UNIVERSITY OF WARSAW

The Atomic Condition in varifold theory

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(work in progress)

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   Φ — measure of the surface

2. The isoperimetric problem

   geometric objects — open sets with fixed volume
   Φ — measure of the boundary

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$$\Phi_F : \mathcal{C} \to \mathbf{R} \cup \{+\infty\}$$

$$\Phi_F(M) = \int_M F(\operatorname{Tan}(M,x))\, \mathrm{d}\mathscr{H}^k(x) \qquad \text{for } M \in \mathcal{C}$$

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$$\Phi_{\mathrm{BH}}(M) = \mathscr{H}^k_\phi(M) \quad \text{for any } \mathscr{C}^1\text{-manifold } M \subseteq \mathbf{R}^n$$

$$\Phi_{\mathrm{HT}} = \text{symplectic measure}$$

Theorem (Almgren, Ann. of Math., 1968)

Let $k \geq 3$ be either a positive integer or ∞. Suppose F is a $\mathscr{C}^k$ integrand (real analytic integrand) which is uniformly elliptic, B is a boundary, $\mathcal{U}$ is an open covering of $\mathbf{R}^n$.
If S is a surface which is F minimal with respect to B and $\mathcal{U}$, then S is $\mathscr{C}^{k-1}$ regular almost everywhere (real analytic almost everywhere).

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$$\mathcal{P}_{\mathrm{rect}} = \left\{ (S, D) : \begin{array}{l} S, D \text{ compact } (\mathscr{H}^k, k)\text{-rectifiable}, \\ D \text{ is a flat disc, } \partial D \text{ is not a retract of } S \end{array} \right\}$$

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$$\mathcal{P}_{\mathbf{Z}} = \left\{ (S, D) : \begin{array}{l} S, D\ k\text{-dimensional integral currents,} \\ \operatorname{spt} D \text{ is a flat disc, } \partial S = \partial D \end{array} \right\}$$

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$$\mathcal{P}_{\mathrm{graph}} = \left\{ (S, D) : \begin{array}{l} D \ k\text{-dimensional flat disc}, S \text{ a graph over } D \\ \text{of some Lipschitz function vanishing on } \partial D \end{array} \right\}$$

Definition

An integrand $f : \operatorname{Hom}(\mathbf{R}^k, \mathbf{R}^{n-k}) \to \mathbf{R}$ is *quasi-convex* if

$$\int_D f(A + \mathrm{D}u(x)) \, \mathrm{d}\mathscr{L}^k(x) \geq \int_D f(A) \, \mathrm{d}\mathscr{L}^k = \mathscr{L}^k(D) f(A)$$

for any domain $D \subseteq \mathbf{R}^k$, $A \in \operatorname{Hom}(\mathbf{R}^k, \mathbf{R}^{n-k})$, and any function $u \in \mathscr{C}^1(\mathbf{R}^k, \mathbf{R}^{n-k})$ supported in D.

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Theorem (Morrey, Pacific J. Math., 1952)

The functional

$$I_f(u, D) = \int_D f(\mathrm{D}u)\,\mathrm{d}\mathscr{L}^k$$

is l.s.c. w.r.t. uniform convergence in $\mathrm{Lip}_L(D, \mathbf{R}^{n-k})$ *for all possible domains* $D \subseteq \mathbf{R}^k$ *iff.* f *is quasi-convex.*

Suppose

$$F : \mathbf{G}(n,k) \to \mathbf{R} \text{ a geometric integrand},$$
$$j : \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^{n-k}) \to \mathrm{Hom}(\mathbf{R}^k, \mathbf{R}^n), \quad j(A)x = (x, Ax)$$

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Then

$$\Phi_F(M) = I_f(u, D)$$

whenever

$$u \in \operatorname{Lip}(\mathbf{R}^k, \mathbf{R}^{n-k}), \quad D \text{ a domain in } \mathbf{R}^k, \quad M = \operatorname{graph} u | D$$

and

$$F \in \operatorname{AE}(\mathcal{P}_{\text{graph}}) \quad \Longrightarrow \quad f \in \operatorname{QC}$$

cf. e.g. De Rosa and Tione, Invent. Math. 2022

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Definition

F_0 is called *extendibly (uniformly/strictly) convex* if there exists a (uniformly/strictly convex) norm Ψ on $\bigwedge_k \mathbf{R}^n$ such that

$$F_0 = \Psi | \mathbf{G}_0(n,k)\,.$$

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Federer's book 1969, §5.1.2

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5. *There exist $F : \mathbf{G}_0(4, 2) \to \mathbf{R}_+ \in \mathrm{semiAE}(\mathcal{P}_{\mathbf{Z}})$ which is not extendibly convex.*
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De Philippis and De Rosa and Ghiraldin, CPAM 2018

$$F \in \mathrm{AC} \quad \Longleftrightarrow \quad \begin{array}{l} \forall \mu \in \operatorname{Prob}(\mathbf{G}(n,k)) \quad \dim \operatorname{im} A_F(\mu) \geq k \\ \text{and if } \dim \operatorname{im} A_F(\mu) = k \text{, then } \mu \text{ is atomic} \end{array}$$

Theorem (De Philippis and De Rosa and Ghiraldin, CPAM 2018)

Let $\mathcal{V}$ be the family of all k-dimensional varifolds in an open set $U \subseteq \mathbf{R}^n$ with positive lower density and total F-variation measure being Radon. Then $\mathcal{V} \subseteq \mathbf{RV}_k(U)$ iff. $F \in \mathrm{AC}$.

In particular, all F-stationary k-varifold with positive density are rectifiable iff. $F \in \mathrm{AC}$.

De Rosa and K., CPAM 2020

$$AC \subseteq AE(\mathcal{P}_{\mathrm{rect}})$$

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$$\mathrm{AC} \subseteq \mathrm{AE}(\mathcal{P}_{\mathrm{rect}})$$

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In case $n - k = 1$, F is extendibly strictly convex iff. $F \in \mathrm{AC}$

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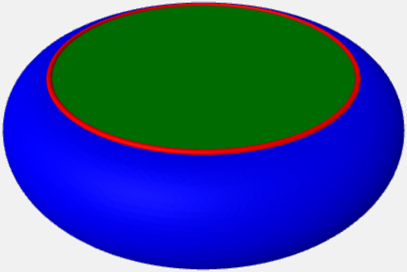

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$$Q_F(T) = \mathrm{id}_{\mathbf{R}^n} - P_F(T)$$

De Rosa and Tione, Invent. Math. 2022

Def. $F \in \mathrm{SAC}$ (*scalar atomic condition*) if

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Note. $F \in \mathrm{SAC}$ iff.

$$\mathcal{G}_F \subseteq H(S) = \mathrm{End}(\mathbf{R}^n) \cap \{A : A \bullet Q_F(S)^\top \geq 0\}$$
$$\text{and} \quad \partial H(S) \cap \mathcal{G}_F = \{P_F(S)\}$$

In particular, $P_F(S)$ is an exposed point of $\operatorname{conv} \mathcal{G}_F$

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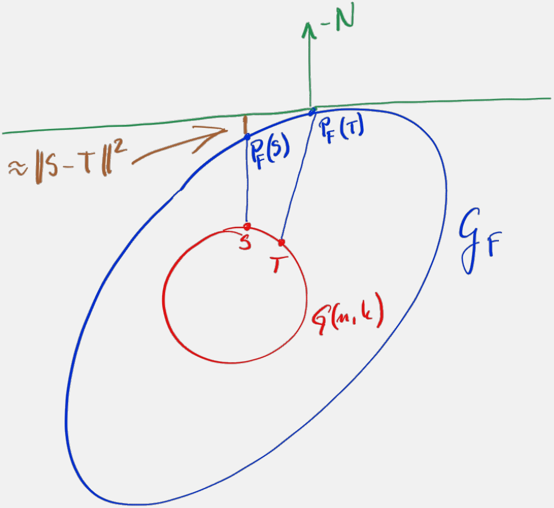
$F \in \mathrm{mUSAC}$ analogously


-N
≈ ‖S − T‖²
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S
T
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G_F

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Note.

$$Q_F(S)^{\top} \in I(S)^{\perp}$$
$$K = \bigcap\{H(N(S)) : S \in \mathbf{G}(n,k)\} = \operatorname{cone} \mathcal{G}_F$$
$$K^{\circ} = \operatorname{cone}\{N(S) : S \in \mathbf{G}(n,k)\}$$

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2. $\mathcal{G}^*$ is the set of extreme points of $\operatorname{conv} \mathcal{G}^*$

Construct two subsets of $\operatorname{End}(\mathbf{R}^n)$

$$\mathcal{G} = \{P(S) : S \in \mathbf{G}(n,k)\} \quad \text{and} \quad \mathcal{G}^* = \{N(S) : S \in \mathbf{G}(n,k)\}$$

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After that

6. Recover F such that $\mathcal{G} = \mathcal{G}_F$

Thank you.