Coalgebra morphisms subsume open maps

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Abstract

We relate two abstract notions of bisimulation, induced by open maps and by coalgebras morphisms, respectively. We show that open maps correspond to coalgebra morphisms for a suitable chosen endofunctor in a category of many sorted sets. This demonstrates that the notion of open-maps bisimilarity is of essentially coalgebraic nature. We find moreover characterizations of subcategories of coalgebras corresponding to some relevant model categories: category of presheaves is isomorphic to the subcategory of *consistent* coalgebras with lax cohomomorphisms; a suitably chosen subcategory of many sorted algebras is equivalent to the full subcategory of presheaves preserving finite products.

1 Introduction

We investigate two different ways of generalizing the standard notion of bisimulation [16]. First of them is a categorical formulation of bisimulation by means of *open maps* (open morphisms) [9,10], enabling a uniform definition of bisimulation equivalence across a range of different models for parallel computations. This setting turned out appropriate for defining, among others, strong and weak bisimilarity [16], trace equivalence, testing equivalence and strong history preserving bisimulation of event structures (see [4], [10] for an overview). Moreover, in [11] it was shown that also the observational equivalence of abstract data types can be captured in this framework. Open maps, in general, constitute a subclass of morphisms satisfying certain *path-lifting* property [10] and two objects A and B in a category are bisimilar if they are related by a span of open maps, representing abstractly a bisimulation.

An alternative direction of abstraction, aiming at a generalisation of bisimulation, is to turn to a category of coalgebras, induced by an endofunctor on

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an arbitrary category. Among multiple references on application of coalgebras to model state-based dynamical systems, we mention [1,18]. Coalgebra morphisms can be understood then as functional bisimulations, while their spans (again) give an abstract notion of a bisimulation. This can be done for different endofunctors, and a relevant special case is the category of transition systems with so called zig-zag morphisms, equivalent to category of coalgebras induced by the powerset functor.

A basic motivation for this work was to clarify relationships between both approaches considered different and not shown to be comparable so far. Since the central notion of bisimulation is captured in both cases by spans of appropriate morphisms, the crucial point is to relate open maps and coalgebra morphisms. To this aim, we define an embedding of any category with a fixed class of open maps to an appropriate category of coalgebras with so called lax cohomomorphisms [6]. A relevant observation is that under this embedding open maps correspond to coalgebra morphisms. We also demonstrate that a relation-based generalisation of a bisimulation for open maps, called *path bisimulation* in [10], is an instance of coalgebra bisimulation. These results indicate that the coalgebraic treatment of bisimulation is the most basic one, subsuming even as abstract setting as open maps.

A crucial role in our development is played by categories of presheaves [9], studied recently intensively in a context of open maps, e.g. in [3]. On one side, presheaves are a uniform and simple model for concurrent systems in which open-maps bisimilarity arises in a particularly natural way. On the other hand, we show that category of presheaves is isomorphic to some subcategory of well-behaved coalgebras, called *consistent* in the sequel. This motivates a wider research in applying presheaves as models of computation, since they appear as a central and fundamental construction especially well suited for studying generalized bisimulations.

In general, category of presheaves has more objects than particular models, like transition systems or event structures, and an interesting question arises, which presheaves correspond to concrete models in each case. For strong bisimilarity of labelled transition systems this question was answered already in [10], while for example presheaves corresponding to event structures have no convenient and simple characterization so far – this sketches one of possible directions of further research, see Section 8.

As the first step in realizing this program, we study closer the observational equivalence of many sorted algebras, captured in the framework of open maps in [11]. Algebraic observational (behavioural) equivalence was investigated during last two decades in parallel with bisimulations of different models of concurrent systems, and it is already well widely accepted that algebraic behavioural equivalence is to be understood as a counterpart of bisimulation for abstract data types. It has also an appropriate notion of bisimulation, namely *correspondences* of algebras, proposed by [19]. Some attempts were also made to present algebraic data types explicitly in terms of coalgebras, but this was successful only in a special case when a signature is monadic in non-observable sorts (cf. [15] or [5] in the case of hidden-sorted algebra).

We show that presheaves corresponding to algebras are precisely those preserving finite products. Moreover, we derive a categorical equivalence of a particular subcategory of algebras and of these presheaves, which gives an alternative coalgebraic presentation of algebras (this equivalence is a slight generalization of the Lawvere's characterization of many sorted algebras [12]). At a more intuitive level, this gives another evidence that the algebraic observational equivalence is of essentially coalgebraic nature.

We conclude the paper with construction of an adjunction between a cocomplete model category and a corresponding category of presheaves, where right adjoint, roughly, gives a (coalgebraic) *behaviour* of an object, while left adjoint returns a universal *realisation* of a behaviour. Some more studies should be devoted to a closer investigation of this behaviour-realization adjunction.

2 Bisimulation by open maps

Let $\mathbb{M}$ be a category of models of computation, in which we choose a (not necessarily full) *path subcategory* $\mathbb{P}$. Any morphism $\alpha : P \to A$ from a path object $P \in |\mathbb{P}|$ is understood as an observable *computation* in A following path P. A morphism $h : A \to B$ between models can be intuitively thought of as a simulation of A in B, since h transforms every computation $\alpha : P \to A$ in A to a computation $\alpha; h : P \to B$ in B. Moreover any morphism $m : P \to Q$ in $\mathbb{P}$ making $\alpha = m; \alpha'$ can be understood as an extension of α to a "longer" computation $\alpha' : Q \to A$ (via m).

Definition 2.1 (Open maps [10]) *A morphism $h : A \to B$ in $\mathbb{M}$ is* $\mathbb{P}$-open *if for any $m : P \to Q$ in $\mathbb{P}$ and two computations $\alpha : P \to A$, $\beta : Q \to B$ in $\mathbb{M}$, whenever the square*

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "m"'] & A \arrow[d, "h"] \\ Q \arrow[r, "\beta"'] & B \end{tikzcd}$$

commutes, i.e. $\alpha; h = m; \beta$, there exists a diagonal *morphism $\gamma : Q \to A$ in $\mathbb{M}$ making two induced triangles commutative, i.e. $\alpha = m; \gamma$ and $\beta = \gamma; h$. Two objects A and B are* $\mathbb{P}$-bisimilar, *denoted by $A \sim_{\mathbb{P}} B$, if they are related* *by a pair of* $\mathbb{P}$*-open maps in* $\mathbb{M}$*:*

$$
\begin{tikzcd}
 & C \arrow[dl] \arrow[dr] & \\
A & & B
\end{tikzcd}
$$

with a common domain C *(a* span *of* $\mathbb{P}$*-open maps).*

We omit the prefix $\mathbb{P}$- when it is obvious from a context. The notion of $\mathbb{P}$-bisimilarity generalises the strong bisimilarity between labelled transition systems, which was shown in [10] to coincide with bisimilarity induced by the full subcategory $Bran_L$ of transition systems consisting of finite sequences of actions:

$$
i \xrightarrow{a_1} s_1 \xrightarrow{a_2} \ldots \xrightarrow{a_n} s_n \tag{1}
$$

This was done in the category TS_L, whose objects are labelled transition systems (over a fixed set of labels L) with one distinguished initial state, and morphisms are those transition-preserving functions from states to states mapping an initial state to initial one.

3 Coalgebraic characterisation of open maps

Consider any locally small category $\mathbb{M}$ together with an arbitrary small path subcategory $\mathbb{P}$. We will define an embedding of $\mathbb{M}$ into the category of coalgebras of some endofunctor on the category $Set^{|\mathbb{P}|}$ of $|\mathbb{P}|$-sorted sets ($|\mathbb{P}|$-indexed sets), where $|\mathbb{P}|$ is the set of objects of $\mathbb{P}$. This endofunctor, called $F_{\mathbb{P}}$ in the sequel as it is determined by $\mathbb{P}$, is defined by the following rule:

$$
\{X_P\}_{P\in|\mathbb{P}|} \longmapsto \{\prod_{Q\in|\mathbb{P}|} \mathcal{P}(X_Q)^{Hom_{\mathbb{P}}(P,Q)}\}_{P\in|\mathbb{P}|}
$$

where X_P, for $P \in |\mathbb{P}|$, denotes a component of a $|\mathbb{P}|$-sorted set X, and $Hom_{\mathbb{P}}(P,Q)$ stands for the set of all morphisms from P to Q in $\mathbb{P}$. Intuitively, this functor resembles $\mathcal{P}(\_)^L$, for a set L of labels (whose coalgebras are L-labelled transition systems), with a difference that the set of states of a $F_{\mathbb{P}}$-coalgebra is indexed by objects of $\mathbb{P}$ and transitions are labelled by appropriate morphisms of $\mathbb{P}$. On morphisms, $F_{\mathbb{P}}$ is defined as usual; for $f : X \to Y$ in $Set^{|\mathbb{P}|}$, $(F_{\mathbb{P}}(f))_P = \prod_{Q\in|\mathbb{P}|} h_Q$, where

$$
h_Q : g \longmapsto \lambda m \in Hom_{\mathbb{P}}(P,Q).\{f_Q(x) : x \in g(m)\}.
$$

Let $\mathcal{C}oalg_{\mathbb{P}}$ denote the category of $F_{\mathbb{P}}$-coalgebras, i.e. pairs $\langle S,t\rangle$ with S an object in $Set^{|\mathbb{P}|}$ and $t : S \to F_{\mathbb{P}}(S)$ a morphism in $Set^{|\mathbb{P}|}$, together with coalgebra morphisms from $\langle S,t\rangle$ to $\langle S',t'\rangle$ being those $\mathbb{P}$-sorted functions $f : S \to S'$ satisfying the usual "cohomomorphism" condition, i.e. making the following square commutative:

$$\begin{tikzcd} F_{\mathbb{P}}(S) \arrow[r, "F_{\mathbb{P}}(f)"] & F_{\mathbb{P}}(S') \\ S \arrow[u, "t"] \arrow[r, "f"] & S' \arrow[u, "t'"'] \end{tikzcd}$$

Observe that since we are in the category of $|\mathbb{P}|$-sorted sets, a coalgebra structure $t : S \to F_{\mathbb{P}}(S)$ on a set S consists of a family of functions

$$\{t_P : S_P \to \prod_{Q\in|\mathbb{P}|} \mathcal{P}(X_Q)^{Hom_{\mathbb{P}}(P,Q)}\}_{P\in|\mathbb{P}|}.$$

Moreover, any element x of $\prod_{Q\in|\mathbb{P}|} \mathcal{P}(S_Q)^{Hom_{\mathbb{P}}(P,Q)}$ is also a $|\mathbb{P}|$-sorted function,

$$x = \{x_Q : Hom_{\mathbb{P}}(P,Q) \to \mathcal{P}(S_Q)\}_{Q\in|\mathbb{P}|}.$$

For convenience, in the following we feel free to omit the subscript and write $x(m)$ for a morphism $m : P \to Q$ in $\mathbb{P}$ instead of $x_Q(m)$; this is justified by the canonical bijection:

$$\prod_{Q\in|\mathbb{P}|} \mathcal{P}(S_Q)^{Hom_{\mathbb{P}}(P,Q)} \cong \prod_{m\in\biguplus_{Q\in|\mathbb{P}|} Hom_{\mathbb{P}}(P,Q)} \mathcal{P}(S_{Q=\mathrm{codomain}(m)}).$$

For a coalgebra $\langle S, t : S \to F_{\mathbb{P}}(S)\rangle$, a triple $\langle \alpha, m, \beta\rangle$, where $\alpha \in S_P$, $\beta \in S_Q$ and $m : P \to Q$ in $\mathbb{P}$, satisfying $\beta \in t_P(\alpha)(m)$ will be called a transition and denoted by $\alpha \overset{m}{\to} \beta$.

Turning back to the category $\mathbb{M}$, each of its objects determines a $F_{\mathbb{P}}$-coalgebra by taking as states all its computations and as transitions all commuting triangles induced by a pair of computations. Formally, for X in $\mathbb{M}$, the corresponding $F_{\mathbb{P}}$-coalgebra has a carrier $\{Hom_{\mathbb{M}}(P,X)\}_{P\in|\mathbb{P}|}$ and transitions

$$\alpha \overset{m}{\to} \beta \iff \begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "m"'] & X \\ Q \arrow[ur, "\beta"'] & \end{tikzcd} \text{ commutes in } \mathbb{M}. \tag{2}$$

To extend this mapping to morphisms, we should relax the requirement on coalgebra morphisms. Following [6], a $|\mathbb{P}|$-sorted function $f : S \to S'$ between carriers of coalgebras $C = \langle S, t\rangle$ and $C' = \langle S', t'\rangle$ is called a *lax cohomomorphism* from C to C' if for each $m : P \to Q$ in $\mathbb{P}$ and $s \in S_P$,

$$\{f_Q(r) : r \in t_P(s)(m)\} \subseteq t'_P(f_P(s))(m).$$

Coalgebras together with their lax cohomomorphisms form a category $\mathcal{C}oalg_{\mathbb{P}}^{lax}$, having $\mathcal{C}oalg_{\mathbb{P}}$ as a strict subcategory. Lax cohomomorphisms are like usual morphisms of transition systems, i.e. transition preserving $|\mathbb{P}|$-sorted functions:

Proposition 3.1 *A $\mathbb{P}$-sorted function $f : S \to S'$ is a lax cohomomorphism $C \to C'$ iff for each transition $\alpha \overset{m}{\to} \beta$ in C, $f_P(\alpha) \overset{m}{\to} f_Q(\beta)$ is a transition in C'.*

Moreover, according to the terminology from [6], the lax cohomomorphisms are induced by $F_{\mathbb{P}}$ endowed by a family of preorders $\sqsubseteq_{X,Y} \subseteq Hom_{Set^{|\mathbb{P}|}}(X, F_{\mathbb{P}}(Y)) \times Hom_{Set^{|\mathbb{P}|}}(X, F_{\mathbb{P}}(Y))$ defined by

$$f \sqsubseteq g \iff f_P(x)(m) \subseteq g_P(x)(m), \text{for each } m : P \to Q \text{ in } \mathbb{P} \text{ and } x \in X_P.$$

Now we are ready to introduce a functor $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}} : \mathbb{M} \to \mathcal{C}oalg^{lax}_{\mathbb{P}}$, acting on an object as in (2) and on a morphism by postcomposing it with computations: for $f : X \to Y$ in $\mathbb{M}$,

$$\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(f)_P : \alpha \longmapsto (\alpha; f) : Hom_{\mathbb{M}}(P, X) \to Hom_{\mathbb{M}}(P, Y).$$

As its name suggests, an object of $\mathbb{M}$ is mapped to a coalgebra representing its whole observable behaviour, expressed in terms of computations and path morphisms. This functor will allow us to relate $\mathbb{P}$-bisimilarity of objects of $\mathbb{M}$ with coalgebraic bisimilarity in $\mathcal{C}oalg_{\mathbb{P}}$. While spans of ($\mathbb{P}$-)open maps represent abstractly a bisimulation in the category $\mathbb{M}$, for coalgebras the same role is played by spans of coalgebra morphisms. It turns out that open maps in $\mathbb{M}$ correspond via $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}$ precisely to coalgebra morphisms:

Proposition 3.2 *A morphism $f : X \to Y$ in $\mathbb{M}$ is $\mathbb{P}$-open iff $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(f)$ is a coalgebra morphism.*

Proof First, notice that coalgebra morphisms $f : \langle S, t\rangle \to \langle S', t'\rangle$ are those lax cohomomorphisms satisfying the following zig-zag condition (analogously to the zig-zag morphisms of transition systems): for each $m : P \to Q$ in $\mathbb{P}$ and $\alpha \in S_P$, whenever $f_P(\alpha) \xrightarrow{m} \beta$ in $\langle S', t'\rangle$ for some $\beta \in S'_Q$, there is some $\gamma \in S_Q$ such that $\alpha \xrightarrow{m} \gamma$ in $\langle S, t\rangle$ and $f_Q(\gamma) = \beta$. Now, recalling the definition of $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}$ on objects (see (2)) and morphisms, the zig-zag condition for $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(f)$ can be spelled out equivalently: for each $m : P \to Q$ in $\mathbb{P}$ and $\alpha : P \to X$ in $\mathbb{M}$, whenever the following diagram commutes

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "m"'] & X \arrow[r, "f"] & Y \\ Q \arrow[urr, "\beta"'] & & \end{tikzcd}$$

for some $\beta : Q \to Y$ in $\mathbb{M}$, there exists some $\gamma : Q \to X$ making the following triangles commute:

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "m"'] & X \\ Q \arrow[ur, "\gamma"'] & \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[d, "f"] \\ Q \arrow[ur, "\gamma"] \arrow[r, "\beta"'] & Y \end{tikzcd}$$

This is obviously equivalent to $\mathbb{P}$-openness of $\mathcal{B}eh^{\mathbb{M}}_{\mathbb{P}}(f)$. $\square$

A very general and intuitive conclusion from the above fact is that spans of open maps, intended to represent very abstractly and uniformly bisimulations for different models of computation, are essentially spans of coalgebra morphisms. This justifies an opinion that a coalgebraic bisimulation captures not only the essence of this notion, but is also widely applicable to many different models. We will further expand on this while studying examples in the following sections.

Not all coalgebras are in the image of the functor. Those which do, are well-behaved in a sense that they are *consistent* with composition of path morphisms:

Definition 3.3 *A $F_{\mathbb{P}}$-coalgebra $\langle S, t\rangle$ is* consistent *if*

(i) $\alpha \stackrel{id_P}{\to} \alpha$*, for each* $P \in |\mathbb{P}|$ *and* $\alpha \in S_P$*,*

(ii) whenever $\alpha \stackrel{m}{\to} \beta$ *and* $\beta \stackrel{n}{\to} \gamma$*, there is a compose transition* $\alpha \stackrel{m;n}{\to} \gamma$ *in* $\langle S, t\rangle$*,*

(iii) for each $\beta \in S_Q$ *and* $m : P \to Q$ *in* $\mathbb{P}$*, there is exactly one* $\alpha \in S_P$ *with* $\alpha \stackrel{m}{\to} \beta$*.*

A coalgebra satisfying only (iii) will be called reachable *in the sequel.*

One easily recognises among requirements for consistent coalgebras some categorical aspects, like the existence of identities or definedness of composition. It turns out that the consistent coalgebra $\mathcal{B}eh_{\mathbb{P}}^{\mathbb{M}}(X)$, for an object X in $\mathbb{M}$, is another way of looking at the small *category of* $\mathbb{P}$*-computations* of X, denoted below by $\int_{\mathbb{P}}^{\mathbb{M}} X$. Its objects are all pairs (P, α), where P is an object of $\mathbb{P}$ and $\alpha : P \to X$ in $\mathbb{M}$. Its morphisms $(P, \alpha) \to (Q, \beta)$ are those morphisms $m : P \to Q$ from $\mathbb{P}$ for which $\alpha = m; \beta$. (This category can be equivalently defined as the comma category of the inclusion $\mathbb{P} \hookrightarrow \mathbb{M}$ and the constant functor $1 \longmapsto X : \mathbf{1} \to \mathbb{M}$.) States of the coalgebra $\mathcal{B}eh_{\mathbb{P}}^{\mathbb{M}}(X)$ are precisely objects of $\int_{\mathbb{P}}^{\mathbb{M}} X$, while its transitions correspond to morphisms.

For the constructions presented in this section to work, $\mathbb{P}$ must be small. However, in the following sections we allow $\mathbb{P}$ to be only essentially small, and implicitely assume working with some its skeleton.

4 Examples

As an example illustrating definitions from the previous section, let us consider the category TS_L of transition systems (with initial states) labelled by a fixed set L of labels (with functions which preserve transitions and initial state). A path subcategory appropriate for strong bisimilarity is the full subcategory $Bran_L$ of finite sequences of transitions. This means that a set of states of a F_{Bran_L}-coalgebra is indexed by the set L^* of all finite sequences of actions, and transitions are labelled by all "embeddings" of a sequence as a prefix of a longer one. Moreover, the functor $\mathcal{B}eh_{Bran_L}^{TS_L}$ takes a transition system T to a coalgebra representing its unfolding, whose states correspond to finite sequences of actions in T starting in the initial state.

What about the category ${TS_L}^0$ of transition systems without a distinguished initial state? This category (whose morphisms are all transition-preserving mappings from states to states) is isomorphic to the category of the coalgebras for a powerset functor $\mathcal{P}(\_)^L$ together with lax cohomomorphisms (for the canonical order endowment of the powerset endofunctor). To find out how ${TS_L}^0$ fits into our framework, by Proposition 3.2 we should find a path subcategory inducing as open maps precisely the $\mathcal{P}(\_)^L$-coalgebra homomorphisms, i.e. all functions $f : S \to S'$ satisfying the following zig-zag property [2]: for any $s \in S$, whenever $f(s) \xrightarrow{a} s'$, then $s \xrightarrow{a} r$, for some $r \in S$ satisfying $f(r) = s'$. As such a path subcategory let us take Single_L, the category having as objects a (unique up to isomorphism) one-state transition system with no transitions and all single-transition systems (i.e. those with two states related by one transition); the only morphisms of Single_L, besides identities, are functions mapping a one-state system to the starting state of a single transition:

objects : morphisms :

Now, consider category of coalgebras for the endofunctor F_{Single_L} on $Set^{L \cup \{*\}}$ (the indexing set $L \cup \{*\}$ corresponds to objects of Single_L, with $*$ representing the one-state system). The carrier set of a F_{Single_L}-coalgebra $\langle S, t \rangle$ corresponding to a transition system T contains the set of its states (S_*) and for each $a \in L$, the set of a-transitions (S_a) in T. Moreover, $S_* \ni s \xrightarrow{f_a} \langle s, a, s' \rangle \in S_a$ iff $\langle s, a, s' \rangle$ is a transition in T. Observe that reachability amounts in this case to the requirement that each transition has precisely one starting state. On the other hand, F_{Single_L}-coalgebras have no explicit information about a final state of a transition – this would require introducing into Single_L extra morphisms, mapping the one-state system to the final state of a single transition. But this would change the class of induced open maps and we would loose the property that $\mathcal{P}(\_)^L$-coalgebra maps coincide with Single_L-open morphisms, and thus correspond (via $\mathcal{B}eh^{{TS_L}^0}_{\mathrm{Single}_L}$) precisely to F_{Single_L}-coalgebra maps.

[2] This is slightly different to the analogous property characterizing $Bran_L$-open morphisms between transition systems with initial states, where a zig-zag condition is only required for reachable states s.

5 Presheaves as coalgebras

Category of presheaves, i.e. set-valued functors was proposed as a general and uniform model for concurrency [10,3]. For an arbitrary category $\mathbb{P}$, category $\widehat{\mathbb{P}}$ of presheaves consists of all contravariant functors $\mathbb{P}^{op} \to Set$, together with natural transformations as morphisms between them.

It is suitable to consider presheaves over subcategory $\mathbb{P}$ of paths. For example, category $\widehat{Bran_L}$ corresponds to synchronisation forests, i.e. collections of *synchronisation trees* (unfoldings of transition systems) (cf. [10]). A presheaf F over $\mathbb{P}$ is a particularly simple model of behaviour (wrt. $\mathbb{P}$), since its value $F(P)$ on P, an object from $\mathbb{P}$, corresponds bijectively, by Yoneda Lemma, to the set of computations $P \to F$ (we treat here $\mathbb{P}$ as a subcategory of $\widehat{\mathbb{P}}$, as it embeds fully and faithfully into $\widehat{\mathbb{P}}$ via Yoneda embedding). Moreover, $F(m)$, for a morphism $m : P \to Q$ in $\mathbb{P}$, describes how "longer" computations from Q restrict uniquely to "shorter" ones (via m). The advantage of considering presheaves $\mathbb{P}^{op} \to Set$ instead of concrete models is that many universal constructions like limits or colimits are point-wise, hence like in Set; however, results obtained for presheaves can not always be translated to concrete categories.

For any category $\mathbb{M}$ and a path subcategory $\mathbb{P}$, we can consider a canonical functor $\mathcal{Y}^{\mathbb{M}}_{\mathbb{P}} : \mathbb{M} \to \widehat{\mathbb{P}}$, mapping an object X of $\mathbb{M}$ to the hom-functor $Hom_{\mathbb{M}}(\_, X) : \mathbb{P}^{op} \to Set$. For transition systems we have a pleasing situation: the subcategory of synchronisation trees is equivalent via such a canonical functor to subcategory of *rooted* presheaves, i.e. those assigning to the initial object in $Bran_L$ (or generally in $\mathbb{P}$) a singleton set.

From now on we consider presheaves over an arbitrary fixed small category $\mathbb{P}$. The behavior functor $\mathcal{B}eh^{\widehat{\mathbb{P}}}_{\mathbb{P}}$ maps presheaves to consistent coalgebras; moreover, we prove that every consistent coalgebra corresponds to some presheaf. This could be expected, since a careful examination of the consistency condition leads to the observation that it guarantees a kind of "functorial" properties of transitions, e.g. preservation of composition (ii) and identities (i).

Proposition 5.1 *Category $\widehat{\mathbb{P}}$ is isomorphic via $\mathcal{B}eh^{\widehat{\mathbb{P}}}_{\mathbb{P}}$ to the full subcategory of consistent $F_{\mathbb{P}}$-coalgebras with lax cohomomorphisms, with the inverse functor $\mathcal{I}nv$ given by:*

$$\mathcal{I}nv(\langle S,t\rangle)(P) = S_P,$$
$$\mathcal{I}nv(\langle S,t\rangle)(m : P \to Q)(\beta \in S_Q) = \alpha \in S_P, \quad \text{when } \beta \in t_P(\alpha)(m),$$
$$\mathcal{I}nv(f : \langle S,t\rangle \to \langle S',t'\rangle) = f : \mathcal{I}nv(\langle S,t\rangle) \to \mathcal{I}nv(\langle S',t'\rangle).$$

Proof Note that in the definition above, the action of $\mathcal{I}nv(\langle S,t\rangle)$ on morphisms is determined uniquely due to the condition (iii) in Definition 3.3. Moreover, $\mathcal{I}nv$ gives always a functor, i.e. $\mathcal{I}nv(\langle S,t\rangle)$ preserves composition and identities, which is guaranteed by condition (ii) and (i), respectively. Both compositions of $\mathcal{B}eh_{\mathbb{P}}^{\widehat{\mathbb{P}}}$ and $\mathcal{I}nv$ result in identities, since they change neither carriers of objects nor morphisms and their only relevant action is to recode an internal structure (a behavior) of an object in one of two ways:

- for a presheaf F, a functional restriction of a "longer" computation $Q \to F$ (which is in $F(Q)$ by Yoneda Lemma) via $m : P \to Q$ to a "shorter" computation $P \to F$, represented by $F(m) : F(Q) \to F(P)$,
- for a coalgebra $\langle S, t\rangle$, the set of all extensions of a shorter computation to a longer one via $m : P \to Q$, represented by a coalgebra structure t.

Concerning morphisms, by Proposition 3.1 it follows that the naturality condition for a $|\mathbb{P}|$-sorted function between two presheaves is equivalent to the lax-cohomomorphism condition wrt. the corresponding coalgebra structures.□

This proposition indicates that consistent $F_{\mathbb{P}}$-coalgebras with lax cohomomorphisms are just another formulation of the category of presheaves over $\mathbb{P}$ and condition on a $\mathbb{P}$-sorted function to be a coalgebra morphism is equivalent to $\mathbb{P}$-openness of this function (treated as a natural transformation) in $\widehat{\mathbb{P}}$. Presheaves seem to be better suited since we don't need to impose any consistency requirements on them. On the other hand, the notion of bisimulation (and bisimilarity) for coalgebras arises more naturally with coalgebra morphisms, and we avoid formulating any explicit condition on morphisms to represent a bisimulation, like the property of openness. Common for both approaches is that we start by choosing a path category $\mathbb{P}$, which stands for all possible observations to be performed on objects.

The isomorphism agrees with functors $\mathcal{Y}_{\mathbb{P}}^{\mathbb{M}}$ and $\mathcal{B}eh_{\mathbb{P}}^{\mathbb{M}}$, i.e. $\mathcal{Y}_{\mathbb{P}}^{\mathbb{M}};\mathcal{B}eh_{\mathbb{P}}^{\widehat{\mathbb{P}}} = \mathcal{B}eh_{\mathbb{P}}^{\mathbb{M}}$.

6 Coalgebraic presentation of many sorted algebras

By Proposition 3.2, spans of $\mathbb{P}$-open maps in $\mathbb{M}$ are mapped to spans of coalgebra morphisms, in the image subcategory $\mathcal{B}eh_{\mathbb{P}}^{\mathbb{M}}(\mathbb{M})$, being usually a proper subcategory of $\mathcal{C}oalg_{\mathbb{P}}^{lax}$. To be able to relate both bisimilarities, it is an important issue to characterize this image subcategory, or equivalently to find a necessary and sufficient condition on a consistent coalgebra (or a presheaf) to correspond to some object from $\mathbb{M}$. More formally, we say that a coalgebra (presheaf) C *corresponds* to an object X from $\mathbb{M}$ if $C \cong \mathcal{B}eh_{\mathbb{P}}^{\mathbb{M}}(X)$ ($C \cong \mathcal{Y}_{\mathbb{P}}^{\mathbb{M}}(X)$).

In this section we proceed the first step along the program sketched above, for possibilities of further research see Section 8. We inspect here closer a relationship between category of many sorted algebras and induced category of presheaves. By the results from previous sections, this will immediately lead to a kind of behavioural presentation of algebras as some subclass of consistent coalgebras.

For the rest of this section we fix an algebraic many sorted signature Σ, with a set S of sorts, and a subset $OBS \subseteq S$ of *observable sorts.* By a behavioural (observational) equivalence in an algebra we mean a contextual indistinguishability (on non-observable sorts) induced by all contexts of observable sorts possibly containing variables of observable sorts. Indistinguishability relations are necessarily partial congruences, from which we derive an equivalence of algebras: two algebras are *observationally equivalent* if their quotients by these indistinguishability relations (understood as corresponding fully-abstract models) are isomorphic [7,2]. The following characterization will be helpful in the sequel:

Lemma 6.1 ([2]) *Two algebras A and B are observationally equivalent iff for some set $X \subseteq Set^{OBS}$ of variables of observable sorts and surjective valuations v_A and v_B of X in the carriers of observable sorts of A and B, respectively, the sets of equalities of observable sorts satisfied in A by v_A and in B by v_B coincide: for all terms t, t' of an observable sort with variables from X,*

$$A[v_A] \models t = t' \quad \textit{iff} \quad B[v_B] \models t = t'.$$

A crucial result for our further considerations is that observational equivalence is (re)definable by means of open maps [11]. There are many possibilities for a path subcategory, but the best suited for our purposes is the full subcategory $\mathbb{T}$ of free algebras over finite OBS-sorted sets. The carrier of each such free algebra $T_\Sigma(X)$ is a set of all terms with variables in X, and computations from $T_\Sigma(X)$ correspond to valuations of variables in X; morphisms in $\mathbb{T}$ are given by substitutions.

We know already that Σ-algebras are mapped to their behaviours in $\widehat{\mathbb{T}}$ via $\mathcal{Y}_{\mathbb{T}}^{Alg_\Sigma}$. Observe that an algebra is always mapped to a functor which preserves finite products, i.e. transports a finite coproduct (diagram) in $\mathbb{T}$ to product (diagram) in Set – for instance, the set of computations from $T_\Sigma(X_1 \uplus X_2)$ (i.e. valuations from $X_1 \uplus X_2$), the coproduct of $T_\Sigma(X_1)$ and $T_\Sigma(X_2)$, bijectively corresponds to pairs of valuations from X_1 and X_2. It turns out (Lemma 6.2) that each presheaf preserving finite products corresponds to some Σ-algebra. (In particular, note that each such presheaf is automatically rooted, i.e. maps initial algebra (empty coproduct) in $\mathbb{T}$ to a singleton.) This fact is closely related to the standard Lawvere's representation ([12]) of models of algebraic theories as finite product preserving functors.

Lemma 6.2 *A presheaf over $\mathbb{T}$ corresponds to a Σ-algebra iff it preserves finite products.*

One way of proving this is by exploiting the Lawvere's characterization. Alternatively, a direct proof can be based on the following method of recovering an algebra from a behaviour. For an arbitrary cocomplete $\mathbb{M}$ and any small path subcategory $\mathbb{P}$, consider a category $\int_{\mathbb{P}}^{\widehat{\mathbb{P}}} F$ of computations of a presheaf F (it is also called *the category of elements* of F). F induces an object in $\mathbb{M}$, as a colimit of the diagram given by the projection functor $\pi_{\mathbb{P}} : \int_{\mathbb{P}}^{\widehat{\mathbb{P}}} F \to \mathbb{P}$, taking (P, α) to P:

$$\mathrm{R}_{\mathbb{P}}^{\mathbb{M}}(F) = \mathrm{Colim}(\textstyle\int_{\mathbb{P}}^{\widehat{\mathbb{P}}} F \xrightarrow{\pi_{\mathbb{P}}} \mathbb{P} \hookrightarrow \mathbb{M}). \tag{3}$$

$\mathrm{R}_{\mathbb{P}}^{\mathbb{M}}(F)$ can be intuitively understood as a reconstruction of an object from a "behaviour" F. By universal properties of colimits, this construction can be extended to a functor $\mathrm{R}_{\mathbb{P}}^{\mathbb{M}} : \widehat{\mathbb{P}} \to \mathbb{M}$ (cf. [14], I.5).

Whenever an algebra A satisfies

$$A = \mathrm{R}_{\mathbb{T}}^{Alg_\Sigma}(F) = \mathrm{Colim}(\textstyle\int_{\mathbb{T}}^{Alg_\Sigma} F \xrightarrow{\pi_{\mathbb{T}}} Alg_\Sigma), \text{ for some } F \in \widehat{T}, \tag{4}$$

then F necessarily corresponds to A, which implies $A \cong \mathrm{R}_{\mathbb{T}}^{Alg_\Sigma}(\mathcal{Y}_{\mathbb{T}}^{Alg_\Sigma}(A))$, or equivalently:

$$A = \mathrm{Colim}(\textstyle\int_{\mathbb{T}}^{Alg_\Sigma} A \xrightarrow{\pi_{\mathbb{T}}} Alg_\Sigma). \tag{5}$$

Following an intuition that a presheaf corresponding to an algebra represents its "behaviour", an algebra A satisfying (4) can be seen as a canonical realisation of behaviour F. Along the same lines, (5) means that A is a canonical realisation of its own behaviour. Let us call an algebra A *colimitive* if it satisfies (5); in general not all algebras are colimitive, and the universal morphism $\mathrm{Colim}(\int_{\mathbb{T}}^{Alg_\Sigma} A \xrightarrow{\pi_{\mathbb{T}}} Alg_\Sigma) \xrightarrow{!} A$ need not necessarily be an isomorphism. The following result, being a direct conclusion from Lemma 6.2, gives a coalgebraic presentation of colimitive algebras:

Corollary 6.3 *The following categories are equivalent (via $\mathcal{Y}_{\mathbb{T}}^{Alg_\Sigma}$ and $\mathrm{R}_{\mathbb{T}}^{Alg_\Sigma}$):*

- *the full subcategory of colimitive Σ-algebras,*
- *the full subcategory of presheaves over $\mathbb{T}$, preserving finite products.*

Note that for $OBS = S$, the subcategory of finite product preserving presheaves is equivalent to the whole category Alg_Σ (all algebras are colimitive then) – in this case Lemma 6.2 together with the corollary above are equivalent to the standard Lawvere's result, since $\mathbb{T}$ is dual to the clasifying category of the empty equational theory (cf. [17]). In this light, our construction can be seen as a kind of specialisation of the latter one, in order to deal with observational equivalence induced by a subset of observable sorts.

We cannot do better than in Corollary 6.3, since for any subcategory of Σ-algebras equivalent to a full subcategory of $\widehat{\mathbb{T}}$, if it contains $\mathbb{T}$ as a subcategory, then $\mathbb{T}$ must be its dense subcategory (for details see [13], section X.6). On the other hand, colimitive algebras are not very rare, and each equivalence class of observational equivalence has a colimitive representative. This follows from the fact that a colimitive algebra A can be explicitly characterized as, roughly speaking, a quotient of the algebra freely generated by its observable carriers, by the smallest congruence induced by all observable equalities in A for some surjective valuation as in Lemma 6.1. The colimitive representatives are unique (up to isomorphism), hence observationally equivalent pairs of algebras are those mapped to isomorphic presheaves (their behaviours). [3]

Proposition 6.4 *Σ-algebras A and B are observationally equivalent if and only if $\mathcal{Y}_{\mathbb{T}}^{Alg_\Sigma}(A) \cong \mathcal{Y}_{\mathbb{T}}^{Alg_\Sigma}(B)$.*

This is in accordance with an idea of a correspondence of [19], being an algebraic counterpart of bisimulation: a correspondence can be seen as a proof of an isomorphism between the representatives.

Corollary 6.3 can be opposed to other attempts of a coalgebraic view of many sorted algebras [15] or hidden-sorted algebras [5]. An advantage of our approach is that we need not to impose any specific requirements on Σ, as it was done in [15] and [5], where only signatures *monadic* in observable sorts were allowed.

7 Bisimulations

As usual in theory of coalgebras, by a *bisimulation* we mean a relation represented by a span of coalgebra morphisms [18]. For $F_{\mathbb{P}}$, bisimulations have an explicit formulation, in analogy to strong bisimulations of transition systems:

Definition 7.1 *A* bisimulation *between two coalgebras $\langle S, t\rangle$ and $\langle S', t'\rangle$ is any $\mathbb{P}$-sorted relation $R = \{R_P\}_{P\in|\mathbb{P}|} \subseteq S \times S'$ such that, if $(\alpha, \beta) \in R_P$ and $m : P \to Q$ in $\mathbb{P}$, then*

- *whenever $\alpha \stackrel{m}{\to} \alpha'$, there exists some $\beta' \in S'$ with $\beta \stackrel{m}{\to} \beta'$ and $(\alpha', \beta') \in R_Q$,*
- *whenever $\beta \stackrel{m}{\to} \beta'$, there exists some $\alpha' \in S$ with $\alpha \stackrel{m}{\to} \alpha'$ and $(\alpha', \beta') \in R_Q$,*

One can easily see that each bisimulation, being a $|\mathbb{P}|$-sorted set, has a coalgebra structure (transitions are all $(\alpha, \beta) \stackrel{m}{\to} (\alpha', \beta')$ with $m : P \to Q$, when $(\alpha, \beta) \in R_P$, $(\alpha', \beta') \in R_Q$, $\alpha \stackrel{m}{\to} \alpha'$ and $\beta \stackrel{m}{\to} \beta'$) and together with two projections forms a span of coalgebra maps. We call a bisimulation R consistent, whenever this coalgebra is so.

In [10] a generalisation of bisimulation was studied for an arbitrary category $\mathbb{M}$ (with a path subcategory $\mathbb{P}$), called there a *path bisimulation*. Similarly as there, we assume in this section that $\mathbb{P}$ and $\mathbb{M}$ have a common initial object.

Definition 7.2 ([10]) *A* path bisimulation *wrt. $\mathbb{P}$ between two objects X, Y in $\mathbb{M}$ is a ($|\mathbb{P}|$-sorted) set R of pairs of computations in X and Y, respectively, with a common domain, such that:*

- *R contains the pair of unique computations ι_X, ι_Y from the initial object,*

[3] This is obviously not the case in general, for arbitrary $\mathbb{M}$ and $\mathbb{P}$.

- *for $(\alpha, \beta) \in R$ and a morphism m in $\mathbb{P}$,*
  - · *if $\alpha = m; \alpha'$, there exists some β' satisfying $\beta = m; \beta'$ and $(\alpha', \beta') \in R$.*
  - · *if $\beta = m; \beta'$, there exists some α' satisfying $\alpha = m; \alpha'$ and $(\alpha', \beta') \in R$.*

Intending to clarify a relationship between a path bisimulation and its coalgebraic counterpart, we derive the following equivalence:

Proposition 7.3 *For any two objects X, Y in $\mathbb{M}$, a ($|\mathbb{P}|$-sorted) relation R is a path bisimulation between X and Y iff it is a coalgebraic bisimulation between $\mathcal{B}eh_{\mathbb{P}}^{\mathbb{M}}(X)$ and $\mathcal{B}eh_{\mathbb{P}}^{\mathbb{M}}(Y)$, containing the pair (ι_X, ι_Y).*

Since a path bisimulation is in general too weak to force existence of a span of open maps, in [10] some strengthening of path bisimulation was proposed – it is called *strong* if it further satisfies:

- if $(\alpha, \beta) \in R$, with $\alpha : Q \to X$, $\beta : Q \to Y$ and $m : P \to Q$ in $\mathbb{P}$, then $(m; \alpha, m; \beta) \in R$.

Proposition 7.4 *A path bisimulation R between X and Y, objects in $\mathbb{M}$, is strong iff it is consistent.*

(Equivalently, one could only require reachability, since other two condition from Definition 3.3 are automatically satisfied.) As a corollary, the following sentences are equivelent:

- X and Y in $\mathbb{M}$ are strong path bisimilar,
- $\mathcal{B}eh_{\mathbb{P}}^{\mathbb{M}}(X)$ and $\mathcal{B}eh_{\mathbb{P}}^{\mathbb{M}}(Y)$ are related by a nonempty consistent bisimulation,
- $\mathcal{Y}_{\mathbb{P}}^{\mathbb{M}}(X)$ and $\mathcal{Y}_{\mathbb{P}}^{\mathbb{M}}(X)$ are related by a span of open maps in the category of non-empty (or rooted) presheaves.

For $\mathbb{M}$ being the category of non-empty (or rooted) presheaves we conclude by Propositions 7.4 and 5.1 that strong path bisimilarity and open-maps bisimilarity coincide – a fact proved already in [10].

8 Future research

We found some relationship between bisimulations defined by open-maps and by coalgebra morphisms. A central role in our development is played by categories of presheaves, being at an intermediate position and linking both approaches: they are uniform models to study open maps as well as they correspond to consistent coalgebras, with open maps corresponding precisely to coalgebra morphisms.

The work presented here is not finished yet, and there rests still much to be done. The most important issue is to search for characterizations of subcategories of coalgebras corresponding to specific models, along the lines of Section 6. A possible continuation is to verify to what extent the result can be moved to partial algebras or regular algebras, whose observational equivalences were also captured in the setting of open maps [11]. For partial algebras, it is promising to study presheaves preserving finite limits, over a path subcategory consisting of finite "initial fragments" of total algebras from $\mathbb{T}$. Of particular interest is to find such a characterisation for event structures with strong history preserving bisimulation induced by a path subcategory Pom_L consisting of finite pomsets [10]. It is known that event structures embed fully and faithfully into $\widehat{Pom_L}$, but is seems difficult to find an exact, simple characterization in this case. An interesting issue is whether there exists any general characterization of a subcategory of presheaves $\widehat{\mathbb{P}}$ meaningful as a model of computation, for a suitably chosen path category $\mathbb{P}$.

Another direction of research is motivated by the intuition about a behaviour-realisation relationship, suggested in previous sections. In the case of a path subcategory $\mathbb{P}$ of a cocomplete category $\mathbb{M}$, the functors $\mathrm{R}_{\mathbb{P}}^{\mathbb{M}}$ and $\mathcal{Y}_{\mathbb{P}}^{\mathbb{M}}$ form an adjunction (this is a special case of Theorem 2, p. 41 in [14]):

$$\widehat{\mathbb{P}} \underset{\mathcal{Y}_{\mathbb{P}}^{\mathbb{M}}}{\overset{\mathrm{R}_{\mathbb{P}}^{\mathbb{M}}}{\rightleftarrows}} \mathbb{M}$$

with the right adjoint giving a behaviour of an object from $\mathbb{M}$ while the left adjoint returning a canonical realisation of a behaviour. It is desirable to closer investigate some concrete examples of this general situation as well as to compare it to other adjunction-based approaches to a relationship between a behaviour and its implementation, for instance in [8].

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