Contextual Coalitional Games

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Abstract. The study of cooperation among agents is of central interest in multi-agent systems research. A popular way to model cooperation is through coalitional game theory. Much research in this area has had limited practical applicability as regards real-world multi-agent systems due to the fact that it assumes *deterministic* payoffs to coalitions and in addition does not apply to multi-agent environments that are *stochastic* in nature. In this paper, we propose a novel approach to modeling such scenarios where coalitional games will be contextualized through the use of logical expressions representing environmental and other state, and probability distributions will be placed on the space of contexts in order to model the stochastic nature of the scenarios. More formally, we present a formal representation language for representing contextualized coalitional games embedded in stochastic environments and we define and show how to compute *expected Shapley values* in such games in a computationally efficient manner. We present the value of the approach through an example involving robotics assistance in emergencies.

1 Introduction

The study of cooperation among agents is of central interest in multi-agent systems research. The reason for this is that more often than not, agents working together perform tasks more efficiently than agents that do not. Although our intuitions tell us this is so, formal models provide a basis for actually proving when and when not this is the case, in addition to providing a basis for efficient implementation of cooperative multi-agent systems.

A popular way to model cooperation is through coalitional games. The key questions in coalitional game theory are related to division of payoff from cooperation so that stability and/or fairness are achieved. Although many of these issues have already been extensively studied in the AI/MAS context [12], most of the research has had limited practical applicability as regards real-world multi-agent systems. The reasons for this, from the modeling point of view, are twofold. Firstly, most work to date assumes deterministic payoffs to coalitions, which is clearly not achievable in many multi-agent systems which are embedded in stochastic environments. Secondly, although some recent work in AI has proposed game models that account for uncertainty, these new developments are highly theoretical and do not take into account computational issues. Specifically, from the computational point of view, the paramount question is how to *concisely* represent a coalitional game when the number of potential coalitions is exponential, precisely $2^n - 1$ where n is the number of agents. Important measures used to asses representations are: *expressiveness*, i.e., does it allow representation of a broad class of games, and *efficiency*, i.e., does it allow for efficient computing of solutions to games from a considered class.

As a motivating example of a real-world multi-agent system, consider a scenario in the emergency services application domain where, as support for rescue missions, one wants to hire configurations of autonomous ground robots (UGVs) and Unmanned Aerial Vehicles (UAVs) from a number of suppliers. Each of the robots, may take on different roles, based on particular sensor capability. In addition, operational efficiency may be affected by particular environmental characteristics which in turn influence the payoffs to coalitions in a contextual manner. Since future environmental characteristics are unknown, the contexts are stochastic in nature.

For instance, in cases where there is wind and rain in the catastrophe areas, it may be the case that only one type of UAV can be used. When the wind is very strong such as during a typhoon, UAVs are useless and one has to depend more on the use of UGVs or other types of vehicles. Use of different configurations of UGVs and UAVs contribute to different hiring costs and differences in resulting quality of usage. One also has to pay a certain fixed fee for keeping equipment ready for immediate use. In cases such as this, one of the main issues is how to distribute the total budget among different suppliers, when a long term contract is being negotiated and many such missions are expected to be carried out under various circumstances. The key factors involved here are dynamic coalition formation, dynamic contexts in which coalitions form, and the stochastic nature in which these contexts occur in the long run.

The research topic is to develop general and computationally efficient frameworks to be able to model such scenarios. Although such scenarios can be modeled to some extent by a number of existing theoretical frameworks that account for uncertainty [8], even for a relatively small number of agents and states in the environment, these models become impractical from a computational point of view and perhaps even a modeling point of view. Thus simplicity is one of our important goals.

Therefore, in this paper we propose a novel approach to modeling such scenarios where coalitional games will be contextualized through the use of logical expressions representing environmental and other states. Probability distributions are placed on the space of contexts in order to model the stochastic nature of the scenarios. More specifically:

- we define contextual coalitional games embedded in stochastic environments and show how to efficiently translate contextual coalitional games into linear combinations of traditional coalitional games;
- we propose a family of formalisms for representing contextualized coalitional games, where each specific formalism is obtained from the general pattern instantiated by fixing a specific representation of traditional coalitional games and a specific logic;

- we define and show how to compute *expected Shapley values* in such games in a computationally efficient manner; and
- we instantiate our general representation and exemplify its use by modeling the informal scenario involving UGVs and UAVs.

The paper is structured as follows. The first section contains notation and preliminary definitions. In the second section we introduce and discuss contextual coalitional games and the Shapley value for such games. In the third section we introduce and motivate our representational viewpoint as well as demonstrate its flexibility and conciseness. We also show that the new representation differs in complexity from conventional games by a factor of the number of states. Next, we instantiate our general definition to Marginal Contribution Nets [7]. Then, we consider computational aspects related to our representation and discuss related work.

2 Preliminaries

A game-theoretical convention for modeling coalitional games is a characteristic function game (CFG) representation. In this approach values of all non-empty coalitions are explicitly listed. Formally, a *coalitional game* is described by a tuple $\mathcal{G} = \langle A, v \rangle$, where $A = \{a_1, \ldots, a_n\}$, is a set of $n = |A|$ agents, and a function $v : 2^A \longrightarrow \Re$ maps any coalition, i.e., a set of agents, to a real value, where it is assumed that $v(\emptyset) = 0$. The coalition of all the agents in the game is called the *grand coalition*.

Example 2.1 (Characteristic function). For $A = \{a_1, a_2, a_3\}$, a sample characteristic function is:

$$
\begin{array}{lll}
v(\{a_1\}) = 0 & v(\{a_2\}) = 0 & v(\{a_3\}) = 1 \\
v(\{a_1, a_2\}) = 1 & v(\{a_1, a_3\}) = 1 & v(\{a_2, a_3\}) = 1 \\
v(\{a_1, a_2, a_3\}) = 2. & &
\end{array}
\qquad \triangleleft
$$

The majority of the best-known solution concepts used with coalitional games have been developed building upon the above CFG representation. Arguably the most famous normative solution concept is the Shapley value. Assuming that the grand coalition is optimal and eventually will form, the Shapley value shows what is the fair division of payoff between agents.[4] Any agent is reimbursed, not only for its performance in the grand coalition, but for its potential marginal contribution to every other coalition. It is assumed that agents join the coalitions in random order and thus all permutations of agents are equally likely. More formally, let $\Pi(A)$ be the set of all permutations of agents in A. For $\pi \in \Pi(A)$ denote by $C_\pi(a_i) \stackrel{\text{def}}{=} \{a_j \mid \pi(a_j) < \pi(a_i)\}$, where $\pi(a_j) < \pi(a_i)$ denotes the fact that agent a_j occurs in π before agent a_i. The *Shapley*

[4] The grand coalition is optimal if its value is at least as large as the sum of the values of any partition of agents into smaller coalition. This assumption ensures that it is a rational choice to form the grand coalition, as is required by the Shapley value as well as many other solution concepts. Nevertheless, the formal analysis is meaningful without the assumption.

value of agent a_i in a game $\mathcal{G} = \langle A, v\rangle$, denoted by $\phi_{\mathcal{G}}(a_i)$, is given by the following expression:

$$\phi_{\mathcal{G}}(a_i) = \frac{1}{n!} \sum_{\pi \in \Pi(A)} \big[v(C_\pi(a_i) \cup \{a_i\}) - v(C_\pi(a_i))\big].$$

Example 2.2 (Shapley value). For the game $\mathcal{G} = \langle A, v\rangle$ defined in Example 2.1, the Shapley values of successive agents are $\phi_{\mathcal{G}}(a_1) = \phi_{\mathcal{G}}(a_2) = \frac{1}{2}$ and $\phi_{\mathcal{G}}(a_3) = 1$. $\lhd$

The importance of the Shapley value comes from the fact that it is the only payoff division scheme that satisfies the following natural "fairness" axioms:

1. *efficiency*: it fully distributes the total payoff available to the agents:
$$\sum_{a \in A} \phi_{\mathcal{G}}(a) = v(A) \tag{1}$$
2. *symmetry*: if agents a_i and a_j are interchangeable, then they have the same payoff:
$$\begin{array}{l}\text{if, for any } C \subseteq A \setminus \{a_i, a_j\}, \text{ one has } v(C \cup \{a_i\}) = v(C \cup \{a_j\}) \\ \text{then } \phi_{\mathcal{G}}(a_i) = \phi_{\mathcal{G}}(a_j)\end{array} \tag{2}$$
3. *dummy*: if an agent a_i does not contribute to any coalition then its value is 0:
$$\text{if, for any } C \subseteq A \setminus \{a_i\}, \text{ one has } v(C) = v(C \cup \{a_i\}) \text{ then } \phi_{\mathcal{G}}(a_i) = 0 \tag{3}$$
4. *linearity*: for any two coalitional games $\mathcal{G} = \langle A, v\rangle$ and $\mathcal{G}' = \langle A, v'\rangle$:
$$\phi_{a*\mathcal{G}+b*\mathcal{G}'}(a_i) = a * \phi_{\mathcal{G}}(a_i) + b * \phi_{\mathcal{G}'}(a_i) \tag{4}$$
where $a, b \in \Re$ and $a * \mathcal{G} + b * \mathcal{G}' \stackrel{\text{def}}{=} \langle A, a * v + b * v'\rangle$.

If a coalitional game is modeled using a CFG representation, computation of the Shapley value as well as many other solution concepts becomes problematic. This is because the number of feasible coalitions grows exponentially in the number of agents. It means that the size of the input renders the computational insights regarding those solution concepts meaningless for larger n.

We will say that a given representation of coalitional games is *fully expressive* iff it allows to represent a characteristic function of any coalitional game. Clearly, the CFG representation is fully expressive.

3 Formalization of Contextual Coalitional Games

In this section, we formally introduce coalitional games with stochastic contexts and their representations.

Definition 3.1. A *contextual coalitional game* (CCG, in short) is a tuple: $\langle A, \mathcal{S}, \vartheta, \mathcal{P}\rangle$, where:

- A is a set of agents in the game;
- $\mathcal{S} \stackrel{\text{def}}{=} \{\sigma_1, \ldots, \sigma_k\}$ is a finite set of *states of the environment* in which the game is played; it is assumed that, in a given moment, the environment is in exactly one state;
- $\vartheta : \mathcal{S} \times 2^A \longrightarrow \mathfrak{R}$ is a mapping, which associates payoffs to coalitions in states;
- $\mathcal{P} = \{p_\sigma \mid \sigma \in S\}$ is a probability distribution on states, where p_σ denotes the probability that state σ materializes. $\lhd$

As before, we assume the payoff 0 for the empty coalition, i.e., for all $s \in \mathcal{S}$ it holds that $\vartheta(s, \emptyset) = 0$.

Definition 3.2. Let $\mathcal{G} = \langle A, \mathcal{S}, \vartheta, \mathcal{P} \rangle$. The *expected value $v_\mathcal{G}(C)$ of a coalition $C \subseteq A$ in game $\mathcal{G}$* is defined by:[5]

$$v_\mathcal{G}(C) \stackrel{\text{def}}{=} \sum_{\sigma \in \mathcal{S}} p_\sigma * \vartheta(\sigma, C). \qquad \lhd$$

The following example illustrates the idea of CCGs.

Example 3.3. A sample contextual coalitional game $\mathcal{G}$ can be given by setting $A = \{a_1, a_2\}$, $\mathcal{S} = \{\sigma_1, \sigma_2\}$, $p_{\sigma_1} = 0.4$ and $p_{\sigma_2} = 0.6$ and

$$\begin{array}{lll} \vartheta(\sigma_1, \{a_1\}) = 2 & \vartheta(\sigma_1, \{a_2\}) = 3 & \vartheta(\sigma_1, \{a_1, a_2\}) = 4 \\ \vartheta(\sigma_2, \{a_1\}) = 2 & \vartheta(\sigma_2, \{a_2\}) = 1 & \vartheta(\sigma_2, \{a_1, a_2\}) = 3. \end{array}$$

Consider coalitional games $\mathcal{G}_1 = \langle A, v_1 \rangle, \mathcal{G}_2 = \langle A, v_2 \rangle$, where:

$$\begin{array}{lll} v_1(\{a_1\}) = 2 & v_1(\{a_2\}) = 3 & v_1(\{a_1, a_2\}) = 4 \\ v_2(\{a_1\}) = 2 & v_2(\{a_2\}) = 1 & v_2(\{a_1, a_2\}) = 3. \end{array}$$

The intuition behind the contextual coalitional game $\mathcal{G}$ is that the coalitional game $\mathcal{G}_1$ takes place when the environment is in the state σ_1 (with probability 0.4) and the coalitional game $\mathcal{G}_2$ takes place when the environment is in the state σ_2 (with probability 0.6). $\lhd$

We can generalize this in the following proposition, showing that contextual coalitional games can be represented as linear combinations of traditional coalitional games. This can be proved by a direct application of Definition 3.2.

Proposition 3.4. Let $\mathcal{G} = \langle A, \mathcal{S}, \vartheta, \mathcal{P} \rangle$ be a CCG. Then $\mathcal{G} = \sum_{\sigma \in \mathcal{S}} p_\sigma * \mathcal{G}_\sigma$, where

$$\mathcal{G}_\sigma \stackrel{\text{def}}{=} \langle A, \vartheta_\sigma \rangle \text{ with } \vartheta_\sigma \stackrel{\text{def}}{=} \vartheta(\sigma, C). \qquad \lhd$$

We then have the following definition of the expected Shapley value for CCGs.

Definition 3.5. Let $\mathcal{G} = \langle A, \mathcal{S}, \vartheta, \mathcal{P} \rangle$ be a CCG. Then *the expected Shapley value for* $\mathcal{G}$ is

$$\Phi_\mathcal{G}(a_i) \stackrel{\text{def}}{=} \Phi_{\sum_{\sigma \in \mathcal{S}} p_\sigma * \mathcal{G}_\sigma}(a_i). \qquad \lhd$$

[5] Throughout the paper we omit the expectation symbol for notational convenience.

4 Representations of Contextual Coalitional Games

A general representation for CCGs considered in this paper is composed of rules of the form:

$$\textit{prerequisite } (\alpha) \mid \textit{coalitional game representation } (\varrho) \tag{5}$$

where the prerequisite α is a formula expressed in some logical language $\mathcal{L}$. We do not fix any particular representation type used for ϱ. CFG is one such conventional game representation type, although in what follows, we will not restrict ourselves to only CFG representations. Intuitively, rule (5) reads as

> "in the states where the prerequisite α is true, the coalitional game is represented by ϱ".[6]

If multiple rules are true at the same time, then coalition values are to be computed additively.

The game consisting of no rules is called the *empty game*. In the empty game the payoff for all coalitions is 0.

This representation is intended to take into account influences or circumstances external to a coalitional game. Such influences are expressed by the "α parts" of rules (5). The formal meaning of α formulas is given by states, where each state materializes with a given probability. Such probability distributions are often given on the basis of statistical data and from other sources (see, e.g., [14]).

Let us now formally define our representation.

Definition 4.1. A *CCG representation* is a tuple $\langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F}\rangle$, where:

- A, $\mathcal{S}$ and $\mathcal{P}$ are as in Definition 3.1;
- $\mathcal{R}$ is a finite set of rules of the form (5) such that for each $(\alpha|\varrho) \in \mathcal{R}$, for the game $\mathcal{G} = \langle A', v\rangle$ that ϱ represents, it holds that $A' \subseteq A$;
- $\mathcal{F} = \{\alpha \mid \text{ there is } (\alpha|\varrho) \in \mathcal{R}\}$, i.e., $\mathcal{F}$ is the set of formulas appearing as prerequisites in rules of $\mathcal{R}$. ◁

Definition 4.2. An *interpretation* of a CCG representation $\langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F}\rangle$ is a tuple $\langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F}, f\rangle$, where:

- A, $\mathcal{S}$, $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{F}$ are as in Definition 4.1;
- $f : \mathcal{F} \longrightarrow 2^{\mathcal{S}}$ is a mapping, which associates to formulas sets of states where they are TRUE. ◁

Remark 4.3. Observe that f appearing in Definition 4.2 should reflect the semantics of a particular logic chosen for expressing prerequisites of rules.

Note also that f provides truth values of formulas in states. Namely, a formula $\alpha \in \mathcal{F}$ is TRUE in a state $\sigma \in \mathcal{S}$ iff $\sigma \in f(\alpha)$. ◁

[6] For notational convenience, we assume that for an instance where the prerequisite α is omitted, this rule should be treated as having α being TRUE.

Representations and their meanings are defined as follows.

Definition 4.4. Given a state $\sigma \in \mathcal{S}$ and an interpretation $\mathcal{I} = \langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F}, f \rangle$, the *meaning* of a rule of $\mathcal{R}$ is defined by

$$(\alpha|\varrho)^{\mathcal{I}}_{\sigma} \stackrel{\text{def}}{=} \begin{cases} \varrho \text{ if } \sigma \in f(\alpha) \\ \emptyset \text{ otherwise,} \end{cases} \tag{6}$$

where $\emptyset$ is the game given by a representation consisting of no rules. ◁

Now we define the value of a coalition C and the Shapley value in a state $\sigma \in \mathcal{S}$ as follows.

Definition 4.5. Let $R = \langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F} \rangle$ be a CCG representation with the set of rules $\mathcal{R} = \{\alpha_1|\varrho_1, \ldots, \alpha_m|\varrho_m\}$ and $\mathcal{I} = \langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F}, f \rangle$ be an interpretation of R.

- For $\sigma \in \mathcal{S}$, by the $(\sigma, \mathcal{I})$-*reduct* of R we understand game $\mathcal{G}^{\mathcal{I}}_{\sigma}$ represented by the set of conventional rules $\{(\alpha_i|\varrho_i)^{\mathcal{I}}_{\sigma} \mid 1 \leq i \leq m\}$.
- The *value of a coalition* $C \subseteq A$ *in state* $\sigma \in \mathcal{S}$ under interpretation $\mathcal{I}$, is defined as $v_{\mathcal{G}^{\mathcal{I}}_{\sigma}}(C)$.
- The *Shapley value* for a_i over R, $\mathcal{I}$ and $\sigma \in \mathcal{S}$, denoted as $\phi^{\mathcal{I}}_{R,\sigma}(a_i)$, is defined as the Shapley value $\phi_{\mathcal{G}^{\mathcal{I}}_{\sigma}}(a_i)$. ◁

Definition 4.6. We say that $R = \langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F} \rangle$ *represents* a CCG $\mathcal{G} = \langle A, \mathcal{S}, \vartheta, \mathcal{P} \rangle$ over an interpretation $\mathcal{I} = \langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F}, f \rangle$ provided that for any coalition $C \subseteq A$ and $\sigma \in \mathcal{S}$ we have that $v_{\mathcal{G}^{\mathcal{I}}_{\sigma}}(C) = \vartheta(\sigma, C)$, where $\mathcal{G}^{\mathcal{I}}_{\sigma}$ is the $(\sigma, \mathcal{I})$-*reduct* of R. ◁

We have the following lemma showing that CCGs can be represented as traditional coalitional games.

Lemma 4.7. Let $R = \langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F} \rangle$ represent a CCG $\mathcal{G} = \langle A, \mathcal{S}, \vartheta, \mathcal{P} \rangle$ over an interpretation $\mathcal{I} = \langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F}, f \rangle$. Then:

$$\mathcal{G} = \sum_{\sigma \in \mathcal{S}} p_{\sigma} \mathcal{G}^{\mathcal{I}}_{\sigma}. \tag{7}$$

Proof. According to Definition 3.2, $v_{\mathcal{G}}(C) = \sum_{\sigma \in \mathcal{S}} p_{\sigma} * \vartheta(\sigma, C)$. By Definition 4.5, for any $C \subseteq A$ and $\sigma \in \mathcal{S}$ we have that $\vartheta(\sigma, C) = v_{\mathcal{G}^{\mathcal{I}}_{\sigma}}(C)$. Therefore,

$$v_{\mathcal{G}}(C) = \sum_{\sigma \in \mathcal{S}} p_{\sigma} * v_{\mathcal{G}^{\mathcal{I}}_{\sigma}}(C),$$

which completes the proof. ◁

Similarly, in the broader contextual coalitional context, we compute the Shapley value for players in state $\sigma \in \mathcal{S}$ using the additivity axiom met by the Shapley value.

Having defined the Shapley value for a game in state $\sigma \in \mathcal{S}$, we are now interested in the value for a contextual coalitional game as a whole. In our stochastic environment this value will be a mapping which takes as input a tuple $\langle \mathcal{I}, R, a_i \rangle$, where $\mathcal{I}$ is an interpretation, R is a CCG representation and $a_i \in A$ is an agent, and returns the expected Shapley value of a_i in the game represented by R over $\mathcal{I}$. This value will be denoted by $\Phi_{R,\mathcal{I}}(a_i)$ and formalized as follows.

Definition 4.8. The *expected Shapley value of a contextual coalitional game represented by* $R = \langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F} \rangle$ over an interpretation $\mathcal{I} = \langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F}, f \rangle$ for player $a_i \in A$ is given by:

$$\Phi_{R,\mathcal{I}}(a_i) \stackrel{\text{def}}{=} \sum_{\sigma \in \mathcal{S}} p_\sigma * \phi^{\mathcal{I}}_{R,\sigma}(a_i). \qquad \lhd$$

Our contextual coalitional game representation is intended to reflect games which are repeated over a longer time period in order to make the stochastic nature of the expected Shapley values practically acceptable. For example, rather then considering a single rescue mission relative to the generic scenario described in the introduction, we would consider a time period where there might be many such missions. The equipment/services' suppliers need to have equipment and staff ready on demand, so they have to know in advance whether their income will be satisfactory. It is reasonable to assume that they receive a fixed fee covering fixed costs such as equipment amortization, maintenance, etc., independently of the number of missions actually carried out. For each mission carried out they then receive additional fees covering resources used, e.g., gas, electricity, repairing, etc. In such scenarios we mainly focus on the distribution of fixed fee, which reflects the importance of equipment and services supplied.

Definition 4.9. Let $\mathbb{P} = \langle \mathbb{R}, \mathbb{I} \rangle$, where $\mathbb{R}$ is a set of representations and $\mathbb{I}$ is a set of interpretations. We say that $\mathbb{P}$ is *fully expressive for CCGs* iff for any CCG $\mathcal{G}$ there is $R \in \mathbb{R}$ and $\mathcal{I} \in \mathbb{I}$ such that R represents $\mathcal{G}$ over $\mathcal{I}$. $\lhd$

Remark 4.10. Recall that any rule of the form $\text{TRUE}|\varrho$ represents ϱ itself. Therefore $\mathbb{P} = \langle \mathbb{R}, \mathbb{I} \rangle$ is fully expressive if the representation type used for righthand sides of rules is fully expressive for conventional games. $\lhd$

By Definition 4.8, the expected Shapley value $\Phi_{R,\mathcal{I}}(a_i)$ is given by $\sum_{\sigma \in \mathcal{S}} p_\sigma * \phi^{\mathcal{I}}_{R,\sigma}(a_i)$. An algorithm for computing $\Phi_{R,\mathcal{I}}(a_i)$ directly from this formula provides the following complexity result.

Theorem 4.11. The complexity of computing the expected Shapley value for the CCG representation $R = \langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F} \rangle$ over an interpretation $\mathcal{I} = \langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F}, f \rangle$ is

$$O\left(|\mathcal{S}| \; * \max_{\rho \in \left\{ (\alpha|\varrho)^{\mathcal{I}}_{\sigma} \;\middle|\; (\alpha|\varrho) \in \mathcal{R} \right\}} \{ g(\rho), h(f) \} \right)$$

where $g(\varrho)$ is the complexity of computing the Shapley value for the representation ϱ and $h(f)$ is the complexity of checking whether a given formula is true in a given state. $\lhd$

Due to the linearity of the Shapley value, every basic rule

$$a_{i_1} \wedge \ldots \wedge a_{i_m} \wedge \neg a_{j_1} \wedge \ldots \wedge \neg a_{j_k} \rightarrow Value$$

can be considered as a separate game. The Shapley values for any agent a_{i_u} and a_{j_w} are respectively:

$$\frac{Value}{m\binom{m+k}{k}} \quad \text{and} \quad \frac{-Value}{k\binom{m+k}{m}} \tag{8}$$

Fig. 1. Ieong and Shoham's method for computing Shapley value

5 Contextual Marginal Contribution Nets

The representation described in the previous section is general in the sense that the context α in a rule can denote a formula of a *given logic* and ϱ denotes a *conventional game representation* CFG.

6 Contextual Marginal Contribution Nets

In this section, we will instantiate the general representation in the following manner by choosing propositional logic as the given logic for α and by using basic Marginal Contribution Nets (abbreviated by MC-nets) of [7] for ϱ. The choice of MC-Nets for ϱ is useful due to the computationally efficient manner in which Shapely values can be computed and also due to the fact that the representation is in logical form.

Formally, given a set of agents, an MC-net is defined as a finite set ϱ of rules of the form

$$\mathbf{P} \rightarrow Value$$

where $Value$ is a real number and *pattern* $\mathbf{P}$ is a Boolean expression with agents as atoms. A coalition C of agents is said to *meet the requirements of* (or shortly *meet*) a given $\mathbf{P}$ (denoted by $C \models \mathbf{P}$) if $\mathbf{P}$ evaluates to TRUE when the values of all Boolean variables that correspond to agents in C are set to TRUE, and the values of all Boolean variables that correspond to agents not in C are set to FALSE. The value $v(C)$ is equal to the sum of all values from rules of which the requirements are met by C. More formally,

$$v(C) \;=\; \sum_{\mathbf{P} \rightarrow Value \in \varrho:\; C \models \mathbf{P}} Value$$

Example 6.1 (MC-nets representation for Example 2.1). The coalitional game from Example 2.1 can be represented with only two rules $a_3 \rightarrow 1$ and $a_1 \wedge a_2 \rightarrow 1$. ◁

Such rules have an interesting interpretation, as they show the marginal contribution to all the coalitions agents can form. The advantages of MC-nets are twofold. Firstly, they allow for representing many important classes of games in a number of rules that is polynomial in n. Secondly, they allow for computing the Shapley value in time linear in the number of rules. However, although the definition of MC-nets is quite general, this latter computational result, as discussed in [7], is limited only to patterns which are conjunctions of literals. More formally, patterns taking the form:

$$a_{i_1} \wedge \ldots \wedge a_{i_m} \wedge \neg a_{j_1} \wedge \ldots \wedge \neg a_{j_k} \tag{9}$$

Following [6] we will call them *basic patterns* and the representation *basic MC-nets*.[7] It will be formally denoted $\langle A, \varrho \rangle$ where ϱ is the set of all the rules. Note, that all the patterns in Example 6.1 are, in fact, basic.

MC-nets are fully expressive [7] even when limited to conjunctions of literals. The linear method of computation of the Shapley value from rules of the form shown in (9) is explained in Figure 1.

Let $\mathcal{V}_0 = \{p_0, \ldots, p_l\}$ be a finite *set of propositional variables*. Variables specify atomic properties of a context by means of a mapping:

$$f_0 : \mathcal{V}_0 \longrightarrow 2^{\mathcal{S}} \tag{10}$$

where $f_0(p)$ is the set of states in which p is TRUE.

Propositional formulas over $\mathcal{V}_0$ are built using $\mathcal{V}_0$ and connectives $\neg, \vee, \wedge, \rightarrow, \equiv$. The set of propositional formulas is denoted by $\mathcal{F}_0$. The mapping f_0 is extended to $\mathcal{F}_0$ in the standard way:

$$\begin{array}{ll} f_0(\neg\alpha) & \stackrel{\text{def}}{=} \mathcal{S} - f_0(\alpha) \\ f_0(\alpha \vee \beta) & \stackrel{\text{def}}{=} f_0(\alpha) \cup f_0(\beta) \\ f_0(\alpha \wedge \beta) & \stackrel{\text{def}}{=} f_0(\alpha) \cap f_0(\beta) \\ f_0(\alpha \rightarrow \beta) & \stackrel{\text{def}}{=} f_0(\neg\alpha) \cup f_0(\beta) \\ f_0(\alpha \equiv \beta) & \stackrel{\text{def}}{=} f_0(\alpha \rightarrow \beta) \cap f_0(\beta \rightarrow \alpha). \end{array} \tag{11}$$

Consequently, the rule representation for contextual MC-nets consists of rules of the form:

$$\alpha | \{\mathbf{p} \rightarrow \mathit{Value}\} \tag{12}$$

where α is a propositional formula over the set of propositional variables $\mathcal{V}_0$, $\mathbf{p}$ is a pattern of the form (9) and Value is a real number.

Definition 6.2. By a *contextual MC-net* we understand any finite set of rules of the form (12). $\lhd$

[7] In the rest of the paper we will assume that every pattern has distinct literals, i.e., $|\{i_1, \ldots, i_m, j_1, \ldots, j_k\}| = m + k$. It can be easily seen that if a conjunction of literals cannot be normalized to this form, i.e., $i_u = j_w$ for some u, w, then removing it does not change the represented game.

Definition 6.3. An interpretation of *contextual MC-nets* is a tuple $\langle A, \mathcal{S}, \mathcal{P}, \mathcal{R}, \mathcal{F}, f_0 \rangle$, where

- A is a finite set of agents in the game;
- $\mathcal{S}, \mathcal{P}$ are as in Definition 4.2;
- $\mathcal{R}$ is a finite set of rules of the form (12);
- $\mathcal{F} \subseteq \mathcal{F}_0$ are propositional formulas over $\mathcal{V}_0$, appearing as prerequisites in $\mathcal{R}$;
- f_0 is defined by (10) and (11). $\lhd$

Since MC-nets are fully expressive, we have the following corollary (cf. Remark 4.10).

Corollary 6.4. The representation of contextual MC-nets is fully expressive. $\lhd$

The complexity of computing the Shapley value for MC-nets is PTIME. Therefore, by Theorem 4.11 we have the following corollary.

Corollary 6.5. The complexity of computing the expected Shapley value for contextual MC-nets is in PTIME in the maximum of size of the representation and the number of states. $\lhd$

7 An Example using Contextual MC-nets

In the following example, we will show how contexts and uncertainty associated with contexts can be used to model stochastic contextual coalitional games.

Example 7.1. Using the scenario considered in the introduction, one can assume that there are states providing values for the propositional variables r, w, s standing for *rain*, *moderate wind* and *strong wind*. Assume that for the rescue missions considered, there is a probability distribution on weather conditions.
The CCG modeling our scenario is $\langle A, \mathcal{S}, \vartheta, \mathcal{P} \rangle$, where

- $A = \{uav_1, uav_2, ugv_1, ugv_2\}$;
- $\mathcal{S} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$;
- $\vartheta(\sigma_1, \{uav_1\}) = 6$, $\vartheta(\sigma_1, \{uav_1, uav_2\}) = 13$, etc.;[8]
- $\mathcal{P}$ is provided in Table 1.

The following contextual MC-net rules are used to model our scenario:

$$\mid \{uav_1 \rightarrow 6, uav_2 \rightarrow 7, ugv_1 \rightarrow 3, ugv_2 \rightarrow 2, ugv_1 \wedge ugv_2 \rightarrow 1\} \tag{13}$$
$$r \wedge w \mid \{uav_1 \rightarrow -6, ugv_1 \rightarrow 2, ugv_2 \rightarrow 2\} \tag{14}$$
$$\neg r \wedge s \mid \{uav_1 \rightarrow -6, uav_2 \rightarrow -7, ugv_1 \rightarrow 4, ugv_2 \rightarrow 3.5\} \tag{15}$$

The first rule defines the basic game which applies in all contexts. The values of this game can be amended by other rules if specific weather condition contexts occur.

[8] We avoid here listing values for all $4 * 15 = 60$ state–coalition pairs. The values are actually given by rules (13)–(15).

Table 1. Probability of various weather conditions .

State	Weather	Literals true in the state	Probability
σ_1	rain and moderate wind	$r, w, \neg s$	0.20
σ_2	rain without wind	$r, \neg w, \neg s$	0.10
σ_3	rain with strong wind	$r, \neg w, s$	0.35
σ_4	no rain with strong wind	$\neg r, \neg w, s$	0.35

Specifically, in the case of rain and moderate wind, uav_1 becomes useless and the importance of ground robots, ugv_1, ugv_2 increases. If the wind becomes strong, both UAVs are grounded and the importance of both ground robots increases even more.

Using formulas from Figure 1, it is easy to check that the Shapley values in the game described by rules (13), (14) and (15) are respectively:

$$\begin{array}{llll} \phi_1(uav_1) = 6 & \phi_1(uav_2) = 7 & \phi_1(ugv_1) = 3.5 & \phi_1(ugv_2) = 2.5 \\ \phi_2(uav_1) = -6 & \phi_2(uav_2) = 0 & \phi_2(ugv_1) = 2 & \phi_2(ugv_2) = 2 \\ \phi_3(uav_1) = -6 & \phi_3(uav_2) = -7 & \phi_3(ugv_1) = 4 & \phi_3(ugv_2) = 3.5. \end{array}$$

By referring to prerequisites of the rules and Table 1, one observes that the rule (13) always holds, the rule (14) holds with probability 0.20, whereas the rule (15) holds with probability 0.35. Thus, the expected Shapley values for the entire game are:

$$\phi(uav_1) = 2.7, \phi(uav_2) = 4.55, \phi(ugv_1) = 5.3, \phi(ugv_2) = 4.125.$$

This means that ugv_1 contributes most value to the coalitional game, while uav_1 contributes the least value. $\lhd$

8 Related Work

Two main streams in the literature on coalitional games are relevant to the ideas contained in this paper. Firstly, there is a body of research where uncertainty is modeled probabilistically and secondly, there is a body of research which focuses on concise representations of coalitional games which enhances computational efficiency in their use.

Regarding the modeling of uncertainty in the context of coalitional games, a short but informative literature review is provided in [8]. Important and relevant recent contributions include [13], [9], [1,2,3] and [8] itself. We focus on [8], where Bayesian Coalitional Games are introduced as a tuple of agents, set of possible worlds (i.e., states), common prior over these worlds, each agent's information partition of the worlds, and their preferences over the distribution of payoffs. An information partition is composed of agents' information sets — subsets of worlds that are undistinguishable from the individual agent's point of view, but where the real world actually resides.[9] If the agent-specific elements are added to our model, we will obtain Bayesian Coalitional Games.

[9] For more details about the information partition method for modeling uncertainty in the non-cooperative game area, see [11].

Nevertheless, the crucial difference between our approach and the others is related to the representation of a coalitional game. As all the other approaches build upon the conventional game theoretical method of representing games (i.e., characteristic function), the number of values to be defined is exponential in the number of agents. This prohibits efficient computation of solution concepts even for a moderate number of agents. Using our approach it is possible to represent many games in a polynomial number of rules.

In this respect our work is related to the literature on alternative representations of coalitional games. The aim of this research is to develop representations for coalitional games that are compact, but still allow for the efficient computation of solution concepts such as Shapley value and coalitional game cores [7], [6], [4], or for finding an optimal arrangement of coalitions in a system [10].

For instance, [5,15] give the characteristic function a specific interpretation in terms of combinatorial structures. The advantage of this method is that the representation can always be guaranteed to be succinct. The disadvantage is that the representation is not fully expressive being incapable of expressing the full space of characteristic function game instances. Many of the other papers propose representations that are fully *expressive* but are not always guaranteed to be succinct [7]. Our work falls under this latter class.

9 Conclusions

In this paper, we proposed a representation for coalitional games which takes into account the stochastic nature of real-world multi-agent scenarios and which relaxes the need for a deterministic payoff to coalitions. The representation is based on the idea of contextualizing coalitional games through the use of logical expressions representing environmental and other state and placing probability distributions on the space of contexts in order to model the stochastic nature of the scenarios. The representation is succinct and intuitive and takes advantage of representational features of logic and its relation to probability. Additionally, we define and show how to compute *expected Shapley values* in such games in a computationally efficient manner. We show the value of the approach through a generic example involving robotics assistance in emergencies.

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