heat kernel centered at (x_0, t_0).

Parabolic case

A parabolic version of Almgren's monotonicity formula was discovered by C. Poon in 1996. More precisely, Poon showed that if u is a bounded solution to

$$\Delta u - u_t = b.\nabla u + cu \tag{1.3}$$

in say $\mathbb{R}^n \times (t_1, t_0)$, then the following quantity

$$N(r) = \frac{r^2 \int_{t=t_0-r^2} |\nabla u(x,t)|^2 G_{x_0,t_0}(x,t)dx}{\int_{t=t_0-r^2} u(x,t)^2 G_{x_0,t_0}(x,t)dx} \tag{1.4}$$

is bounded where G_{x_0,t_0} is the backward heat kernel centered at (x_0, t_0).In the case of heat equation, one obtains that N is monotonically increasing.

Parabolic case

A parabolic version of Almgren's monotonicity formula was discovered by C. Poon in 1996. More precisely, Poon showed that if u is a bounded solution to

$$\Delta u - u_t = b.\nabla u + cu \tag{1.3}$$

in say $\mathbb{R}^n \times (t_1, t_0)$, then the following quantity

$$N(r) = \frac{r^2 \int_{t=t_0-r^2} |\nabla u(x,t)|^2 G_{x_0,t_0}(x,t)dx}{\int_{t=t_0-r^2} u(x,t)^2 G_{x_0,t_0}(x,t)dx} \tag{1.4}$$

is bounded where G_{x_0,t_0} is the backward heat kernel centered at (x_0, t_0).In the case of heat equation, one obtains that N is monotonically increasing. Using this, he was able to show that a bounded solution u to (1.3) satisfies the backward in time sucp.

Parabolic case

A parabolic version of Almgren's monotonicity formula was discovered by C. Poon in 1996. More precisely, Poon showed that if u is a bounded solution to

$$\Delta u - u_t = b.\nabla u + cu \tag{1.3}$$

in say $\mathbb{R}^n \times (t_1, t_0)$, then the following quantity

$$N(r) = \frac{r^2 \int_{t=t_0-r^2} |\nabla u(x,t)|^2 G_{x_0,t_0}(x,t)dx}{\int_{t=t_0-r^2} u(x,t)^2 G_{x_0,t_0}(x,t)dx} \tag{1.4}$$

is bounded where G_{x_0,t_0} is the backward heat kernel centered at (x_0, t_0).In the case of heat equation, one obtains that N is monotonically increasing. Using this, he was able to show that a bounded solution u to (1.3) satisfies the backward in time sucp. More precisely if

$$sup_{Q_r(x_0,t_0)}|u| = O(r^k) \tag{1.5}$$

for all $k > 0$, where $Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0]$, then $u \equiv 0$.

Remark 1: One requires u to be bounded or have some "Tychonoff" type growth assumption. If that is not the case, there is example by Frank Jones where one has a solution to the heat equation which is supported in a strip.

Remark 1: One requires u to be bounded or have some "Tychonoff" type growth assumption. If that is not the case, there is example by Frank Jones where one has a solution to the heat equation which is supported in a strip.

Subsequently, space like strong unique continuation properties for parabolic equations with Lipschitz principal part was shown by Escauriaza-Fernandez (and also Escauriaza-Fernandez-Vessella) in 2004.

Remark 1: One requires u to be bounded or have some "Tychonoff" type growth assumption. If that is not the case, there is example by Frank Jones where one has a solution to the heat equation which is supported in a strip.

Subsequently, space like strong unique continuation properties for parabolic equations with Lipschitz principal part was shown by Escauriaza-Fernandez (and also Escauriaza-Fernandez-Vessella) in 2004. In fact they showed,

Theorem

Let u solve

$$\mathrm{div}(A(x,t)\nabla u) = u_t + Vu$$

in $Q_r(x_0, t_0)$, where A is Lipschitz in x and $1/2$ hölder in time. Then if u vanishes to infinite order at (x_0, t_0), then $u(\cdot, t_0) \equiv 0$.

Remark 1: One requires u to be bounded or have some "Tychonoff" type growth assumption. If that is not the case, there is example by Frank Jones where one has a solution to the heat equation which is supported in a strip.

Subsequently, space like strong unique continuation properties for parabolic equations with Lipschitz principal part was shown by Escauriaza-Fernandez (and also Escauriaza-Fernandez-Vessella) in 2004. In fact they showed,

Theorem

Let u solve

$$\operatorname{div}(A(x,t)\nabla u) = u_t + Vu$$

in $Q_r(x_0, t_0)$, where A is Lipschitz in x and $1/2$ hölder in time. Then if u vanishes to infinite order at (x_0, t_0), then $u(\cdot, t_0) \equiv 0$.

Remark : The regularity assumption on A was lowered later on by Koch and Tataru. Now for the backward unique continuation property of the Poon type, one requires an additional decay assumption on the derivative of the principal part.

Remark 1: One requires u to be bounded or have some "Tychonoff" type growth assumption. If that is not the case, there is example by Frank Jones where one has a solution to the heat equation which is supported in a strip.

Subsequently, space like strong unique continuation properties for parabolic equations with Lipschitz principal part was shown by Escauriaza-Fernandez (and also Escauriaza-Fernandez-Vessella) in 2004. In fact they showed,

Theorem

Let u solve

$$\mathrm{div}(A(x,t)\nabla u) = u_t + Vu$$

in $Q_r(x_0, t_0)$, where A is Lipschitz in x and $1/2$ hölder in time. Then if u vanishes to infinite order at (x_0, t_0), then $u(\cdot, t_0) \equiv 0$.

Remark : The regularity assumption on A was lowered later on by Koch and Tataru. Now for the backward unique continuation property of the Poon type, one requires an additional decay assumption on the derivative of the principal part. Such conditions are somewhat optimal (Wu-Zhang).

Remark 1: One requires u to be bounded or have some "Tychonoff" type growth assumption. If that is not the case, there is example by Frank Jones where one has a solution to the heat equation which is supported in a strip.

Subsequently, space like strong unique continuation properties for parabolic equations with Lipschitz principal part was shown by Escauriaza-Fernandez (and also Escauriaza-Fernandez-Vessella) in 2004. In fact they showed,

Theorem

Let u solve

$$\mathrm{div}(A(x,t)\nabla u) = u_t + Vu$$

in $Q_r(x_0, t_0)$, where A is Lipschitz in x and $1/2$ hölder in time. Then if u vanishes to infinite order at (x_0, t_0), then $u(\cdot, t_0) \equiv 0$.

Remark : The regularity assumption on A was lowered later on by Koch and Tataru. Now for the backward unique continuation property of the Poon type, one requires an additional decay assumption on the derivative of the principal part. Such conditions are somewhat optimal (Wu-Zhang).

Sublinear Elliptic equations

Recently in 2017, unique continuation property for sublinear equations of the type

$$\mathrm{div}(A(x)\nabla u) + f_p(x,u) + Vu, \tag{1.6}$$

where the sublinearity f_p is modelled on $|v|^{p-2}v(\ 1 \leq p < 2)$ has been studied by Soave and Weth.

Sublinear Elliptic equations

Recently in 2017, unique continuation property for sublinear equations of the type

$$\mathrm{div}(A(x)\nabla u) + f_p(x, u) + Vu, \tag{1.6}$$

where the sublinearity f_p is modelled on $|v|^{p-2}v(\, 1 \leq p < 2)$ has been studied by Soave and Weth. A weak unique continuation property was established in their work by an adaptation of the frequency function approach of Garofalo and Lin.

Sublinear Elliptic equations

Recently in 2017, unique continuation property for sublinear equations of the type

$$\operatorname{div}(A(x)\nabla u) + f_p(x, u) + Vu, \tag{1.6}$$

where the sublinearity f_p is modelled on $|v|^{p-2}v(\ 1 \leq p < 2)$ has been studied by Soave and Weth. A weak unique continuation property was established in their work by an adaptation of the frequency function approach of Garofalo and Lin.

Remark: It is to be mentioned that their work was motivated by an older work of Parini and Weth in 2015 where Neumann problem for such sublinear equations was studied and where among other results, the authors studied the nodal set or the zero set of the so called "least energy" solutions.

Sublinear Elliptic equations

Recently in 2017, unique continuation property for sublinear equations of the type

$$\operatorname{div}(A(x)\nabla u) + f_p(x,u) + Vu, \tag{1.6}$$

where the sublinearity f_p is modelled on $|v|^{p-2}v(\, 1 \leq p < 2)$ has been studied by Soave and Weth. A weak unique continuation property was established in their work by an adaptation of the frequency function approach of Garofalo and Lin.

Remark: It is to be mentioned that their work was motivated by an older work of Parini and Weth in 2015 where Neumann problem for such sublinear equations was studied and where among other results, the authors studied the nodal set or the zero set of the so called "least energy" solutions.

Subsequently strong unique continuation for (1.6) was established by Ruland in 2018 (for $1 < p < 2$) by means of new Carleman estimates which are tailored for such sublinear operators.

Some motivation

The study of (1.6) is partly motivated by its connection to the porous medium equation

$$w_t - \Delta |w|^{m-2} w = 0. \tag{1.7}$$

The study of (1.6) is partly motivated by its connection to the porous medium equation

$$w_t - \Delta |w|^{m-2} w = 0. \tag{1.7}$$

In fact a solution to (1.6) gives rise to a time independent solution of (1.7) (when $f_p = |v|^{p-2}v$) by a change of variable of the type

$$w = c_p |u|^{p-2} u.$$

Remark The class of sublinear equations that we consider also include

$$-\Delta v = v_t + \lambda_+ (v^+)^{p-1} - \lambda_- (v^-)^{p-1}, \text{ where } \lambda_+, \lambda_- > 0, \ p \in [1, 2),$$

which corresponds to the two phase membrane problem.

Finally, I would like to mention that the regularity of the nodal set of solutions to such sublinear equations based on Weiss type monotonicity and blow up arguments has been studied by Soave and Terracini (2018).

Remarks

- We can not linearize as

$$\mathrm{div}(A(x)\nabla u) + Vu = 0$$

  and apply the linear unique continuation results because even in the model case, $V = |u|^{p-2}$ need not be in L^p for any p near the zero set of u as $p \in (1,2)$.
- The sign assumption on the sublinearity is quite crucial because otherwise unique continuation fails. In fact Soave and Weth in 2018 gave a counterexample to show unique continuation is not true for

$$\Delta v = |v|^{p-2}v, \;\; p \in (1,2).$$

Remarks

- We can not linearize as

$$\mathrm{div}(A(x)\nabla u) + Vu = 0$$

and apply the linear unique continuation results because even in the model case, $V = |u|^{p-2}$ need not be in L^p for any p near the zero set of u as $p \in (1, 2)$.

- The sign assumption on the sublinearity is quite crucial because otherwise unique continuation fails. In fact Soave and Weth in 2018 gave a counterexample to show unique continuation is not true for

$$\Delta v = |v|^{p-2}v, \ \ p \in (1, 2).$$

More precisely if one takes $u(t) = c_p t^{\frac{2}{2-p}}$ for $t > 0$ and $u \equiv 0$ for $t < 0$ with an appropriately chosen c_p, then it solves

$$u''(t) = |u|^{p-2}u.$$

Related developments

Related developments

- Strong unique continuation has been extended to fractional sublinear equations recently by Tortone (2020).

Related developments

- Strong unique continuation has been extended to fractional sublinear equations recently by Tortone (2020).
- Strong unique continuation for sublinear Baouendi-Grushin type operators has been obtained by B-Garofalo-Manna and B-Manna.

Space like strong unique continuation(Our results)

Space like strong unique continuation(Our results)

Theorem (B-Manna (2019))

Let u be a solution to

$$\operatorname{div}(A(x,t)\nabla u) + Vu + f_p((x,t),u) - u_t = 0$$

in $Q_r(x_0, t_0)$ where A is Lispchitz in space and time and the sublinear term f satisfies similar structural conditions as in the elliptic case($1 < p < 2$). Now if u vanishes to infinite order in space at (x_0, t_0), then $u(\cdot, t_0) \equiv 0$.

Outline of the proof

Outline of the proof

By change of variable, $t \to -t$, we instead consider the backward parabolic sublinear equation,

$$\operatorname{div}(A(x,t)\nabla u) + Vu + f_p((x,t),u) + u_t = 0.$$

Outline of the proof

By change of variable, $t \to -t$, we instead consider the backward parabolic sublinear equation,

$$\mathrm{div}(A(x,t)\nabla u) + Vu + f_p((x,t),u) + u_t = 0.$$

Now following Escauriaza-Fernandez, we let

$$\theta(t) = t^{1/2}\left(\log \frac{1}{t}\right)^{1+\beta/2}. \tag{1.8}$$

Outline of the proof

By change of variable, $t \to -t$, we instead consider the backward parabolic sublinear equation,

$$\mathrm{div}(A(x,t)\nabla u) + Vu + f_p((x,t),u) + u_t = 0.$$

Now following Escauriaza-Fernandez, we let

$$\theta(t) = t^{1/2}\left(\log\frac{1}{t}\right)^{1+\beta/2}. \tag{1.8}$$

Then one solves the following ODE in time,

$$\frac{d}{dt}\log\left(\frac{\sigma}{t\dot{\sigma}}\right) = \frac{\theta(\gamma t)}{t},\ \ \sigma(0) = 0,\ \ \dot{\sigma}(0) = 1,$$

where $\gamma > 0$ and $0 \le \gamma t \le 1$. It turns out that the solution σ is such that $\sigma \sim t$.

Outline of the proof

By change of variable, $t \to -t$, we instead consider the backward parabolic sublinear equation,

$$\operatorname{div}(A(x,t)\nabla u) + Vu + f_p((x,t),u) + u_t = 0.$$

Now following Escauriaza-Fernandez, we let

$$\theta(t) = t^{1/2}\left(\log\frac{1}{t}\right)^{1+\beta/2}. \tag{1.8}$$

Then one solves the following ODE in time,

$$\frac{d}{dt}\log\left(\frac{\sigma}{t\dot{\sigma}}\right) = \frac{\theta(\gamma t)}{t},\ \sigma(0) = 0,\ \dot{\sigma}(0) = 1,$$

where $\gamma > 0$ and $0 \leq \gamma t \leq 1$. It turns out that the solution σ is such that $\sigma \sim t$.

Also let $G = \frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}}$ and $F_p((x,t),s) = \int_0^s f_p((x,t),s)$.

The main Carleman estimate

The main Carleman estimate

Theorem

Let $u \in C_0^\infty(B_2 \times (0, \frac{1}{2\gamma}))$ *be a solution to*

$$\operatorname{div}(A\nabla u) + \partial_t u + f((x,t),u) = g \tag{1.9}$$

where $A(0,0) = \mathbb{I}$*. Then there are universal constants* $\delta_0, c_0, N_0 > 0$ *and* $\tilde{C}$ *such that for* $\alpha \geq \tilde{C}$ *and* $\delta \leq \delta_0$*, the following inequality holds with* $\gamma = \frac{\alpha}{\delta^2}$*,*

$$\alpha \int_{\mathbb{R}^{n+1}_+} \sigma^{-\alpha} \frac{\theta(\gamma t)}{t} |u|^2 G dX + \int_{\mathbb{R}^{n+1}_+} \sigma^{1-\alpha} \frac{\theta(\gamma t)}{t} |\nabla u|^2 G dX \tag{1.10}$$

$$+ c_0 \alpha \int_{\mathbb{R}^{n+1}_+} \sigma^{-\alpha} F(X,u) G dX$$

$$\leq N_0 \int_{\mathbb{R}^{n+1}_+} \sigma^{1-\alpha} |g|^2 G dX + e^{N_0 \alpha} \gamma^{\alpha + N_0} \int_{\mathbb{R}^{n+1}_+} (u^2 + t|\nabla u|^2 + F(X,u)) dX.$$

Contd.

Step1: Without loss of generality, one can assume that $(x_0, t_0) = (0, 0)$. We also assume first that u vanishes to infinite order in both space and time. Then by applying the Carleman estimate to truncated u combined with regularity estimates for the sublinear PDE, we conclude that $u(\cdot, 0) \equiv 0$.

Contd.

Step1: Without loss of generality, one can assume that $(x_0, t_0) = (0, 0)$. We also assume first that u vanishes to infinite order in both space and time. Then by applying the Carleman estimate to truncated u combined with regularity estimates for the sublinear PDE, we conclude that $u(\cdot, 0) \equiv 0$.

Step 2: Vanishing to infinite order in space $\implies$ vanishing to infinite order in space and time is shown by means of "shifted in time" version of our main Carleman estimate. This idea goes back to a work of Fernandez where using this, an equivalence between the two notions of vanishing was established for linear parabolic equations. An alternate approach in the linear case due to Alessandrini and Vessella is based on using the local asymptotics of solutions. Such an approach however is not quite suitable to our sublinear situation because of different scaling properties of the PDE.

Strong Backward uniqueness

Strong Backward uniqueness

We study backward uniqueness for

$$\mathrm{div}(A(x,t)\nabla u) + u_t + Vu + f_p((x,t),u) = 0 \text{ in } R^n \times (-1,0].$$

Strong Backward uniqueness

We study backward uniqueness for

$$\operatorname{div}(A(x,t)\nabla u) + u_t + Vu + f_p((x,t),u) = 0 \text{ in } R^n \times (-1,0].$$

Again by change of variable $t \to -t$, we instead consider solutions u to the following backward parabolic sublinear equation

$$\operatorname{div}(A(x,t)\nabla u) + u_t + Vu + f_p((x,t),u) = 0 \qquad \text{in } R^n \times [0,1]. \quad (1.11)$$

Strong Backward uniqueness

We study backward uniqueness for

$$\text{div}(A(x,t)\nabla u) + u_t + Vu + f_p((x,t),u) = 0 \text{ in } R^n \times (-1,0].$$

Again by change of variable $t \to -t$, we instead consider solutions u to the following backward parabolic sublinear equation

$$\text{div}(A(x,t)\nabla u) + u_t + Vu + f_p((x,t),u) = 0 \qquad \text{in } R^n \times [0,1]. \quad (1.11)$$

Similar to the linear case as in the work of Wu-Zhang, we assume that A satisfies,

$$|\nabla_x A(x,t)| \leq \frac{K}{1+|x|}, \; |\partial_t A(x,t)| \leq K.$$

In the case when $1 < p < 2$, we prove the following.

In the case when $1 < p < 2$, we prove the following.

Theorem (Arya-B(2020))

i) Assume $p \in (1,2)$ and u a solution to (1.11) satisfies the following Tychonoff type growth assumption

$$|u(x,t)| \leq N e^{N|x|^2}$$

for some $N > 0$. Now if u vanishes to infinite order in space at $(0,0)$, then $u \equiv 0$.

Carleman Estimate

Theorem

Let $u \in C_0^\infty(\mathbb{R}^n \times (0,T))$ *be a solution of*

$$\operatorname{div} A(x,t)\nabla u) + u_t + f_p((x,t),u) + Wu = g.$$

Then the following estimate holds with $G = e^{2\gamma(t^{-K}-1) - \frac{b\langle x\rangle^2 + K}{t}}$ *for some universal* $C > 0$,

$$K \int (u^2 + |\nabla u|^2) G dxdt \;+\; \gamma K \int \frac{|u|^p G}{t^{K+1}} dxdt \;\leq\; C \left(\int |u|^p e^{-2\gamma - \frac{\frac{b}{2}\langle x\rangle^2 + K}{t}} dxdt + \int g^2 G dxdt \right)$$

where K, γ *are large enough depending only on* $n, \lambda, \Lambda, p, M, T$.

Carleman Estimate

Theorem

Let $u \in C_0^\infty(\mathbb{R}^n \times (0,T))$ *be a solution of*

$$\operatorname{div} A(x,t)\nabla u) + u_t + f_p((x,t),u) + Wu = g.$$

Then the following estimate holds with $G = e^{2\gamma(t^{-K}-1) - \frac{b\langle x\rangle^2 + K}{t}}$ *for some universal* $C > 0$*,*

$$K\int (u^2 + |\nabla u|^2) G dxdt \;+\; \gamma K \int \frac{|u|^p G}{t^{K+1}} dxdt$$
$$\leq \; C\left(\int |u|^p e^{-2\gamma - \frac{\frac{b}{2}\langle x\rangle^2 + K}{t}} dxdt + \int g^2 G dxdt\right)$$

where K, γ *are large enough depending only on* $n, \lambda, \Lambda, p, M, T$*.*

- Note here G is different from Gaussian and $b = \frac{1}{8\Lambda}$.

Sketch Of the Proof

- First we obtain the above Carleman estimate, which is a generalization of a Carleman estimate by Wu and Zhang (2017) to the sublinear case.

Sketch Of the Proof

- First we obtain the above Carleman estimate, which is a generalization of a Carleman estimate by Wu and Zhang (2017) to the sublinear case.
- From my work with Manna, we have that $u(x, 0) = 0$ for all $x \in \mathbb{R}^n$ and also that u vanishes to infinite order in time.

Sketch Of the Proof

- First we obtain the above Carleman estimate, which is a generalization of a Carleman estimate by Wu and Zhang (2017) to the sublinear case.
- From my work with Manna, we have that $u(x,0) = 0$ for all $x \in \mathbb{R}^n$ and also that u vanishes to infinite order in time.
- Now we proceed like Wu and Zhang (2017).
  For $t < 0$, extend u by 0, the principal coefficients a^{ij} by $a^{ij}(x,0)$ and W by 0 and note that u is solution in the extended region.

Sketch Of the Proof

- First we obtain the above Carleman estimate, which is a generalization of a Carleman estimate by Wu and Zhang (2017) to the sublinear case.
- From my work with Manna, we have that $u(x,0) = 0$ for all $x \in \mathbb{R}^n$ and also that u vanishes to infinite order in time.
- Now we proceed like Wu and Zhang (2017).
  For $t < 0$, extend u by 0, the principal coefficients a^{ij} by $a^{ij}(x,0)$ and W by 0 and note that u is solution in the extended region. Define

$$v(x,t) = u(rx, r^2(t - 1/2))$$

for r sufficiently small, we can ensure that

$$|v(x,t)| \leq Ce^{\frac{b}{8}|x|^2}.$$

Note that v vanishes to infinite order at $t = 1/2$.

Sketch of Proof Contd.

- Define smooth function η as following

$$\begin{cases} \eta(t) \equiv 1 \text{ for } t < 3/4 \\ \eta(t) \equiv 0 \text{ for } t > 7/8 \end{cases}$$

Sketch of Proof Contd.

- Define smooth function η as following

$$\begin{cases} \eta(t) \equiv 1 \text{ for } t < 3/4 \\ \eta(t) \equiv 0 \text{ for } t > 7/8 \end{cases}$$

- Using cut-offs in space (thanks to Tychonoff type growth assumption), we can put ηv in Carleman estimate.

Sketch of Proof Contd.

- Define smooth function η as following

$$\begin{cases} \eta(t) \equiv 1 \text{ for } t < 3/4 \\ \eta(t) \equiv 0 \text{ for } t > 7/8 \end{cases}$$

- Using cut-offs in space (thanks to Tychonoff type growth assumption), we can put ηv in Carleman estimate.
- Using the fact that v is solution and definition of η for $l \in (1/2, 3/4)$ we have

$$e^{2\gamma(l^{-K}-1)} \int_{\frac{1}{2} \leq t \leq l} (v^2 + |\nabla v|^2) e^{-\frac{b\langle x\rangle^2 + K}{t}} dxdt$$

$$\leq C\left(e^{-2\gamma} \int_{\frac{1}{2} \leq t \leq 1} \frac{|\eta v|^p}{t} e^{\frac{-\frac{b}{2}\langle x\rangle^2 + K}{t}} dxdt \right.$$

$$\left. + e^{2\gamma((\frac{3}{4})^{-K}-1)} \int_{\frac{3}{4} \leq t \leq 1} (|v|^2 + |v|^{p-1}) e^{-\frac{b\langle x\rangle^2 + K}{t}} dxdt \right)$$

Sketch of Proof Contd.

- Dividing by $e^{2\gamma(l^{-k}-1)}$ and letting $\gamma \longrightarrow \infty$ we get $v(x,t) = 0$ for $\frac{1}{2} \leq t \leq l$. Now by going back to the original u by scaling back, we obtain that $u(\cdot, t) \equiv 0$ for $0 \leq t \leq t_0$ for some $t_0 > 0$ universal.

Sketch of Proof Contd.

- Dividing by $e^{2\gamma(l^{-k}-1)}$ and letting $\gamma \longrightarrow \infty$ we get $v(x,t) = 0$ for $\frac{1}{2} \leq t \leq l$. Now by going back to the original u by scaling back, we obtain that $u(\cdot,t) \equiv 0$ for $0 \leq t \leq t_0$ for some $t_0 > 0$ universal.
- As t_0 is universal, we can now keep spreading the zero set.

Case when $p = 1$

Theorem

Let u be a solution to the backward parabolic sublinear equation

$$\Delta u + u_t + Vu + f_p((x,t),u) = 0 \qquad \text{in } R^n \times [0,1].$$

where $1 \leq p < 2$ and $||V||_\infty \leq M$.
Assume u is bounded. Now if u vanishes to infinite order in space-time at $(0,0)$, then $u \equiv 0$.

Case when $p = 1$

Theorem

Let u be a solution to the backward parabolic sublinear equation

$$\Delta u + u_t + Vu + f_p((x,t),u) = 0 \qquad \text{in } R^n \times [0,1].$$

where $1 \leq p < 2$ and $||V||_\infty \leq M$.
Assume u is bounded. Now if u vanishes to infinite order in space-time at $(0,0)$, then $u \equiv 0$.

- Our result is also valid for

$$-\Delta u = u_t + \lambda_+(u^+)^{p-1} - \lambda_-(u^-)^{p-1}, \text{ where } \lambda_+, \lambda_- > 0,\ p \in [1,2).$$

Over here, we note that when $p = 1$, $(u^+)^{p-1} = \chi_{\{v>0\}}$ and $(u^-)^{p-1} = \chi_{\{v<0\}}$.

Sketch of Proof

- Such a result is proven by a two parameter Weiss type Monotonicity.

Sketch of Proof

- Such a result is proven by a two parameter Weiss type Monotonicity.
- Following Soave and Terracini (2018) and Poon (1996), we let

$$H(R) = \int_{t=R^2} u^2 G dx$$

$$I(R) = R^2 \int_{t=R^2} |\nabla u|^2 G dx - \frac{2R^2}{p} \int_{t=R^2} |u|^p G dx$$

$$W_\gamma(R) = \frac{I(R)}{R^{2\gamma}} - \frac{\gamma}{2R^{2\gamma}} H(R)$$

where $G = \frac{1}{|t|^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$.

Sketch of Proof

- Such a result is proven by a two parameter Weiss type Monotonicity.
- Following Soave and Terracini (2018) and Poon (1996), we let

$$H(R) = \int_{t=R^2} u^2 G dx$$

$$I(R) = R^2 \int_{t=R^2} |\nabla u|^2 G dx - \frac{2R^2}{p} \int_{t=R^2} |u|^p G dx$$

$$W_\gamma(R) = \frac{I(R)}{R^{2\gamma}} - \frac{\gamma}{2R^{2\gamma}} H(R)$$

where $G = \frac{1}{|t|^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$.

- For γ sufficiently large, depending also on the L^∞ norm of u, we have that

$$W_\gamma'(R) \geq 0 \text{ for a.e. } R \in (0,1).$$

Sketch of Proof Contd.

- Assume on contrary that u is not zero. So, for some $R > 0$, $H(R) \neq 0$. We, then, choose $\gamma > 0$ large enough such that

$$W_\gamma(R) < 0$$

hold. Then from the monotonicity of W_γ, we must have that $W_\gamma(0+) < 0$. However since u vanishes to infinite order at $(0,0)$ in space-time we get $W_\gamma(0+) \geq 0$. This leads to a contradiction and thus finishes the proof of the Theorem.

Further directions

- Can one lower the regularity assumption on the principal part in time for the validity of space like strong unique continuation?

Further directions

- Can one lower the regularity assumption on the principal part in time for the validity of space like strong unique continuation?
- Extend the backward uniqueness result in the case when $p = 1$ to variable coefficients.

Further directions

- Can one lower the regularity assumption on the principal part in time for the validity of space like strong unique continuation?
- Extend the backward uniqueness result in the case when $p = 1$ to variable coefficients.
- Is the vanishing to infinite order in space-time equivalent to vanishing to infinite order in space when $p = 1$?

Further directions

- Can one lower the regularity assumption on the principal part in time for the validity of space like strong unique continuation?
- Extend the backward uniqueness result in the case when $p = 1$ to variable coefficients.
- Is the vanishing to infinite order in space-time equivalent to vanishing to infinite order in space when $p = 1$?
- Regularity of the nodal set.

References

- E. Parini & T. Weth, *Existence, unique continuation and symmetry of least energy nodal solutions to sublinear Neumann problems*, Math. Z, **280** (2015), 707-732.
- N. Soave & T. Weth, *The unique continuation property of sublinear equations*, SIAM J. Math. Anal. **50**(4) (2018), 3919-3938.
- A. Ruland, *Unique Continuation for Sublinear Elliptic Equations Based on Carleman Estimates*, J. Differential Equations **265** (2018) 6009-6035.
- N. Soave & S. Terracini, *The nodal set of solutions to some elliptic problems: sublinear equations, and unstable two-phase membrane problem*, Adv. Math. **334** (2018), 243-299.
- A. Banerjee & R. Manna *Space like strong unique continuation for sublinear parabolic equations.* J. Lond. Math. Soc. (2) **102** (2020), no. 1, 205-228.

References contd.

- A. Banerjee, N. Garofalo & R. Manna *Carleman estimates for Baouendi-Grushin operators with applications to quantitative uniqueness and strong unique continuation*, arXiv:1903.08382, to appear in Applicable Analysis.
- V. Arya & A. Banerjee *Strong backward uniqueness for sublinear parabolic equations.* NoDEA Nonlinear Differential Equations Appl. **27** (2020).
- G. Tortone, *The nodal set of solutions to some nonlocal sublinear problems*, arXiv:2004.04652.
- A. Banerjee & R. Manna *Carleman estimates for a class of variable coefficient degenerate elliptic operators with applications to unique continuation.* arXiv:2011.12624, to appear in Discrete Contin. Dyn. Syst, Series A.

Thank you all for your kind attention