Metoda HMM:
Ukryty Model Markowa

Przykład

- Jak oszacować średnią temperaturę w przeszłym okresie?
- Na podstawie pierścieni drzew
- Macierz A: p-wo zmian temperatur z roku na rok
- Macierz B: wpływ temp. na grubość pierścienia
- Stan początkowy: [0.6, 0.4]
- **Problem 1:**
  Dany jest ciąg obserwacji
  O = (S,M,S,L)
  Jaki był najbardziej prawdopodobny ciąg temperatur w tych latach?

$$\begin{array}{c} \\ H \\ C \end{array} \begin{array}{c} \begin{array}{cc} H & C \end{array} \\ \left[\begin{array}{cc} 0.7 & 0.3 \\ 0.4 & 0.6 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ H \\ C \end{array} \begin{array}{c} \begin{array}{ccc} S & M & L \end{array} \\ \left[\begin{array}{ccc} 0.1 & 0.4 & 0.5 \\ 0.7 & 0.2 & 0.1 \end{array}\right] \end{array}$$

$$\pi = \left[\begin{array}{cc} 0.6 & 0.4 \end{array}\right].$$

Notacje

$$
\begin{aligned}
T &= \text{the length of the observation sequence} \\
N &= \text{the number of states in the model} \\
M &= \text{the number of observation symbols} \\
Q &= \{q_0, q_1, \ldots, q_{N-1}\} = \text{the states of the Markov process} \\
V &= \{0, 1, \ldots, M-1\} = \text{set of possible observations} \\
A &= \text{the state transition probabilities} \\
B &= \text{the observation probability matrix} \\
\pi &= \text{the initial state distribution} \\
\mathcal{O} &= (\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_{T-1}) = \text{observation sequence.}
\end{aligned}
$$

$$\mathcal{O}_i \in V \text{ for } i = 0, 1, \ldots, T-1$$

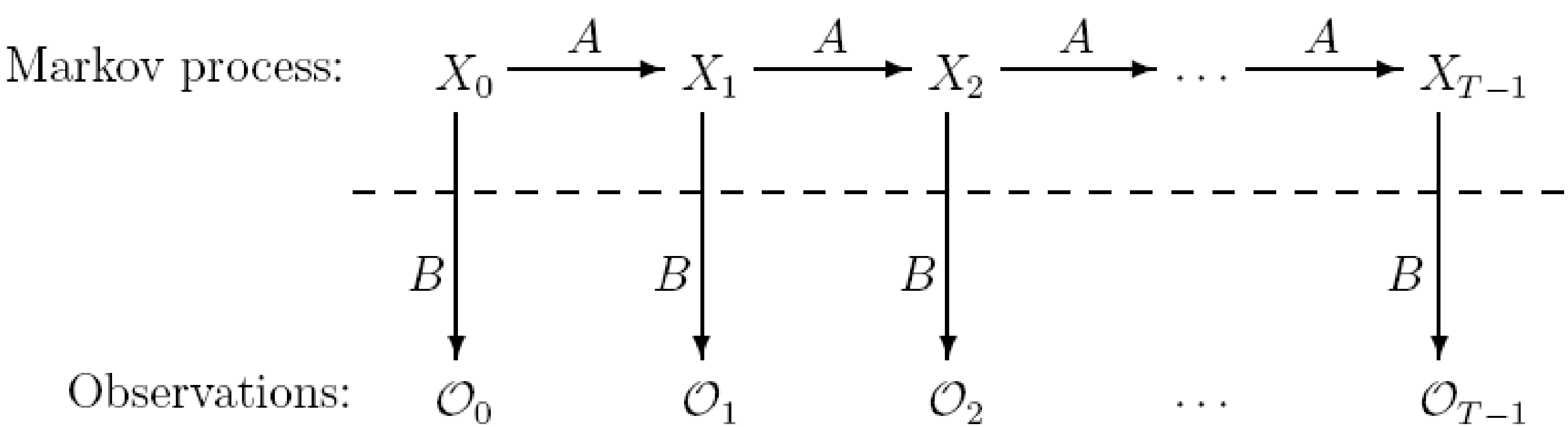
HMM to trójka: $\lambda = (A, B, \pi)$.

The matrix $A = \{a_{ij}\}$ is $N \times N$ with

$$a_{ij} = P(\text{state } q_j \text{ at } t+1 \,|\, \text{state } q_i \text{ at } t)$$

$B = \{b_j(k)\}$ is an $N \times M$ with

$$b_j(k) = P(\text{observation } k \text{ at } t \,|\, \text{state } q_j \text{ at } t).$$



Przykład obliczenia w HMM

- Niech $X = (x_0, x_1, x_2, x_3)$

$$O = (O_0, O_1, O_2, O_3)$$

$$P(X) = \pi_{x_0} b_{x_0}(\mathcal{O}_0) a_{x_0,x_1} b_{x_1}(\mathcal{O}_1) a_{x_1,x_2} b_{x_2}(\mathcal{O}_2) a_{x_2,x_3} b_{x_3}(\mathcal{O}_3).$$

- Np.

$$P(HHCC) = 0.6(0.1)(0.7)(0.4)(0.3)(0.7)(0.6)(0.1) = 0.000212.$$

3 problemy (Rabiner, 1989)

- **Problem 1: Oszacowanie**
  - Dany jest model $\lambda = (A, B, \pi)$ i ciąg obserwacji O;
  - Znaleźć $P(\mathrm{O} \mid \lambda)$
- **Problem 2: Ciąg stanów**
  - Dany jest model $\lambda = (A, B, \pi)$ i ciąg obserwacji O;
  - Znaleźć Q^*: $P(Q^* \mid O, \lambda) = \max_Q P(Q \mid O, \lambda)$
- **Problem 3: Uczenie się**
  - Dany jest ciąg obserwacji O, wymiary M,N;
  - Znaleźć model $\lambda = (A, B, \pi)$ maksymalizując $P(\mathrm{O} \mid \lambda)$

Problem1: Oszacowanie
Metoda 1

$$\begin{aligned} P(\mathcal{O}\,|\,\lambda) &= \sum_X P(\mathcal{O}, X\,|\,\lambda) \\ &= \sum_X P(\mathcal{O}\,|\,X, \lambda) P(X\,|\,\lambda) \\ &= \sum_X \pi_{x_0} b_{x_0}(\mathcal{O}_0) a_{x_0,x_1} b_{x_1}(\mathcal{O}_1) \cdots a_{x_{T-2},x_{T-1}} b_{x_{T-1}}(\mathcal{O}_{T-1}). \end{aligned}$$

- Czasochłonna metoda: $2TN^T$ mnożeń

Problem1: Oszacowanie

Metoda 2: „alpha-pass”

- Dla i = 0, …,T-1 oraz t = 0,…, N-1 definiujemy:

$$\alpha_t(i) = P(\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_t, x_t = q_i \,|\, \lambda). \tag{8}$$

- Liczymy indukcyjnie:

1. Let $\alpha_0(i) = \pi_i b_i(\mathcal{O}_0)$, for $i = 0, 1, \ldots, N-1$

2. For $t = 1, 2, \ldots, T-1$ and $i = 0, 1, \ldots, N-1$, compute

$$\alpha_t(i) = \left[\sum_{j=0}^{N-1} \alpha_{t-1}(j) a_{ji}\right] b_i(\mathcal{O}_t)$$

3. Then from (8) it is clear that

$$P(\mathcal{O} \,|\, \lambda) = \sum_{i=0}^{N-1} \alpha_{T-1}(i).$$

Problem2: Ciąg stanów „beta-pass”

- Dla i = 0, …,T-1 oraz t = 0,…, N-1 definiujemy:

$$\beta_t(i) = P(\mathcal{O}_{t+1}, \mathcal{O}_{t+2}, \ldots, \mathcal{O}_{T-1} \,|\, x_t = q_i, \lambda).$$

- Liczymy indukcyjnie:

1. Let $\beta_{T-1}(i) = 1$, for $i = 0, 1, \ldots, N-1$.
2. For $t = T-2, T-1, \ldots, 0$ and $i = 0, 1, \ldots, N-1$ compute

$$\beta_t(i) = \sum_{j=0}^{N-1} a_{ij} b_j(\mathcal{O}_{t+1}) \beta_{t+1}(j).$$

For $t = 0, 1, \ldots, T-2$ and $i = 0, 1, \ldots, N-1$, define

$$\gamma_t(i) = P(x_t = q_i \,|\, \mathcal{O}, \lambda). = \frac{\alpha_t(i)\beta_t(i)}{P(\mathcal{O} \,|\, \lambda)}$$

Znaleźć ciąg stanów

- P: Czy wystarczy wziąć

$$q_t^* = \arg\max_i \gamma_t(i)$$

jako najlepszy ciąg?

- Odp.: **NIE** (przykład)

$\gamma_t(i)$	0	1	2	3
$P(H)$	0.188170	0.519432	0.228878	0.803979
$P(C)$	0.811830	0.480568	0.771122	0.196021

state	probability	normalized probability
$HHHH$	.000412	.042743
$HHHC$	.000035	.003664
$HHCH$	.000706	.073274
$HHCC$	.000212	.021982
$HCHH$	.000050	.005234
$HCHC$	.000004	.000449
$HCCH$	.000302	.031403
$HCCC$	.000091	.009421
$CHHH$	.001098	.113982
$CHHC$	.000094	.009770
$CHCH$	.001882	.195398
$CHCC$	.000564	.058619
$CCHH$	.000470	.048849
$CCHC$	.000040	.004187
$CCCH$	.002822	.293096
$CCCC$	.000847	.087929

Problem2: ciąg stanów
Algorytm Viterbi'ego

$$\delta_t(i) \equiv \max_{q1q2\cdots qt-1} p(q_1 q_2 \cdots q_{t-1}, q_t = S_i, O_1 \cdots O_t \mid \lambda)$$

- Initialization:

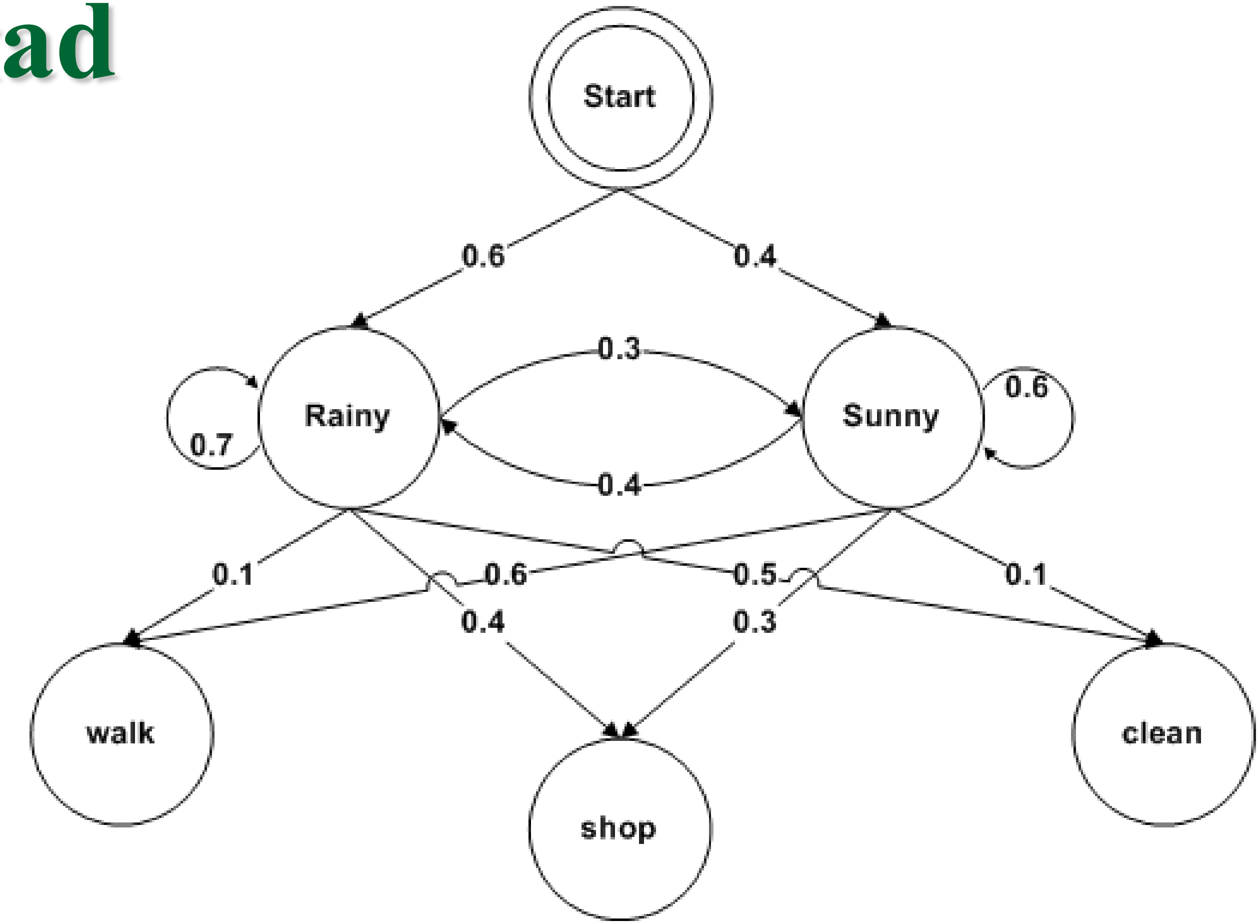
  $\delta_1(i) = \pi_i b_i(O_1),\ \psi_1(i) = 0$
- Recursion:

  $\delta_t(j) = \max_i \delta_{t-1}(i) a_{ij} b_j(O_t),\ \psi_t(j) = \mathrm{argmax}_i\ \delta_{t-1}(i) a_{ij}$
- Termination:

  $p^* = \max_i \delta_T(i),\ q_T^* = \mathrm{argmax}_i\ \delta_T(i)$
- Path backtracking:

  $q_t^* = \psi_{t+1}(q_{t+1}^*),\ t = T-1, T-2, \ldots, 1$

Przykład



Ciąg obserwacji: ['walk', 'shop', 'clean']
Ścieżka Viterbiego: ['Sunny', 'Rainy', 'Rainy', 'Rainy']

Problem 3: „uczenie się"
Algorytm Baum-Welch

- Definiujemy

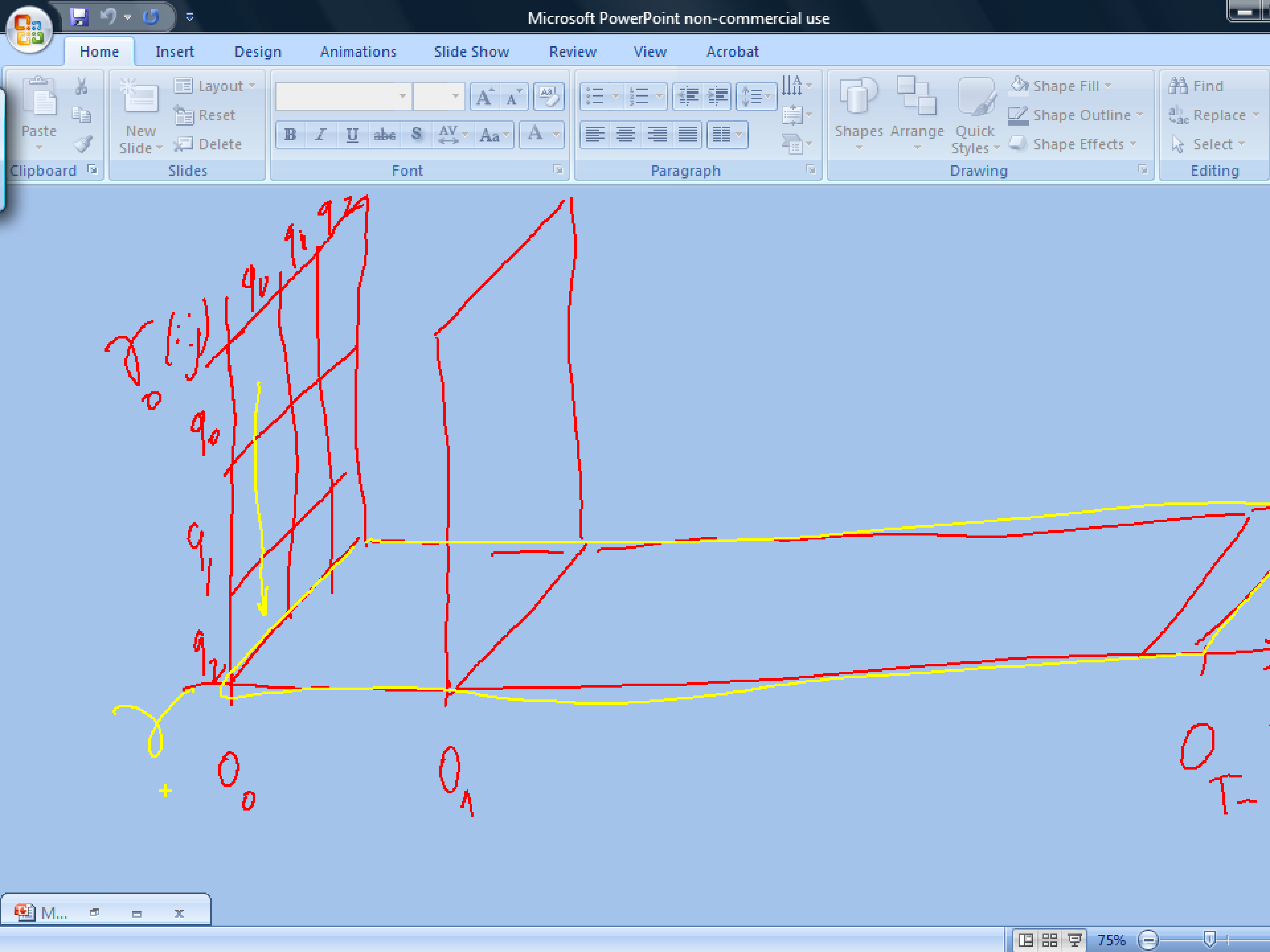
$$\gamma_t(i,j) = \frac{\alpha_t(i) a_{ij} b_j(\mathcal{O}_{t+1}) \beta_{t+1}(j)}{P(\mathcal{O} \mid \lambda)} \equiv P(q_t = S_i, q_{t+1} = S_j \mid O, \lambda)$$

- Wówczas

$$\gamma_t(i) = \sum_{j=0}^{N-1} \gamma_t(i,j).$$

- **Algorytm EM (Baum-Welch)**

1. Initialize, $\lambda = (A, B, \pi)$.
2. Compute $\alpha_t(i)$, $\beta_t(i)$, $\gamma_t(i,j)$ and $\gamma_t(i)$.
3. Re-estimate the model $\lambda = (A, B, \pi)$.
4. If $P(\mathcal{O} \mid \lambda)$ increases, goto 2.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Shape Fill
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Shape Effects
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Algorytm Baum-Welch
- Re-estymacja modelu

For $i = 0, 1, \ldots, N-1$, let

$$\pi_i = \gamma_0(i)$$

For $i = 0, 1, \ldots, N-1$ and $j = 0, 1, \ldots, N-1$, compute

$$a_{ij} = \sum_{t=0}^{T-2} \gamma_t(i,j) \Bigg/ \sum_{t=0}^{T-2} \gamma_t(i)$$

For $j = 0, 1, \ldots, N-1$ and $k = 0, 1, \ldots, M-1$, compute

$$b_j(k) = \sum_{\substack{t \in \{0,1,\ldots,T-2\} \\ \mathcal{O}_t = k}} \gamma_t(j) \Bigg/ \sum_{t=0}^{T-2} \gamma_t(j)$$

Przykład (Cave & Neuwirth)

- HMM:
  - 2 stany
  - 27 symboli (26 i spacja)
  - Obserwacje: 50000 pierwszych liter pewnego tekstu z korpusu Browna.

$$\pi = \begin{bmatrix} 0.51316 & 0.48684 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.47468 & 0.52532 \\ 0.51656 & 0.48344 \end{bmatrix}$$

- Experyment:
  - Początkowo: [1/2,1/2], …
  - 100 iteracji

$$\pi = \begin{bmatrix} 0.00000 & 1.00000 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.25596 & 0.74404 \\ 0.71571 & 0.28429 \end{bmatrix}$$

	Initial		Final	
a	0.03735	0.03909	0.13845	0.00075
b	0.03408	0.03537	0.00000	0.02311
c	0.03455	0.03537	0.00062	0.05614
d	0.03828	0.03909	0.00000	0.06937
e	0.03782	0.03583	0.21404	0.00000
f	0.03922	0.03630	0.00000	0.03559
g	0.03688	0.04048	0.00081	0.02724
h	0.03408	0.03537	0.00066	0.07278
i	0.03875	0.03816	0.12275	0.00000
j	0.04062	0.03909	0.00000	0.00365
k	0.03735	0.03490	0.00182	0.00703
l	0.03968	0.03723	0.00049	0.07231
m	0.03548	0.03537	0.00000	0.03889
n	0.03735	0.03909	0.00000	0.11461
o	0.04062	0.03397	0.13156	0.00000
p	0.03595	0.03397	0.00040	0.03674
q	0.03641	0.03816	0.00000	0.00153
r	0.03408	0.03676	0.00000	0.10225
s	0.04062	0.04048	0.00000	0.11042
t	0.03548	0.03443	0.01102	0.14392
u	0.03922	0.03537	0.04508	0.00000
v	0.04062	0.03955	0.00000	0.01621
w	0.03455	0.03816	0.00000	0.02303
x	0.03595	0.03723	0.00000	0.00447
y	0.03408	0.03769	0.00019	0.02587
z	0.03408	0.03955	0.00000	0.00110
space	0.03688	0.03397	0.33211	0.01298

Eksperyment z 12 stanami

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*	*M*	*N*	*O*	*P*	*Q*	*R*	*S*	*T*	*U*	*V*	*W*	*X*	*Y*	*Z*	*SP*
V	*				*				*						*						*						
SP																											*
C						*	*			*	*																
LL											*													*			
FL	*									*					*		*										
VF																											
VP										*							*										
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6		*	*	*		*	*	*		*		*	*	*		*	*	*	*			*	*		*	*	
7			*	*			*				*	*	*	*		*		*	*	*		*			*		
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9						*					*	*	*	*		*		*	*		*		*	*			
10	*				*				*						*												
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12			*			*	*									*			*				*				

Table 2: Cave and Neuwirth's 12 states result interpretation

Oznaczenia

V	Vowel
SP	Space
C	Consonant
FL	First Letter
LL	Last Letter
VF	Vowel Follower
VP	Vowel Proceeder
CP	Consonant Follower.