Geometric bordism and cobordism

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Chapter 1

Differential Topology

We'll discuss briefly embeddings of manifolds, tubular neighborhoods and transversality - crucial ideas for establishing a link between topology of manifolds and homotopy theory.

1.1 Smooth manifolds

A topological manifold manifold with boundary is a Hausdorff, space with countable basis and locally homeomorphic to a cartesian half-space $\mathbb{R}^n_+ := \mathbb{R}_+ \times \mathbb{R}^{n-1} = \{(x_1, ..., x_n)|x_1 \geq 0\}$. A chart on a manifold M is an open subset $U \subset M$ equipped with homeomorphism $h\colon U \to \mathbb{R}^n_+$. An atlas on M is a set of maps $\{(U_i, h_i)\}_{i\in I}$ such that $\{U_i\}_{i\in I}$ is a covering of M. The set of points whose images under some chart are in $\mathbb{R}^{n-1}$ constitute a boundary of the manifld, and is denoted ∂M. The space ∂M is also a topological manifold (without boundary points). Closed manifold is compact, connected manifold with no boundary points. When we say "manifold" we usually mean a manifold with boundary points.

A smooth structure on M is a maximal atlas $\{U_i, h_i\}_{i\in I}$ such that all transition functions between charts $h_j h_i^{-1}$ are smooth. A smooth manifold is a topological manifold with fixed maximal smooth atlas. A map between smooth manifolds $f\colon M \to N$ is smooth if compositions with charts (and their inverses) are smooth maps between subsets of cartesian spaces. The space ∂M is also a smooth manifold (without boundary points). Smooth manifolds and smooth maps from a category in which exist direct sums and finite products.

Each smooth manifold is equipped with the vector bundle of tangent vectors (tangent bundle) and derivative of any smooth map $f\colon M \to N$ induces a morphism of the tangent bundles.

$$\begin{tikzcd} TM \arrow[r, "Df"] \arrow[d, "p_M"'] & TN \arrow[d, "p_N"] \\ M \arrow[r, "f"] & N \end{tikzcd} \tag{1.1}$$

The total space of the tangent bundle is a smooth manifold and projection is a submerssion. The derivative Df is a smooth map.

1.2 Tubular Neighborhoods

For a complete treatment of the subject we refer to Lang [10]. Let V, W be manifolds and let $i\colon V \to W$ be an immersion. A differential $Di\colon TV \to TW$ induces then a monomorphism $Di\colon TV \to i^*TW$ over V.

Definition 1.2.1. *The quotient bundle $\nu(i) = i^*TW/TV$ is called a normal bundle to the immersion $i\colon V \to W$.*

If $i\colon V \hookrightarrow W$ is an inclusion of a submanifold then a normal bundle $\nu(i)$ is called a normal bundle of V in W and is denoted by $\nu(V,W)$. Next we define a tubular neighborhood of V in W.

Definition 1.2.2. *A tubular neighborhood of a submanifold $V \subset W$ consists of*

a) a vector bundle $\pi\colon E \to V$ and an open neighborhood Z of its zero-section.

b) a diffeomorphism $h\colon Z \to U$ where U is an open neighborhood of V in W such that diagram:

$$\begin{tikzcd} Z \arrow[rr, "h"] & & U \\ & V \arrow[ul, "s_0"] \arrow[ur, "\subset"'] & \end{tikzcd} \tag{1.2}$$

commutes ($s_0\colon V \to Z$ is the zero-section).

Proposition 1.2.1. *Let $\xi = (p\colon E \to M)$ be a smooth vector bundle over a manifold M. A normal bundle of the zero-section $s_0(M) \hookrightarrow E$ is canonically isomorphic to ξ. If $E \to W$ is a tubular neighborhood of V in W, then the bundle E is isomorphic to the normal bundle $\nu(V,W)$.*

Proof. The first conclusion follows from the exact sequence: $0 \to p^*\xi \to TE \xrightarrow{Dp} p^*TM \to 0$ which has a canonical splitting Ds_0 over the zero-section. The tubular neighborhood is obviously the whole $E(\xi)$ with the identity map. The second follows from the first immediately. $\square$

Theorem 1.2.1. *Let V be a closed submanifold of W. There exists a tubular neighborhood of V in W.*

Sketch of the proof. We choose a Riemannian metric on W. This metric defines the map $\nu(V) \to (TV)^\perp$ and thus the splitting of the sequence: $0 \to TV \to TW|_V \to \nu(V) \to 0$. The Riemannian metric defines also an exponential map: $TW \supset Z_0 \to^{\exp} W$. This exponential map being an identity on the zero-section is thus a diffeomorphism on an open subset $Z \subset Z_0 \cup (TV)^\perp$; $s_0(V) \subset Z$ (here we need the assumption that V is a closed submanifold). $\square$

Proof of Theorem 1.2.1 gives us a canonical method of construction of tubular neighborhood after a Riemannian metric is chosen. It is important for our purposes to have a tubular neighborhood defined over the whole bundle and not only on a neighborhood of its zero-section.

Definition 1.2.3. *A vector bundle $p\colon E \to X$ is* compressible *iff for every open neighborhood Z of the zero-section $s_0(X) \subset E$ there exists an open neighborhood Z_1, $s_0(X) \subset Z_1 \subset Z \subset E$ and a homeomorphism (a diffeomorphism, if the bundle is smooth) $h\colon E \to Z_1$ such that the following diagram commutes:*

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & Z_1 \arrow[dl, "p"] \\ & X & \end{tikzcd} \tag{1.3}$$

Definition 1.2.4. *A smooth map $F\colon X \times \mathbb{R} \to Y$ is an* isotopy *(of embeddings) if for each $t \in \mathbb{R}$ the map $F(-,t)$ is an embedding and $F(-,t) = id$ for $t \notin [0,1]$. We often denote $f_t := F(-,t)$.*

Lemma 1.2.1. *Any vector bundle over a manifold is compressible.*

Proof. See Lang [10;VII.4.]. It is important to notice that in this case the compressions are diffeomorphisms isotopic in E to the identity. $\square$

Thus having chosen a Riemannian metric on the manifold W we have the canonical construction of the total tubular neighborhood:

$$\nu(V,W) \to (TV)^{\perp} \xrightarrow{\text{compression}} Z \xrightarrow{\exp} W$$

We shall see that total tubular neighborhoods are unique up to an isomorphism of vector bundles.

Definition 1.2.5. *Let $p\colon E \to V$ be a vector bundle; $V \subset W$ and $Z \to W$ a tubular neighborhood of V in W. The isotopy $F\colon Z \times \mathbb{R} \to W$ is called an isotopy of tubular neighborhoods iff each map $f_t\colon Z \to W$ is a tubular neighborhood.*

Theorem 1.2.2. *Let $h\colon E \to W$ and $g\colon E' \to W$ be total tubular neighborhoods of V in W. Then there exists a vector bundle isomorphism $\lambda\colon E \to E'$ and an isotopy of tubular neighborhoods $f_t\colon E \to W$ such that $f_0 = h$ and $f_1 = g \circ \lambda$. Moreover, if E and E' are endowed with Riemannian metric, then λ can be chosen to be an isometry.*

Remark 1.2.1. Any two canonical tubular neighborhoods are isotopic. This follows easily from the fact that any two Riemannian metrics are homotopic through Riemannian metrics (thus in this case λ is an identity). We will use this fact later.

ADD: tubular nbhds in manifolds with boundary $(W, \partial W)$**.**

1.3 Embeddings of Manifolds

Throughout these note we'll consider proper maps between topological spaces i.e. maps $f\colon X \to Y$ such that for every compact subset $K \subset Y$ its inverse image $f^{-1}(K)$ is compact. If f is a proper map, then f is closed i.e. images of closed sets are closed. Moreover, a map $f\colon X \to Y$ between locally compact spaces is closed if and only if it is closed and its fibers are compact.

Definition 1.3.1. *A smooth map $f\colon V \to W$ is called a proper embedding if its derivative Df_x is monomorphism for each $x \in V$ and f is an injective, proper map .*

If $f\colon V \to W$ is a proper embedding then $f(V) \subset W$ is a closed submanifold of W. From now on we assume that all embeddings under consideration are proper. We recall Whitney's embedding theorem:

Theorem 1.3.1. *Let $\varepsilon\colon V \to \mathbb{R}$ be a smooth function such that for every $x \in V$, $\varepsilon(x) > 0$. Let $f\colon V \to \mathbb{R}^q$ be a proper map which is an embedding of a neighborhood U of a closed set $A \subset V$. If $2 \cdot \dim V < q$ then there exists $g\colon V \to \mathbb{R}^q$ such that $g|_A = f|_A$, g is an ε-approximation of f, and g is a proper embedding.*

Proof. See Narasimhan [12, 2.15]. $\square$

Remark 1.3.1. $\mathbb{R}^q$ may be replaced with any manifold N with $\dim N > 2 \cdot \dim V$.

Definition 1.3.2. *We will say that two proper embeddings $f_0, f_1\colon V \to W$ are properly isotopic iff there exists an isotopy $F\colon V \times \mathbb{R} \to W$ such that F_t is proper for each t, $F_0 = f_0$ and $F_1 = f_1$.*

Note that the map F doesn't have to be proper! However, a map $F\colon V \times \mathbb{R} \to W$ is a proper isotopy if and only if the map $(F, \mathrm{pr}_2)\colon V \times \mathbb{R} \to W \times \mathbb{R}$, $(F, \mathrm{pr}_2)(\nu, t) = (F(\nu, t), t)$ is a proper embedding. From the Whitney theorem it follows:

Theorem 1.3.2. *If $q > 2 \cdot \dim V + 4$ then any two proper embedding $i_0, i_1\colon V \to \mathbb{R}^q$ are properly isotopic. Any two such isotopies are themselves properly isotopic leaving the endpoints fixed.*

The last theorem enables us to identify the normal bundles to different embeddings of the same manifold.

1.4 Transversality

Definition 1.4.1. *Let $Z \subset X$ be a submanifold and $g\colon Y \to X$ a map of manifolds. We say that the map g is transversal to the submanifold Z on a set $A \subset Y$ iff for every $a \in A$ such that $g(a) \in Z$ the following equality holds:*

$$T_{g(a)}Z + Dg(T_aY) = T_{g(a)}X$$

where $Dg\colon TY \to TX$ denotes the differential of g. If $g\colon Y \to X$ is transversal to Z on Y we say simply that it is transversal to Z.

Definition 1.4.2. *Let $f\colon Z \to X$, $\quad g\colon Y \to X$ be maps. We say that f and g are transversal ($f \pitchfork g$) iff $f \times g\colon Z \times Y \to X \times X$ is transversal to the "diagonal submanifold" $\triangle(X) \hookrightarrow X \times X$.*

Theorem 1.4.1. *If $g\colon Y \to X$ is transversal to $Z \subset X$ and $g^{-1}(Z) \neq \emptyset$ then $g^{-1}(Z)$ is a submanifold and the differential $Dg\colon TY \to TX$ induces a canonical isomorphism of bundles:*

$$\nu(g^{-1}(Z), Y) \xrightarrow{\simeq} g^*\nu(Z, X).$$

Proof. From the implicit function theorem we know that $g^{-1}(Z) \subset Y$ is a submanifold of the same codimension as $Z \subset X$ i.e. $\dim X - \dim Z = \dim Y - \dim g^{-1}(Z)$. Consider the diagram of tangent maps

$$\begin{tikzcd}
T(g^{-}1(Z)) \arrow[r] \arrow[drr] & \ker Dg \arrow[r] \arrow[dr] & TY|g^{-1}(Z) \arrow[r, "Dg"] \arrow[d] & TX|Z \arrow[r] \arrow[d] & \nu(Z,X) \arrow[dl] \\
& & g^{-1}(Z) \arrow[r, "g"] & Z &
\end{tikzcd} \tag{1.4}$$

From the transversality condition the inclusion $T(g^{-}1(Z)) \subset \ker Dg$ must be an isomorphism on each fiber, thus an isomorphism of bundles. Thus

$$\nu(g^{-1}(Z), Y) = (TY|g^{-1}(Z))/T(g^{-1}(Z)) \xrightarrow{\simeq} g^*(TX|Z)/TZ \simeq g^*\nu(Z, X).$$

□

Remark 1.4.1. If maps $f\colon Z \to X$ and $g\colon Y \to X$ are transversal then their pullback $V = (f \times g)^{-1}(\triangle X)$ is a submanifold of $Z \times Y$.

We recall the Thom transversality theorem.

Theorem 1.4.2. *Let $g\colon Y \to X$ be a map which is transversal to $Z \subset X$ on a closed set $A \subset Y$. Let $\varepsilon\colon Y \to \mathbb{R}$, $\quad \varepsilon > 0$ be a smooth function. Then there exists a ε-approximation $g'\colon Y \to X$ such that $g'|_A = g|_A$ and g' is transversal to Z on Y.*

Notice that from this theorem it follows that for any map $g\colon Y \to X$ as above we can find a map $g'\colon Y \to X$ which is homotopic to g relative A and transversal to Z (if δ is small enough then g and g' must be homotopic.) The transversality theorem holds in a slightly more general form (Karoubi [9]).

Theorem 1.4.3. *Let $f\colon Z \to X$ and $g\colon Y \to X$ be maps, and $\delta\colon Y \to \mathbb{R}$, $\quad \delta > 0$ a smooth function. Then there exists a δ-approximation $g'\colon Y \to X$ of g such that $g' \pitchfork f$.*

Chapter 2

Generalized homology and cohomology

2.1 Axioms

Let $\mathcal{T}_*$ denote category of (some) well-pointed spaces, including CW-complexes and $\mathcal{A}$ the category of abelian groups. Let $\Sigma\colon \mathcal{T}_* \to \mathcal{T}_*$ be the reduced suspension functor.

Definition 2.1.1 (Homology Theory)**.** *A sequence of covariant functors $\tilde{h}_n\colon \mathcal{T}_* \to \mathcal{A}$, $n \in \mathbb{Z}$ and natural transformations $\sigma_n\colon \tilde{h}_n \to \tilde{h}_{n+1} \circ \Sigma$ is called a* homology theory *if the following conditions are satisfied:*

1. *(Homotopy) Functors h_n factor through the homotopy category $\mathcal{T}_{*h}$,*
2. *(Exactness) For every map $X \xrightarrow{f} Y \xrightarrow{j} C(f)$ the sequence $\tilde{h}_n(X) \xrightarrow{f_*} \tilde{h}_n(Y) \xrightarrow{j_*} \tilde{h}_n(C(f))$ is exact,*
3. *(Suspension) Natural transformations $\sigma_n\colon \tilde{h}_n \to \tilde{h}_{n+1} \circ \Sigma$ are isomorphisms.*

Definition 2.1.2 (Cohomology Theory)**.** *A sequence of contravariant functors $\tilde{h}^n\colon \mathcal{T}_* \to \mathcal{A}$, $n \in \mathbb{Z}$ and natural transformations $\sigma_n\colon \tilde{h}_n \to \tilde{h}_{n+1} \circ \Sigma$ is called a* cohomology theory *if the following conditions are satisfied:*

1. *(Homotopy) Functors h^n factor through the homotopy category $\mathcal{T}_{*h}$,*
2. *(Exactness) For every cofibred sequence $X \xrightarrow{f} Y \xrightarrow{j} C(f)$ induces an exact sequence*

$$\tilde{h}^n(C(f)) \xrightarrow{f_*} \tilde{h}^n(Y) \xrightarrow{j_*} \tilde{h}^n(X),$$

3. *(Suspension) Natural transformations $\sigma_n\colon \tilde{h}^n \to \tilde{h}^{n+1} \circ \Sigma$ are isomorphisms.*

The homotopy axiom and exactness axiom define each (co-)homology in each dimension as an half-exact functor. The suspension axiom links them. The exactness axiom together with the suspension axiom imply that homology (resp. cohomology) functor applied to the right Puppe sequence lead for every map $f\colon X \to Y$ to long exact sequences of homology (resp. cohomology) groups: :

$$\dots \xrightarrow{\partial_{n+1}} \tilde{h}_n(X) \xrightarrow{f_*} \tilde{h}_n(Y) \xrightarrow{j_*} \tilde{h}_n(C(f)) \xrightarrow{\partial_n} \tilde{h}_{n-1}(X) \xrightarrow{f_*} \dots$$

$$\dots \xleftarrow{\delta_n} \tilde{h}^n(X) \xleftarrow{f^*} \tilde{h}_n(Y) \xleftarrow{j^*} \tilde{h}^n(C(f)) \xleftarrow{\delta_{n-1}} \tilde{h}^{n-1}(X) \xleftarrow{f^*} \dots$$

Uwaga 2.1.1. Homotopy groups $\pi_n(-)$ do not form a homology theory. Neither exactness nor suspension axiom is satisfied.

If one considers (co-)homology on infinite complexes it is useful to have satisfied the following additivity axiom:

Definition 2.1.3 (Additivity axiom). *A (co-)homology theory $\tilde{h}_*(-)$ (resp. $\tilde{h}^*(-)$) satisfies additivity axiom if for any family of pointed spaces $\{X_i\}_{i\in J}$, inclusions induce an isomorphism*

$$\bigoplus_{i\in J} \tilde{h}^*(X_i) \simeq \tilde{h}^*(\bigvee_{i\in J} X_i) \quad \textit{resp.} \quad \tilde{h}^*(\bigvee_{i\in J} X_i) \simeq \prod_{i\in J} \tilde{h}^*(X_i)$$

.

Note that for finite wedges the above isomorphisms follow from other axioms.

Any functor defined on pointed spaces can be extended to unpointed space by composition with functor which ads to each unpointed space a poin $X \mapsto X \sqcup \{pt\} =: X^+$. Thus any (co-)homology defined on pointed spaces extends to unpointed spaces. Having a (co-) homology theory defined on pointed spaces we can define it on pairs $h_n(X, A) := \tilde{h}_n(X/A)$ (note that $X/\emptyset = X^+$.) Such theory satisfies the usual Eilenberg-Steenrod (without Dimension Axiom!) axioms only for the Borsuk pairs (i.e. closed cofibrations).

Let $h^n : \mathcal{T}_2 \to \mathcal{A}$ be a sequence of functors (defined for $n \in \mathbb{Z}$) from the category of (some) pairs of topological spaces, containing all CW-pairs, to t he category of abelian groups which satisfies the Eilenberg-Steenrod axioms, with perhaps exception of the dimension axiom. It obviously defines a theory on pointed spaces $\tilde{h}_*(X) := h_*(X, x_0)$ which satisfies the above axioms.

Definition 2.1.4. *Natural transformation of (co-)homology theories $h_n \to k_n$ consists of sequence of natural transformations $T_n : h_n \to k_n$ which commute with suspension isomorphism in h_n and k_n.*

Theorem 2.1.1. *If $T_n : h^n \to k^n$ is a natural transformation of (co-)homology theories such that for each $n \in \mathbb{Z}$ homomorphism $T_n(S^0) : h^n(S^0) \to k^n(S^0)$ is an isomorphism then for every finite CW-complex X an $n \in \mathbb{Z}$, $T_n(X)$ is an isomorphism. If theories satisfy the additivity axiom then $T_n(X)$ is an isomorphism for any CW-complex.*

Proof. Inductive proof based on the 5-lemma. For inifinite complexes apply Milnor's lemma. □

Proposition 2.1.1. *Let h^* be a cohomology theory. Then, under additional assumptions which may depend on the category on which cohomology theory is defined, the following assertion hold:*

1. *there exists the Mayer-Vietoris exact sequence for (homotopy) push-out diagrams;*
2. *for two pointed spaces X_1, X_2 there is a split exact sequence*

$$0 \to \tilde{h}^*(X_1 \wedge X_2) \to h^*(X_1 \times X_2) \to h^*(X_1) \times h^*(X_2) \to 0.$$

3. *any self map of a sphere $f : S^n \to S^n$ of degree d induces a homomorphism $f^* : \tilde{h}^*(S^n) \to \tilde{h}^*(S^n)$ such that $f^*(u) = d \cdot u$ for each $u \in \tilde{h}^*(S^n)$.*

Proposition 2.1.2. *If $F : \mathcal{A}^* \to \mathcal{A}^*$ is an endofunctor of the category of graded abelian groups which is exact (i.e. preserves exact sequences) then its composition with any (co-)homology theory is again (co-)homology theory. If a functor is contravariant then it switches homology to cohomology and vice versa.*

2.2 Homology and cohomology of inifinite complexes

Definition 2.2.1. *For an arbitrary sequence of maps* $\mathbf{f} := \{X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots\}$ *we define its telescope as*

$$\mathrm{Tel}(\mathbf{f}) := X_1 \times [0,1] \cup_{f_1} X_2 \times [1,2] \cup_{f_2} \dots$$

(in the pointed case we collapse to point the real line passing through $x_0 \in X_0$*.)*

Proposition 2.2.1. *If all maps* $\mathbf{f} := \{X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots\}$ *are cofibration then projection*

$$\mathrm{Tel}(f) \to \mathrm{colim}\ \{X_1 \to X_2 \to \dots\}$$

is a homotopy equivalence.

Zad. 2.2.1. Let X_i and Tel be as above, we consider pointed case:

- T is a homotopy pushout of the following diagram : T_1, T_2,

$$\begin{tikzcd} \bigvee_i X_i \arrow[r, "f^{odd}"] \arrow[d, "f^{ev}"] & \bigvee_i X_{2i+1} \arrow[d] \\ \bigvee_i X_{2i} \arrow[r] & T \end{tikzcd} \tag{2.1}$$

- Homotopy cofiber of embedding $\bigvee_i X_i \hookrightarrow T$is homotopy equivalen to $T/(\bigvee_i X_i) \simeq \bigvee_i \Sigma X_i$
- If F is a contravariant additive half-exact functor then homomorphism in the Mayer-Vietoris sequence for the functor $F$and above diagram $F(\bigvee_i X_{2i+1}) \oplus F(\bigvee_i X_{2i}) \to F(\bigvee X_i)$ w ciągu Mayera-Vietorisa is given

  $$(a_1, a_3, \dots) \oplus (a_2, a_4, \dots) \mapsto (a_1 - F(f_1)(a_2), -a_2 + F(f_2)(a_3), a_3 - F(f_3)(a_4), \dots)$$

  where $a_j \in F(X_j)$ and $f_n \colon X_n \to X_{n+1}$.

Theorem 2.2.1 (Milnor Lemma[1])**.** *For an arbitrary sequence of maps* $\mathbf{f} := \{X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots\}$.

1. *If* F *is a contravariant half-exact functor then tehre exists a short exact sequence of the pointed sets:*
$$0 \to \lim{}^1 F(\Sigma X_i) \to F(\mathrm{Tel}) \to \lim F(X_i) \to 0.$$
2. *If* F *is a covariant additive half-exact functor then* colim $F(X_i) \simeq F(\mathrm{Tel})$.

Proof. Zadanie. Wsk. Wykorzystaj ciąg Mayera-Vietorisa dla pary T_1, T_2. □

[1] John Willard Milnor (Orange, NJ (USA) 1931 –)

2.3 Multiplicative structures

Multiplicative structures in (co-)homology are not axiomatized. Let h_* (resp. $h^*(-)$) be a (co-)homology theory defined on pairs od spaces. We'll adopt the following definition:

Definition 2.3.1. *A multiplicative structure on a (co-)homology theory h^* is a collection of natural transformations, defined for every $p, q \in \mathbb{Z}$*

$$h_p(X, A)\otimes h_q(Y, B) \xrightarrow{\times} h_{p+q}(X\times Y, X\times B\cup A\times Y) \quad \textit{resp.} \quad h^p(X, A)\otimes h^q(Y, B) \xrightarrow{\times} h^{p+q}(X\times Y, X\times B\cup A\times Y)$$

and an element $1 \in h^0(pt)$ which satisfy standard conditions for the cross-product in singular cohomology.[2] *Moreover assume that $h^*(pt) \otimes h^*(pt) \xrightarrow{\times} h^*(pt)$ is a ring structure with unit $1 \in h^0(pt)$. The ring $h^*(pt)$ is called the coefficient ring and often denoted h^*.*

For every pair (X, A) cohomology $h^*(X, A)$ is a graded module over $h^*(pt)$ and induced homomorphisms are h^*-module homomorphisms. . For two subsets $A, B \subset X$ the cup-product $\cup$ is defined as usual as composition

$$h^p(X, A) \otimes h^q(X, B) \xrightarrow{\times} h^{p+q}(X \times X, X \times B \cup A \times X) \xrightarrow{\Delta^*} h^p(X, A \cup B),$$

where $\Delta\colon X \to X \times X$ is the diagonal map. If $A, B = \emptyset$, then $h^*(X) := h^*(X, \emptyset)$ is a ring with unit. Note that commutation of cross product with the boundary operator in the exact sequence of pair implies that the suspension homomorphism in the reduced cohomology $\sigma\colon \tilde{h}^q(X) \to \tilde{h}^{q+1}(\Sigma X)$ is given by the formula $\sigma(u) = \sigma(1) \times u$ where $1 \in \tilde{h}^0(S^0) = h^0(pt)$.

Proposition 2.3.1. *For every multiplicative cohomology theory*

$$\tilde{h}^*((\mathbb{R}^n)^\bullet) = \tilde{h}^*(S^n) = h^*(D^n, S^{n-1}) = h^*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$

is a free graded h^*-module with one generator $\iota_n := \sigma^n(1) \in \tilde{h}^n(S^n)$.

2.4 Classical (co-)homology

Definition 2.4.1 (Dimension Axiom)**.** *A (co-)homology theory $\tilde{h}_n$ (resp. $\tilde{h}^n$) satisfies dimension axiom if $\tilde{h}_n(S^0) = 0$ (resp. $\tilde{h}^n(S^0) = 0$) for $n \neq 0$. Theories satisfying the dimension axiom are called* classical.

Example 1. Singular (co-)homology theory with arbitrary coefficients is classical.

Theorem 2.4.1 (Eilenberg-Steenrod)**.** *If $\tilde{h}^n$ and $\tilde{k}^n$ are classical (co-)homology theories on the category of finite CW-complexes then any homomorphism $\tau : \tilde{h}^0(S^0) \to \tilde{k}^0(S^0)$ extends in the unique way to a natural transformation of (co-)homology theories i.e. there exists unique $\Theta : \tilde{h}^0(X) \to \tilde{k}^0(X)$ such that $\Theta(S^0) = \theta$.*

Corollary 2.4.1 (Eilenberg-Steenrod)**.** *If $\tilde{h}_*$ (resp. $\tilde{h}^*$) is a classical (co-)homology theory such that $\tilde{h}_0(S^0) = \tilde{h}^0(S^0) =: A$, then the theory $\tilde{h}_*$ (resp.$\tilde{h}^*$) on the category of finite CW-complexes is equivalent to the singular (co-)homology with coefficients in the group A.*

Proof of Theorem 2.4.1 is based on lemmas.

[2] cf. E. H. Spanier *Algebraic Topology* 5.6.2-5.

Lemma 2.4.1. *A homomorphism $\tau : \tilde{h}^0(S^0) \to \tilde{k}^0(S^0)$ extends uniquely to a natural transformation on the kategory of CW-complexes homotopy equivalent to finite wedges of spheres of the same dimension.*

We'll prove that the groups $\tilde{h}^n(X)$ are naturally isomorphic to homology groups of the cellular chain complex with coeffcients in h^*. For every $n \geq 0$ we define

$$C_n(X; h^*) := h^n(X^{(n)}, X^{(n-1)}) := \tilde{h}^n(X^{(n)}/X^{(n-1)}) \simeq \tilde{h}^n(\bigvee_{i \in J_n} S_i^n) \simeq \tilde{h}^0(\bigvee_{i \in J_n} S_i^0).$$

We define the boundary homomorphisms $\partial_n : C_n(X; h^*) \to C_{n-1}(X; h^*)$ as the boundary homomorphisms in the exact sequence of the triple $(X^{(n)}, X^{(n-1)}, X^{n-2})$ (or equivalently a pair $(X^{(n)}/X^{(n-2)}, X^{n-1})/X^{n-2}$):

$$\ldots \tilde{h}^n(X^{(n)}, X^{(n-2)}) \to \tilde{h}^n(X^{(n)}, X^{(n-1)}) \xrightarrow{\partial_n} \tilde{h}^{n-1}(X^{(n-1)}, X^{(n-2)}) \to \tilde{h}^{n-1}(X^{(n)}, X^{(n-2)}) \to \ldots$$

Lemma 2.4.2. $\partial_{n-1} \circ \partial_n = 0$.

Definition 2.4.2. *We define cellular (co-) homology functors (with coefficients in $\tilde{h}^*$) on the category of CW-complexes and cellular maps. as (co-)homology of the (co-)chain complex $\{C_n(X; \tilde{h}^*), \partial_n\}$ (resp. $\{C^n(X; \tilde{h}^*), \partial_n\}$)*

$$H_n(X; \tilde{h}^*) := \ker \partial_n \; / \operatorname{im} \partial_{n+1}$$

A cellular map $f : X \to Y$ indusces a homomorphism of chain complexes $f\colon C_n(X; \tilde{h}^*) \to C_n(Y; \tilde{h}^*)$, hence of their homology groups $f_* : H_n(X; \tilde{h}^*) \to H_n(Y; \tilde{h}^*)$.

Lemma 2.4.3. *If $\tilde{h}^*$ is a classical (co-)homology, then on the category of CW-complexes and cellular maps tehre exists a natural equivalnence $\tilde{h}^n(X) \simeq H_n(X; \tilde{h}^*)$ i.e. the classical (co-)homology is uniquely defined by its coefficients $\tilde{h}^0(S^0)$.*

Proof. Proof of Lemma 2.4.3 is based on "diagram chasing" and splits into steps:

Step 1. Inclusion of skeleton $j : X^{(n+1)} \hookrightarrow X$ induces natural isomorphisms

$$h_n(X^{(n+1)}, X^{(n-2)}) \simeq h_n(X^{(n+1)}) \xrightarrow{\simeq} h_n(X).$$

Step 2. $h_n(X^{(n+1)}, X^{(n-2)}) \simeq H_n(X; \tilde{h}^*)$. Consider a diagram:

$$\ldots \tilde{h}^{n+1}(X^{(n+1)}, X^{(n)}) \xrightarrow{\partial_{n+1}} \tilde{h}^n(X^{(n)}, X^{(n-1)}) \xrightarrow{\partial_n} \tilde{h}^{n-1}(X^{(n-1)}, X^{(n-2)}) \ldots$$

$$\tilde{h}$$

∎

2.5 The Leray-Hirsch Theorem

Definition 2.5.1 (Spanier Sec. 5.7)**.** *A* fiber-bundle pair *with a base space B consists of a* total pair *of spaces $(E, \dot{E})$, a* fiber pair *of spaces $(F, \dot{F})$ and a projection $E \xrightarrow{p} B$ such that there exists an open covering $\mathcal{V}$ of the space B and for each $V \in \mathcal{V}$ there is a homeomorphism over V of pairs $V \times (F, \dot{F}) \simeq (p^{-1}(V), p^{-1}(V) \cap \dot{E})$.*

Any vector bundle $E \to B$ leads to a fiber-bundle pair, where fiber is $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ i.e. $(E, \dot{E}) \to B$ where $\dot{E} = E \setminus s_0(B)$. When we pick-up a Riemannian metric on the vector bundle then we get a a fiber-bundle pair with fiber (D^n, S^{n-1}). Another important example is projectivization $(\mathbb{P}(E \oplus \theta^1), \mathcal{P}(E)) \to B$; here fiber-pair is $(\mathbb{P}(E_x \oplus \mathbb{F}), \mathbb{P}(E_x))$.

In fact for every locally-trivial bundle $\dot{E} \to B$ whose fiber is a sphere S^{n-1} we can construct a fiber-bundle pair $(E, \dot{E}) \to B$ with fiber (D^n, S^{n-1}). The total space E is obtained by construction a cone over each fiber $\dot{E}_x \subset E$.

If h^* is a multiplicative cohomology theory then for any fiber-bundle pair cohomology $h^*(E, \dot{E})$ has a graded $h^*(B)$-module structure (in fact local triviality is not relevant here). The multiplication by elements of $h^*(B)$ is defined via the cup product and the induced homomorphism p^*:

$$h^p(B) \otimes h^q(E, \dot{E}) \xrightarrow{p^* \otimes id} h^p(E) \otimes h^q(E, \dot{E}) \xrightarrow{\cup} h^{p+q}(E, \dot{E})$$

Theorem 2.5.1 (Leray-Hirsch)**.** *Let $(E, \dot{E}) \to B$ be a fiber-bundle pair over a CW-complex (with finite trivialization covering). Suppose $u_1, \dots u_n \in h^*(E, \dot{E})$ are homogeneous elements such that for each point $b \in B$ the restrictions $i_b^*(u_1), \dots i_b^*(u_n) \in h^*(F, \dot{F})$ form a free basis of $h^*(F, \dot{F})$ over h^*. Then the elements $u_1, \dots u_n \in h^*(E, \dot{E})$ form a free basis of $h^*(E, \dot{E})$ over $h^*(B)$.*

We begin with proving a version of the Künneth formula in general cohomology, which is a special case of the above theorem.

Proposition 2.5.1 (Künneth formula)**.** *For every finite CW–complex B and a pair $(F, \dot{F})$ such that $h^*(F, \dot{F})$ is a free h^*–module, the cross product homomorphism*

$$\bigoplus_{p+q=n} h^p(B) \otimes_{h^*} h^q(F, \dot{F}) \xrightarrow{\times} h^n(B \times (F, \dot{F}))$$

is an isomorphism for every $n \in \mathbb{Z}$.

Proof. Both graded groups $h^*(-) \otimes h^*(F, \dot{F})$ and $h^*(- \times (F, \dot{F}))$ can be extended to cohomology theories (i.e. relative groups and boundary homomorphisms defined). Exactness is preserved because $h^*(F, \dot{F})$ is a free h^*-module (it is enough to assume that it is flat). The cross product defines a natural transformation which is obviously isomorphism for $B = pt$. Hence it is an isomorphism for any finite CW-complex B. □

Proof of Thm. 2.5.1. We consider a homomorphism:

$$\bigoplus_{i=1}^{n} h^*(B) \ni (z_1, \dots z_k) \mapsto \sum_{i=1}^{n} p^*(z_i) \cup u_i \in h^*(E, \dot{E})$$

and first prove (using the Kunneth formula) that it is an isomorphism for trivial fibre-bundle pair $B \times (F, \dot{F}) \to B$. General case follows by induction on skeleta, applying the Mayer-Vietoris exact sequence. □

Note that the Leray-Hirsch theorem asserts that if the elements $u_1, \dots u_n \in h^*(E, \dot{E})$ exist then $h^*(B)$-module $h^*(E, \dot{E})$ is the isomorphic to $h^*(B \times F, B \times \dot{F})$ thus it doesn't distinguish between isomorphism classes of bundles.

2.6 The Thom isomorphism and the Gysin exact sequence

Note that any choice of orientation *or* of the n-dimensional real vector space $\mathbf{V}$ in the sense of linear algebra defines a generator of h^*-module $h^*(\mathbf{V}^\bullet)$. Formally, we pick up an isomorphism $A\colon \mathbf{V} \to \mathbb{R}^n$ which maps the selected orientation to the canonical orientation and define $[or]_{h^*} := A^*(\iota_n)$. This is well defined because the real linear group has two path components. Elements $\pm\iota_n \in h^n(\mathbf{V}^\bullet)$ we call h^* – orientations of $\mathbf{V}$. It may happen that they coincide (e.g. for $h^*(-;\mathbb{Z}_2)$).

Definition 2.6.1. *An h^*-orientation of an n–dimensional real vector bundle $E \to X$ is an element $U_E \in \tilde{h}^n(\mathrm{Th}(E))$ such that for inclusion of every fiber $i_x\colon p^{-1}(x) \to \mathrm{Th}(E)$ the restriction $i_x^*(U_E) \in \tilde{h}^n(p^{-1}(x)^\bullet)$ is an h^*-orientation of the vector space $p^{-1}(x)$. The element U_E is also called the Thom class of the bundle $E \to X$. A bundle which has a Thom class is called h^*-orientable.*

Zad. 2.6.1. If $U_E \in \tilde{h}^n(\mathrm{Th}(E))$ is the Thom class of the vector bundle $E \to Y$ and $f\colon X \to Y$ is a map then $\hat{f}^*(U_E) \in \tilde{h}^n(\mathrm{Th}(f^!E))$ is the Thom class of the induced bundle.

Zad. 2.6.2. Prove the following assertions concerning orientation of vector bundles:

1. Trivial bundle has the Thom class in any multiplicative cohomology theory.
2. Every vector bundle is orientable in the singular cohomology with $\mathbb{Z}_2$–coefficients (any $\mathbb{Z}_2$–algebra) and the Thom class is unique.
3. There is a bijection between geometric orientations of the vector bundle and its Thom classes in singular cohomology with $\mathbb{Z}$–coefficients (any ring of characteristic $\neq 2$).
4. If in a cohomology theory h^* every vector bundle is orientable then the ring h^* has characteristic 2 i.e. cohomology theory has values in $\mathbb{Z}_2$-vector spaces.
5. If h^* has characteristic $\neq 2$ then any h^*–orientable bundle is geometrically orientable (i.e. $h^*(-;\mathbb{Z})$–orientable.)

Zad. 2.6.3. For every two vector bundles $E_i \to X_i$ with Thom classes $U_{E_i} \in \tilde{h}^{n_i}(\mathrm{Th}(E_i))$ the element $U_{E_1} \times U_{E_2} \in \tilde{h}^{n_1+n_2}(\mathrm{Th}(E_1 \oplus E_2))$ (cf. Zad **??**) is the Thom class of the Whitney sum $E_1 \oplus E_2$. Infer that orientability is a stable property i.e. if $E \oplus \theta^k$ is h^*-orientable then E is h^*-orientable.

Definition 2.6.2. *For an h^*-oriented n-dimensional vector bundle $E \to X$ we define its Euler class $e(E) := s_0^*(U_E) \in h^n(X)$ where $s_0\colon (X, \emptyset) \to (\mathrm{Th}(E), \infty)$ is the zero-section.*

Zad. 2.6.4. If a vector bundle has everywhere non-vanishing section then its Euler class in any cohomology theory vanishes.

Zad. 2.6.5. Let $E \to S^n$ be a real oriented n-dimensional vector bundle over sphere ($n > 1$). Prove two interpretations of its Euler class in singular cohomology:

1. Let $\gamma_E\colon S^{n-1} \to SO_n$ be gluing function of the bundle and consider its composition with projection $p\colon SO_n \to S^{n-1}$. Then $e(E) = \pm\deg(p\gamma_E) \in h^n(S^n) \simeq \mathbb{Z}$.
2. Prove that $E \to S^n$ has a section $s\colon S^n \to E$ such that $s(x) \neq 0$ for all $x \neq 1$. Consider a map locally defined by the section s: $f_s\colon (U, U \setminus \{1\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. Then $e(E) = \pm\deg(f_s) \in h^n(S^n) \simeq \mathbb{Z}$.

Let us note that the Thom class 2.6.1 can be defined for any fiber-bundle pair $(E, \dot{E}) \to B$ whose fiber is either $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ or (D^n, S^{n-1}). It is an element $U_E \in h^n(E, \dot{E})$ whose restriction to each fiber $U|(E_x, \dot{E}_x) \in h^n(E_x, \dot{E}x)$ is a generator. We obtain the following consequence of the Leray-Hirsch Theorem 2.5.1.

Corollary 2.6.1 (Thom Isomorphism)**.** *Let $U_E \in \tilde{h}^n(\mathrm{Th}(E))$ be the Thom class of the fiber-bundle pair $(E, \dot{E}) \to B$. Then multiplication by the Thom class: $\Phi_U \colon h^q(B) \to h^{q+n}(E, \dot{E})$ is an isomorphism $h^*(B)$-modules.*

For an h^*-orientable vector bundle $E \to B$ it means that cohomology of the Thom space $\tilde{h}^*(\mathrm{Th}(E))$ is a free graded $h^*(B)$-module generated by U_E - it follows immediately from Thm. 2.5.1 applied to the fiber-bundle pair $(\mathbb{P}(E \oplus \theta^1), \mathbb{P}(E)) \to B$. .

Remark 1. The Thom isomorphism for one dimensional trivial bundle is the suspension isomorphism.

An important consequence of the Thom isomorphism is the Gysin exact sequence relating cohomology of a total space and base space of h^*-oriented sphere bundle $p \colon \dot{E} \to B$.

Theorem 2.6.1 (The Gysin sequence)**.** *Let $p \colon \dot{E} \to B$ be h^*-oriented sphere bundle and $e(E) \in h^n(B)$ be its Euler class. Then the following sequence is exact:*

$$\cdots \to h^{q-n}(B) \xrightarrow{e(E)\cup -} h^q(B) \xrightarrow{p^*} h^q(\dot{E}) \xrightarrow{\delta} h^{q-n+1}(B) \to \dots$$

2.7 Oriented cohomology theories

A complex oriented cohomology theory is a generalized cohomology theory h^* which is multiplicative and it has a natural choice of the Thom class for every complex vector bundle (considered as a real bundle.) Before we give a precise definition we'll explain relation between real and complex vector bundles.

Obviously for every n-dimensional complex vector bundle $E \to X$ we can forget about complex structure and consider it as $2n$-dimensional real vector bundle $E_{\mathbb{R}} \to X$ (*a realification*) . Conversely, having a n-dimensional real vector bundle we can construct its *complexification* $E \otimes_{\mathbb{R}} \mathbb{C} \to X$.

Zad. 2.7.1. Consider realification and complexification as a pair of adjoint functors between categories of complex (reps. real) vector bundles over a fixed space.

For every complex vector space $\mathbf{V}$ we define a map $\mathbb{P}(\mathbf{V}_{\mathbb{R}} \oplus \mathbb{R}) \ni [\mathbf{v}, t]_{\mathbb{R}} \mapsto [\mathbf{v}, ti]_{\mathbb{C}} \in \mathbb{P}(\mathbf{V} \oplus \mathbb{C})$ which maps $\mathbb{P}(\mathbf{V}_{\mathbb{R}})$ to $\mathbb{P}(\mathbf{V})$ and on the quotients

$$\mathbb{P}(\mathbf{V}_{\mathbb{R}} \oplus \mathbb{R})/\mathbb{P}(\mathbf{V}_{\mathbb{R}}) \to \mathbb{P}(\mathbf{V} \oplus \mathbb{C})/\mathbb{P}(\mathbf{V})$$

it is bijective, thus a homeomorphism. For every complex vector bundle $E \to X$ the map

$$\mathbb{P}(E \oplus \theta^1_{\mathbb{R}})/\mathbb{P}(E_{\mathbb{R}}) \to \mathbb{P}(E \oplus \theta^1_{\mathbb{C}})/\mathbb{P}(E)$$

is defined fiberwise and it is a local homeomorphism, thus a homeomorphism. Hence the Thom space can be constructed out of the complex projectivization

Note that a complex vector space $\mathbf{V}$ has a uniquely defined h^*-orientation i.e. a generator $\iota_{\dim \mathbf{V}} \in \tilde{h}^{\dim \mathbf{V}}(\mathbf{V}^\bullet)$ which is defined by an $\mathbb{C}$-linear isomorphism $\mathbf{V} \to \mathbb{C}^n$. Any two such isomorphisms are homotopic because the complex linear group is path connected.

Definition 2.7.1. *We say that a multiplicative cohomology theory h^* is complex oriented if to every complex vector bundle $E \to X$ a Thom class $U_E \in \tilde{h}^*(\mathrm{Th}(E))$ is assigned in such way that:*

1. *For each $x \in X$, $i_x^*(U_E) = \iota_{\dim E}$*

2. *The classes U_E should be natural under pull-backs: if $f\colon Y \to X$, $U_{f^! E} = \hat{f}^* U_E$;*

3. *Multiplicativity: $U_{E_1 \oplus E_2} = U_{E_1} U_{E_2}$.*

Zad. 2.7.2. The singular cohomology theory is complex oriented.

Remark 2. There are many other examples of the complex oriented theories playing important role in algebraic topology e.g. complex K-theory, Morava K-theories, complex cobordism.

2.8 Cohomology of projective spaces

Now we'll apply the Thom isomorphism theorem to complex oriented cohomology to calculate cohomology rings of the complex projective spaces. If h^* is a complex oriented cohomology then for any complex projective space $\mathbb{P}(\mathbf{V})$ we define an element $x_{\mathbf{V}} \in h^2(\mathbb{P}(\mathbf{V}))$ as the Euler class of the dual tautological bundle $H^*_{\mathbf{V}} \to \mathbb{P}(\mathbf{V})$. Let $d := \dim_{\mathbb{C}} \mathbf{V}$. The projective space $\mathbb{P}(\mathbf{V})$ can be covered by d contractible sets (affine charts), thus $x^d_{\mathbf{V}} = 0$ and there is a homomorphism $\chi_{\mathbf{V}}\colon h^*[x_{\mathbf{V}}]/(x^d_{\mathbf{V}}) \to h^*(\mathbb{P}(\mathbf{V}))$ where $h^*[x_{\mathbf{V}}]$ denotes graded polynomial ring with single generator in gradation 2.

Theorem 2.8.1. *For a complex vector space $\mathbf{V}$ of dimension d the map*

$$\chi_{\mathbf{V}}\colon h^*[x_{\mathbf{V}}]/(x^d_{\mathbf{V}}) \to h^*(\mathbb{P}(\mathbf{V}))$$

is an isomorphism.

Proof. Using Prop. **??** and Cor. 2.6.1 we proceed by induction on dimension of the vector space $\mathbf{V}$. For $\dim \mathbf{V} = 1$ theorem is tautologically true. (For $\dim \mathbf{V} = 1$ it also obviously holds because $\mathbb{P}(\mathbb{C} \oplus \mathbb{C}) = S^2$ and its cohomology can be calculated using suspension isomorphism: $h^*[x_1]/(x_1^2) \xrightarrow{\simeq} h^*(S^2)$.

Suppose

$$h^*[x_{\mathbf{V}}]/(x^d_{\mathbf{V}}) \to h^*(\mathbb{P}(\mathbf{V}))$$

is an isomorphism, and we prove it for the space $\mathbf{V} \oplus \mathbb{C}$. Note that $x_{\mathbf{V}\oplus\mathbb{C}} \in h^2(\mathbb{P}(\mathbf{V}\oplus\mathbb{C}))$ corresponds under homomorphism induced by homeomorphism described in Prop. **??** to the Thom class of the bundle $H_{\mathbf{V}} \to \mathbb{P}(\mathbf{V})$ (apply **??**.3.)

Consider the following diagram

$$\begin{tikzcd}
I(x_{\mathbf{V}\oplus\mathbb{C}}) \arrow[r, "\chi_{\mathbf{V}\oplus\mathbb{C}}"] & \tilde{h}^*(\mathbb{P}(\mathbf{V}\oplus\mathbb{C})) \arrow[r, "h^*_{\mathbf{V}}", "\simeq"'] & \tilde{h}^*(\mathrm{Th}(H_{\mathbf{V}})) \\
h^*[x_{\mathbf{V}}]/(x^d_{\mathbf{V}}) \arrow[u, "\varphi_{\mathbf{V}}", "\simeq"'] \arrow[r, "\chi_{\mathbf{V}}", "\simeq"'] & h^*(\mathbb{P}(\mathbf{V})) \arrow[ur, "\Phi_{\mathbf{V}}", "\simeq"'] &
\end{tikzcd} \tag{2.2}$$

where $I(x_{\mathbf{V}\oplus\mathbb{C}}) \subset h^*[x_{\mathbf{V}\oplus\mathbb{C}}]/(x^{d+1}_{\mathbf{V}\oplus\mathbb{C}}$ is an ideal generated by element $x_{\mathbf{V}\oplus\mathbb{C}}$ and $\varphi_{\mathbf{V}}$ is h^*-module homomorphism defined as $\varphi(x^k_{\mathbf{V}}) := x^{k+1}_{\mathbf{V}\oplus\mathbb{C}}$.

We have to prove that the diagram commutes. Note that the Thom isomorphism $\Phi_{\mathbf{V}}$ is a homomorphism of $h^*(\mathbb{P}(\mathbf{V}))$-modules, thus for each $z \in h^*(\mathbb{P}(\mathbf{V}))$, $\Phi_{\mathbf{V}}(z) = p^*_{\mathbf{V}}(z) \cup \Phi_{\mathbf{V}}(1)$ where $p_{\mathbf{V}}\colon H_{\mathbf{V}} \to \mathbb{P}(\mathbf{V})$ and $\cup\colon h^*(H_{\mathbf{V}}) \otimes \tilde{h}^*(\mathrm{Th}(H_{\mathbf{V}})) \to \tilde{h}^*(\mathrm{Th}(H_{\mathbf{V}}))$. By inductive assumption $h^*(\mathrm{Th}(H_{\mathbf{V}}))$

is a truncated polynomial algebra generated by $x_{\mathbf{V}}$. Since $x_{\mathbf{V}\oplus\mathbb{C}}|\mathbb{P}(\mathbf{V}) = x_{\mathbf{V}}$, applying again Prop. **??** we see that the following commutative diagram

$$\begin{tikzcd}
\tilde{h}^*(\mathbb{P}(\mathbf{V}\oplus\mathbb{C}))\otimes\tilde{h}^*(\mathrm{Th}(H_{\mathbf{V}})) \arrow[r, "h_{\mathbf{V}\oplus\mathbb{C}}\otimes id"] \arrow[dd, "i^*\otimes id"] & \tilde{h}^*(\mathrm{Th}(H_{\mathbf{V}}))\otimes\tilde{h}^*(\mathrm{Th}(H_{\mathbf{V}})) \arrow[d] \arrow[dr, "\cup"] & \\
& h^*(H_{\mathbf{V}})\otimes\tilde{h}^*(\mathrm{Th}(H_{\mathbf{V}})) \arrow[r, "\cup"] & \tilde{h}^*(\mathrm{Th}(H_{\mathbf{V}})) \\
\tilde{h}^*(\mathbb{P}(\mathbf{V}))\otimes\tilde{h}^*(\mathrm{Th}(H_{\mathbf{V}})) \arrow[r, "id\otimes id"] & h^*(\mathbb{P}(\mathbf{V}))\otimes\tilde{h}^*(\mathrm{Th}(H_{\mathbf{V}})) \arrow[u, "p_{\mathbf{V}}^*\otimes id", "\simeq"'] \arrow[ur, "\cup"] &
\end{tikzcd} \tag{2.3}$$

shows that $\Phi_{\mathbf{V}}(x_{\mathbf{V}}^k) = p_{\mathbf{V}}^*(x_{\mathbf{V}}^k)\cup\Phi_{\mathbf{V}}(1) = x_{\mathbf{V}\oplus\mathbb{C}}^k\Phi_{\mathbf{V}}(1) = x_{\mathbf{V}\oplus\mathbb{C}}^{k+1}$ for $0\le k<d$. (Note that $(p^*)^{-1} = s_0^*$.) $\square$

Remark 3. One can prove that if in a multiplicative cohomology theory the above description of the cohomology algebra of the complex projective space holds (with some choice of generators compatible with restrictions to subspaces) then the theory is complex oriented. In other words, orientations of the linear bundles permit to construct orientation for any complex vector bundle.

Remark 4. Analogous theorem holds for h^*-cohomology of real projective spaces if all real vector bundles are h^*-orientable (then $\deg x_{\mathbf{V}} = 1$.)

Zad. 2.8.1. Prove Thm. 2.8.1 using the Gysin sequence Thm. 2.6.1

Chapter 3

Spectra, cohomology & homology

Ważna metoda badania przestrzeni przy pomocy narzędzi algebraicznych polega na rozpatrywaniu klas homotopii odwzorowań z danej przestrzeni (lub do danej przestrzeni) do (lub z) innej "testowej" przestrzeni. Grupy homotopii to klasy homotopii odwzorowań ze sfer w daną przestrzeń. Teraz rozdziale zdefiniujemy *grupy kohomologii* jako zbiory odwzorowań z danej przestrzeni do przestrzeni Eilenberga-MacLane'a. Okazują się one łatwiejsze do oblicznia, niż pozornie prościej zdefiniowane grupy homotopii. Ciąg przestrzeni Eilenberga-MacLane'a dla ustalonej grupy abelowej A i ciąg sfer stanowią ważne przykłady prespektrów, od wprowadzenia których rozpoczniemy nasze rozważania.

3.1 Prespectra

Definition 3.1.1. *A prespectrum consists of a sequence indexed by integers of pointed and maps $\mathbb{E} = (E_n, \epsilon_n)$, where $\epsilon_n : \Sigma E_n \to E_{n+1}$. Prespectrum is called an Ω-spectrum if for each n the adjoint map $\hat{\epsilon}_n : E_n \to \Omega E_{n+1}$ is a (weak) homotopy equivalence. A map of prespectra is a sequence of maps $f_n : E_n \to E'_n$ such that $f_{n+1} \circ \epsilon_n = \epsilon'_n \circ \Sigma f_n$.*

Example 2. Sphere spectrum $\mathbb{S} = (S^n, \epsilon_n)$ where e $\epsilon_n : \Sigma S^n \to S^{n+1}$ is a homeomorphism.

For any prespectrum $\mathbb{E}$ and space X we define:

- The smash product prespectrum $(X \wedge \mathbb{E})_n := X \wedge E_n$. In particular if $\mathbb{E} = \mathbb{S}$ is the sphere spectrum then $Y \wedge \mathbb{S}$ is the suspension spectrum defined by the space Y.
- Let X be a compact space. The function prespectrum $F(X, \mathbb{E})_n := F(X, E_n)$ the space of pointed maps with compact-open topology and the structure maps are defined as follows $\bar{\epsilon}_n([t, f])(x) := \epsilon_n([t, f(x)])$. Multiple application of the exponential law:

  $$F(S^1 \wedge F(X, E_n), F(X, E_{n+1}) = F(F(X, E_n), F(S^1, F(X, E_{n+1})) = F(F(X, E_n), F(S^1 \wedge X, E_{n+1}))$$

  proves that they are continuos.

Example 3. Drugi przykład jest wyznaczony przez ciąg przestrzeni Eilenberga–MacLane'a. Dla dowolnej grupy abelowej A rozpatrzmy ciąg przestrzeni $K(A, n)$, przyjmując $K(A, n) = \{pt\}$ dla $n < 0.$, $K(A, 0) = A$, wybierając model $K(A, 1)$ i dalej indukcyjnie budując przestrzenie $K(A, n)$ przez doklejenie do zawieszenia $\Sigma K(A, n-1)$ komórek wyższych wymiarów, zabijając wyższe grupy homotopii. Odwzorowania $\Sigma K(A, n-1) \to K(A, n)$ są zanurzeniami. Można sprawdzić, że odwzorowania dołączone $K(A, n) \to \Omega K(A, n+1)$

są (słabymi) homotopijnymi równoważnościami. To prespektrum oznaczamy $\mathbb{K}(A)$ i nazywamy spektrum Eilenberga-MacLane'a.

For any space X, prespectrum $\mathbb{E}$ and integer n we define set of homotopy classes

$$\{X, \mathbb{E}\}_n := \operatorname{colim}_k [S^k X, E_{k-n}] := \{X, \mathbb{E}\}^{-n}$$

where colimit is taken with respect to maps:

$$[\Sigma^k X, E_{n+k}] \xrightarrow{\Sigma} [\Sigma^{k+1} X, \Sigma E_{n+k}] \simeq [\Sigma^{k+1} X, \Sigma E_{n+k}] \xrightarrow{\epsilon_*} [\Sigma^{k+1} X, E_{n+k+1}]$$

If $\mathbb{E}$ is a suspension spectrum of a space Y, and $n = 0$ we obtain set of stable homotopy classes of maps $X \to Y$. Note that for any space X, a compact space Y and prespectrum $\mathbb{E}$ we have natural isomorphism:

$$\{X \wedge Y, \mathbb{E}\} \simeq \{X, F(Y, \mathbb{E})\}$$

3.2 Homology and cohomology

For any prespectrum $\mathbb{E} = (E_n, \epsilon_n)$ one defines groups

$$\tilde{\mathbb{E}}^n(X|Y) := \{X, Y \wedge \mathbb{E}\}_{-n} = \operatorname{colim}_k [\Sigma^k X, Y \wedge E_{n+k}]$$

Theorem 3.2.1. *For every prespectrum $\mathbb{E} = (E_n, \epsilon_n)$ sequence of functors $\tilde{\mathbb{E}}^n(X|Y) \colon \mathcal{S}p_{\cdot} \times \mathcal{S}p_{\cdot} \to \mathcal{A}b$ is a cohomology theory with respect to X. The sequence of functors $\tilde{\mathbb{E}}_n(X|Y) := \tilde{\mathbb{E}}^{-n}(X|Y)$ is a homology theory with respect to Y.*

Lemma 3.2.1. *For every pointed map $Y \xrightarrow{f} Y' \xrightarrow{j} C(f)$ and a space Z consider the diagramm:*

$$\begin{tikzcd}
{[Z, Y]_{\cdot}} \arrow[r, "f_\#"] \arrow[d, "\Sigma"'] & {[Z, Y']_{\cdot}} \arrow[r, "i_\#"] \arrow[d, "\Sigma"'] & {[Z, C(f)]_{\cdot}} \arrow[d, "\Sigma"'] \\
{[\Sigma Z\, ,\, \Sigma Y]_{\cdot}} \arrow[r, "(\Sigma f)_\#"] & {[\Sigma Z\, ,\, \Sigma Y']_{\cdot}} \arrow[r, "(\Sigma i)_\#"] & {[\Sigma Z\, ,\, \Sigma C(f)]_{\cdot}}
\end{tikzcd}$$

If $i_\#([g]) = *$, then there exists $\tilde{g} \colon \Sigma Z \to \Sigma Y$ such that $(\Sigma f)_\#([\tilde{g}]) = [\Sigma g]$.

Proof. Istotnie, jeśli $H \colon C(Z) \to C(f)$ jest homotopią $i \circ g \sim *$ to definiuje ono odwzorowanie $\tilde{g} \colon \Sigma Z \to \Sigma Y$, takie, że $\Sigma f \circ \tilde{g} \sim \Sigma g$. $\square$

Proof of Thm. 3.2.1.
Aksjomat homotopii jest oczywiście spełniony, bo w definicji funktorów mamy zbiory klas homotopii przekształceń.

Aksjomat dokładności. Trzeba sprawdzić, że dla dowolnych odwzorowań $X' \xrightarrow{f} X \xrightarrow{i} C(f)$ oraz $Y \xrightarrow{f'} Y' \xrightarrow{i'} C(f)$ ciągi:

$$\tilde{\mathbb{E}}^n(C(f)|Y) \xrightarrow{i^*} \tilde{\mathbb{E}}^n(X|Y) \xrightarrow{f^*} \tilde{\mathbb{E}}^n(X'|Y)$$

$$\tilde{\mathbb{E}}^n(X|Y) \xrightarrow{f'^*} \tilde{\mathbb{E}}^n(X|Y') \xrightarrow{i'^*} \tilde{\mathbb{E}}^n(X|C(f))$$

są dokładne.

Dokładność pierwszego ciągu wynika stąd, że dla dowolnej przestrzeni z wyróżnionym punktem Z ciąg zbiorów z wyróżnionym punktem $[C(f), Z]_{\cdot} \xrightarrow{i^*} [X', Z]_{\cdot} \xrightarrow{f^*} [X, Z]_{\cdot}$ jest dokładny oraz faktu, że granica prosta ciągów dokładnych jest ciągiem dokładnym (**zad.**) Drugi ciąg na poziomie niestabilnym nie musi być dokładny (**zad.** Podać przykład.) ale po przejściu do granicy staje się dokładny na mocy Lematu 3.2.1.

Aksjomat zawieszenia. Trzeba wykazać, że istnieją naturalne równoważności funktorów:

$$\begin{aligned}[\text{kohomo}]\colon\quad & \tilde{\mathbb{E}}^n(X|Y) \simeq \tilde{\mathbb{E}}^{n+1}(\Sigma X|Y) \quad \text{oraz} \\ [\text{homo}]\colon\quad & \tilde{\mathbb{E}}^n(X|\Sigma Y) \simeq \tilde{\mathbb{E}}^{n+1}(X|Y) \quad \text{(zmiana znaku gradacji!)}.\end{aligned}$$

Ad [kohomo]

$$\tilde{\mathbb{E}}^n(X|Y) := \operatorname{colim}_k [\Sigma^k X, Y \wedge E_{n+k}] \simeq \operatorname{colim}_k [\Sigma^{k+1} X, Y \wedge E_{n+k+1}] \simeq \tilde{\mathbb{E}}^{n+1}(\Sigma X|Y).$$

Ad [homo]

$$\begin{aligned}\tilde{\mathbb{E}}^n(X|\Sigma Y) := \operatorname{colim}_k [\Sigma^k X, \Sigma Y \wedge E_{n+k}] \simeq \operatorname{colim}_k [\Sigma^k X, Y \wedge \Sigma E_{n+k}] \xrightarrow{\epsilon_\cdot} \\ \xrightarrow{\epsilon_\cdot} \operatorname{colim}_k [\Sigma^k X, Y \wedge E_{n+k+1}] \simeq \tilde{\mathbb{E}}^{n+1}(X|Y)\end{aligned}$$

Wystarczy więc pokazać, że odwzorowanie

$$\operatorname{colim}_k [\Sigma^k X, Y \wedge \Sigma E_{n+k}] \xrightarrow{\epsilon_\cdot} \operatorname{colim}_k [\Sigma^k X, Y \wedge E_{n+k+1}]$$

jest bijekcją co wynika z przemienności odpowiedniego diagramu...

□

Dla ustalonego prespektrum $\mathbb{E} = (E_n, \epsilon_n)$ będziemy rozważać teorię kohomologii

$$\tilde{\mathbb{E}}^n(X) := \tilde{\mathbb{E}}^n(X|S^0) =: \{X, \mathbb{E}\}_{-n}$$

oraz teorię homologii

$$\tilde{\mathbb{E}}_n(Y) := \tilde{\mathbb{E}}^{-n}(S^0|Y) =: \{S^0, Y \wedge \mathbb{E}\}_n$$

3.3 Multiplicative structures in homology and cohomology

Jeśli dane są prespektra $\mathbb{E}$, $\mathbb{F}$ oraz rodzina odwzorowań $m_{p,q}\colon E_p \wedge F_q \to G_{p+q}$ spełniająca odpowiednie warunki zgodności (takie rodziny oznaczamy $\mathbb{E} \wedge \mathbb{F} \to \mathbb{G}$, choć nie definiowaliśmy smash-produktu prespektrów), to dla dowolnych przestrzeni punktowanych X, Y oraz Z, W definiuje ona produkty

$$\times\colon \tilde{\mathbb{E}}^p(X) \times \tilde{\mathbb{F}}^q(Y) \to \tilde{\mathbb{G}}^{p+q}(X \wedge Y) \quad \text{[cohomology cross-product]} \tag{3.1}$$

$$\times\colon \tilde{\mathbb{E}}_p(X) \times \tilde{\mathbb{F}}_q(Y) \to \tilde{\mathbb{G}}_{p+q}(X \wedge Y) \quad \text{[homology cross-product]} \tag{3.2}$$

$$/ : \tilde{\mathbb{E}}^p(X \wedge Y) \times \tilde{\mathbb{F}}_q(Y) \to \tilde{\mathbb{G}}^{p-q}(X) \quad \text{[/ slant-product]} \tag{3.3}$$

$$\backslash : \tilde{\mathbb{E}}^p(X) \times \tilde{\mathbb{F}}_q(X \wedge Y) \to \tilde{\mathbb{G}}_{q-p}(Y) \quad [\ \backslash \text{ slant-product}] \tag{3.4}$$

Example 4. Jeśli $\mathbb{S}$ jest prespektrum sfer, to dla dowolnego prespektrum $\mathbb{E}$ istnieje rodzina odwzorowań $\mathbb{S} \wedge \mathbb{E} \to \mathbb{E}$ zadana przez strukturalne odwzorowania prespektrum $S^p \wedge E_q \to E_{p+q}$.

Konstrukcja slant-produktu 3.3/: Będziemy stosowali oznaczenie $f\colon X \to E_{+p}$ dla przekształcenia reprezentowanego przez $f\colon \Sigma^k X \to E_{k+p}$. Niech $f\colon X\wedge Y \to E_{+p}$ oraz $g\colon S^0 \to Y \wedge F_{-q}$. Definiujemy odwzorowanie

$$X \xrightarrow{\iota\wedge g} X\wedge Y\wedge F_{-q} \xrightarrow{f\wedge\iota} E_{+p}\wedge F_{-q} \xrightarrow{m_{p,-q}} G_{p-q}.$$

Pozostałe produkty konstruuje się analogicznie. Jeśli dane są odwzorowania $m_{p,q}\colon E_p\wedge E_q \to E_{p+q}$ spełniające warunki homotopijnej łączności oraz jedność $S^0 \to E_0$ to spektrum nazywa się multyplikatywne. W takim przypadku dla dowolnej przestrzeni $\mathbb{E}^*(-)$ jest funktorem do kategorii pierścieni z gradacją. W szczególności teoria kohomologii wyznaczona przez spektrum sfer jest teorią multyplikatywną a odwzorowanie opisane w Ex. 4 zadaje na teorii kohomologii wyznaczonej przez dowolne spektrum $\mathbb{E}$ strukturę π^*_S-modułu.

W stabilnej teorii kohomotopii zachodzi spektakularne:

Twierdzenie (G. Nishida). *Jeśli $q>0$ to dowolny element $x\in\tilde{\pi}_q(S^0)$ jest nilpotentny.*

Zauważmy, że nilpotentność elementu $f\colon S^{k+q} \to S^k$ oznacza, że dla pewnego n odwzorowanie $f\wedge ...\wedge f\colon S^{k+q}\wedge ...\wedge S^{k+q} \to S^k\wedge ...\wedge S^k$ jest stabilnie homotopijne z odwzorowaniem stałym.

Zad. 3.3.1 (do kontemplacji)**.** Sformułować precyzyjnie warunki jakie musi spełniać odwzorowanie $\mathbb{E}\wedge\mathbb{F}\to\mathbb{G}$ aby działania w (ko-)homologiach przez nie wyznaczone spełniały algebraiczne własności struktury multyplikatywnej takie jak w singularnej teorii kohomologii (np. antyprzemienność) (p. Spanier, [Adams]).

Twierdzenie (a la Kunneth)**.** *Jeśli $E^*()$jest multyplikatywną teorią kohomologii a Y jest przestrzenią taką, że $E^*(Y)$ jest płaskim E^*-modułem to dla dowolnego skończonego CW-kompleksu X iloczyny zadają izomorfizmy:*

$$\times\colon \tilde{\mathbb{E}}^*(X)\otimes_{\mathbb{E}^*}\tilde{\mathbb{E}}^*(Y) \xrightarrow{htp} \tilde{\mathbb{E}}^*(X\wedge Y)$$

$$\times\colon \tilde{\mathbb{E}}_*(X)\otimes_{\mathbb{E}^*}\tilde{E}_*(Y) \xrightarrow{htp} \tilde{\mathbb{E}}_*(X\wedge Y).$$

Proof. Te homomorfizmy są transformacjami teorii (ko-) homologii, które są izomorfizmami dla $X=S^0$, a więc dla dowolnego skończonego CW-kompleksu. □

3.4 The Thom Spectra

Definition 3.4.1. *Let $\mathbb{F}=\mathbb{R},\mathbb{C}$. A $\mathbb{F}$- vector bundle spectrum (VBS) $\xi=(\xi_n,\epsilon_n)$ consists of a sequence of vector bundles ξ_n over spaces B_n (defined for $n>k_0$) and pull-back diagrams (or at least inclusions):*

$$\begin{tikzcd} \xi_n\oplus\theta^1_{\mathbb{F}} \arrow[r, "\widetilde{\varepsilon}_n"] \arrow[d] & \xi_{n+1} \arrow[d] \\ B_n \arrow[r, "\varepsilon_n"] & B_{n+1} \end{tikzcd} \tag{3.5}$$

where $\theta^1_{\mathbb{F}}$ denotes one dimensional trivial vector bundle and $\widetilde{\varepsilon}_n$ is an isomorphism on fibers. We define $\dim\xi =: \dim\xi_k - k$.

A definition of maps between spectra of vector bundles and their homotopy classes is analogous to those for spectra of complexes. It is convenient to consider VBS of dimension 0 i.e. such that their k-stage is a k-dimensional vector bundle. Every vector bundle η defines a 0-dimensional spectrum.

Let η , ξ be 0-dimensional spectra of vector bundles. We denote by $[\eta, \xi]$ the set of homotopy classes of maps $\eta \to \xi$. We write $\mathcal{SV}_h^0$ for the homotopy category of 0-dimensional spectra of vector bundles. There is a natural functor from the homotopy category of vector bundles to the homotopy category of spectra : $\mathcal{V}_h \to \mathcal{SV}_h$ associating with a bundle η a spectrum $\{\eta\}_k := \eta \oplus \theta^{k-\dim \eta}$, $k \geq \dim \eta$. We extend the functor $\theta : \mathcal{V} \to \mathcal{V}$, $\theta(\eta) = \eta \oplus \theta^1$, $\theta(f) = f \oplus \mathrm{id}$ to the category $\mathcal{SV}_h$ putting $[\theta(\xi)]_k = \xi_{k-1} \oplus \theta^1$. The following diagram commutes:

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\theta"] \arrow[d] & \mathcal{V} \arrow[d] \\ \mathcal{SV}_h \arrow[r, "\theta"] & \mathcal{SV}_h. \end{tikzcd} \tag{3.6}$$

Definition 3.4.2. *If η is a n-dimensional vector bundle and ξ is a 0-dimensional spectrum of vector bundles we define:*

$$\{\eta, \xi\} := \operatorname{colim}_k [\eta \oplus \theta^k, \xi_{n+k}].$$

A stable ξ-structure on η is an element of $\{\eta, \xi\}$ i.e. an equivalence class of morphisms $\eta \oplus \theta^k \to \xi_{n+k}$. We'll also say that η is ξ-oriented.

We describe in the above terms the stable reduction of the structural group of the vector bundle. Assume that the group homomorhisms $G_n \to GL_n(\mathbb{F})$ are defined in such way that the following diagram commutes:

$$\begin{tikzcd} G_n \arrow[r] \arrow[d] & GL_n(\mathbb{F}) \arrow[d] \\ G_{n+1} \arrow[r] & GL_{n+1}(\mathbb{F}) \end{tikzcd} \tag{3.7}$$

This sequence defines the spectrum of vector bundles: $\gamma(G) = \{i_h^* \gamma_n, i^* \epsilon_n\}$, where $i_n \colon BG_n \to BGL_n(\mathbb{F})$ and γ_n denotes a classifying bundle over $BGL_n(\mathbb{F})$. The stable reduction of the group of the n-dimensional bundle η to the group G is the $\gamma(G)$ structure on η.

Consider now the Thom space functor from the homotopy category of vector bundles to the homotopy category of pointed spaces: $\mathcal{V} \xrightarrow{\mathrm{Th}} \mathcal{S}p$. We use the notation $\mathrm{Th}(\xi) = B^\xi$ where ξ is a bundle over B.

Theorem 3.4.1. *There exists an extension of the Thom-space functor from the category of vector bundles to the category of spectra of vector bundles i.e. there exists a commutative diagram:*

$$\begin{tikzcd} \mathcal{V}_h \arrow[r, "\mathrm{Th}"] \arrow[d] & \mathcal{T}op_h \arrow[d] \\ \mathcal{SV}_h \arrow[r, "\mathrm{Th}"] & \mathcal{S}_h \end{tikzcd} \tag{3.8}$$

Proof. We define $\mathrm{Th}(\xi)_k = \mathrm{Th}(\xi_k)$. If $\epsilon_k \colon \xi_k \oplus \theta^1 \to \xi_{k+1}$ then $\Sigma \,\mathrm{Th}(\xi_k) = \mathrm{Th}(\xi_k \oplus \theta^1) \xrightarrow{\mathrm{Th}\, \epsilon_k} \mathrm{Th}\, \xi_{k+1}$. $\square$

3.5 The stable normal bundle

Let V be a n-dimensional manifold. An embedding $i\colon V \to \mathbb{R}^{n+l}$ composed with the natural embeddings of the euclidean spaces defines a sequence of embeddings $i_k : V \to \mathbb{R}^{n+k}$ defined for $k \geq l$. For the normal bundles the following isomorphism holds: $\nu(i_k) \simeq \nu(i) \oplus \theta^{k-l}$ thus $\dim \nu(i_k) = k$.

We define the bundle spectrum $\nu^S(i) := \theta^l(\{\nu(i)\})$ i.e. $\nu^S(i)_k := \nu(i_k)$ dla $k \geq l$ and call it the stable normal bundle. Note that $\dim \nu^S(i) = 0$.

The stable normal bundle is defined by the manifold V uniquely up to stable isomorphism. For any isotopy $F : V \times \mathbb{R} \to \mathbb{R}^q$ between proper embeddings $i_0, i_1 : V \to \mathbb{R}^q$ the normal bundle of $F\colon V \times \mathbb{R} \to \mathbb{R}^q \times \mathbb{R}$, $\widetilde{F}(v,t) = (F(v,t), t)$ defines a homotopy class of isomorphism $\nu(i_0) \simeq \nu(i_1)$. If q is sufficiently large (as in Theorem 1.3.2) then any two isotopies between i_0 and i_1, being themselves isotopic through isotopies, define the same homotopy class of isomorphism $\nu(i_0) \simeq \nu(i_1)$.

Corollary 3.5.1. *Let $i : V \to \mathbb{R}^p$ and $j : V \to \mathbb{R}^q$ be embeddings. Then there is a canonical stable homotopy class of isomorphisms of stable normal bundles $\nu^S(i) \to \nu^S(j)$.*

The class of all stable normal bundles of embeddings of V in $\mathbb{R}^q$ will be denoted by $-\tau(V)$.

Definition 3.5.1. *Let V be an n-dimensional manifold and ξ be a spectrum of vector bundles, $\dim \xi = 0$. A ξ-structure on $-\tau(V)$ is family of ξ-structures on stable normal bundles to all embeddings of V, such that for every $\nu^S(i) \to \xi$, $\nu^S(j) \to \xi$, and the canonical homotopy class of isomorphisms $\nu^S(i) \to \nu^S(j)$ the following diagram commutes in $\mathcal{SV}_h$:*

$$\begin{tikzcd} \nu^S(i) \arrow[dd] \arrow[dr] & \\ & \xi \\ \nu^S(j) \arrow[ur] & \end{tikzcd} \tag{3.9}$$

Now consider a more general situation. $f\colon V \to Y$ a proper smooth map and $n = \dim V$. Let $p\colon E \to Y$ be a smooth vector l-dimensional bundle, thus $\dim E = n + l$. We consider liftings of f with respect to p:

$$\begin{tikzcd} & E \arrow[d, "p"] \\ V \arrow[ur, "i_l"] \arrow[r, "f"] & Y \end{tikzcd} \tag{3.10}$$

such that i_l is an embedding (note that i_l must be proper). Note that any two liftings of f are homotopic (not necessary via emebeddings!).

As before every embedding i_l determines a bundle spectrum such that $\nu^S(i_l)_k = \nu(i_k)$ for $k \geq l$ where i_k is the composition $V \xrightarrow{i_l} E \hookrightarrow E \oplus \theta^{k-l}$.

Proposition 3.5.1. *Let*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ V \arrow[ur, shift left, "i_l"] \arrow[ur, shift right, "i'_l"'] \arrow[r, "f"] & Y \end{tikzcd} \tag{3.11}$$

be two embedding-liftings the same map f. Then there is a canonical homotopy class of isomorphism $\nu^S(i_l) \to \nu^S(i'_l)$.

Outline of the proof. A fiberwise homotopy between i_l and i'_l gives us a map: $H : V \times \mathbb{R} \to E \oplus \theta^1$ (note that we can choose a proper homotopy.) According to the Whitney theorem we can take an embedding $\widetilde{H} : V \times \mathbb{R} \to E \oplus \theta^1$ (we assume that $\dim E$ is bog enough, or add more trivial summands) arbitrarily close to H (and thus homotopic to H), such that $\widetilde{H}|_{V\times\{i\}} = H|_{V\times\{i\}}, i = 0, 1$. The normal bundle to $\widetilde{H}$ gives the required isomorphism determined by isotopies $H\colon V \times \mathbb{R} \to E \oplus \theta^1$ for which the following diagram commutes homotopically:

$$\begin{tikzcd} & E \oplus \theta^1 \arrow[d, "p"] \\ V \times \mathbb{R} \arrow[ur, "H"] \arrow[r, "f\times id"'] & Y \times \mathbb{R} \end{tikzcd} \tag{3.12}$$

To establish the uniqueness of the homotopy class of such isomorphism we have to repeat the previous argument. □

The last proposition allows us to define a stable normal bundle of a map f with respect to p. We'll denote it $\nu^S(f, p)$. If E is a trivial bundle, then we denote $\nu^S(f)$ for short and call is stable normal bundle to f.

Proposition 3.5.2. *Suppose $i : V \to \mathbb{R}^l$ is an embedding and $f : V \to Y$ a proper map. The diagonal $(f, i)\colon V \to Y \times \mathbb{R}^l$ is then an embedding -lifting' of f whose normal bundle satisfies the equality:*

$$\nu(f, i) \oplus \theta^l = f^*\tau(Y) \oplus \nu(i) \oplus \theta^l.$$

Proof.

$$\nu(f, i) \oplus \theta^l = \nu(f, i) \oplus \tau(V) \oplus \nu(i) = (f^*\tau(Y) \oplus \theta^l_V)/\tau(V) \oplus \tau(V) \oplus \nu(i) = f^*\tau(Y) \oplus \nu(i) \oplus \theta^l_V.$$

□

For this reason the class of the stable normal bundles $\nu^S(f)$ to the map f is also denoted by $f^*\tau(Y) - \tau(V)$, where $\tau(-)$ denotes the stable tangent bundle of a manifold. Definition of ξ-structure on $\nu(f)$ is analogous to Def. 3.5.1.

Definition 3.5.2. *ξ-orientation of f is a ξ-structure on $\nu^S(f)$.*

Proposition 3.5.3. *Let $f : V \to Y$ and $g : V_1 \to Y$ be proper maps and suppose that there is a diffeomorphism $\varphi : V \to V_1$ over Y. The diffeomorphism φ induces a canonical isomorphism of the stable normal bundles $\nu^S(f) \to \nu^S(g)$.*

Proof. Let $i : V \to Y \times \mathbb{R}^q$, $\quad j : V \to Y \times \mathbb{R}^q$ be embeddings lifting f and g respectively. The canonical homotopy class of isomorphism $\nu^S(i) \to \nu^S(j)$ is the composition of the canonical homotopy class of isomorphism $\nu^S(i) \to \nu^S(j\varphi)$ with an isomorphism $\nu^S(j\varphi) \to \nu^S(j)$ (induces by φ in obvious way). □

Chapter 4

The Spanier-Whitehead duality

4.1 Dwoistość Spaniera-Whiteheada[1]

Poniżej znajdują się rozwiązania zadań z rozdziału 8. (seria F) w [15]. Będziemy zakładać, że wszystkie rozpatrywane przestrzenie są skończonymi CW-kompleksami z punktami bazowymi. Niech X oraz X^* będą takimi kompleksami.

Definition 4.1.1. Dualnością n-wymiarową *nazwiemy element* $u \in \{X^* \wedge X,\ S^0\}_{-n}$ *spełniający dwa następujące warunki:*

- *przekształcenie, które odwzorowuje element* $\{\alpha\} \in \{S^0,\ X^*\}_q \cong \{S^q,\ X^*\}$ *na element* $u \circ (\{\alpha\} \wedge \{1_X\}) \in \{S^q \wedge X,\ S^0\}_{-n} \cong \{X,\ S^0\}_{q-n}$ *jest izomorfizmem*
$$D_u\colon \{S^0,\ X^*\}_q \cong \{X,\ S^0\}_{q-n};$$
- *przekształcenie, które odwzorowuje element* $\{\beta\} \in \{S^0,\ X\}_q \cong \{S^q,\ X\}$ *na element* $u \circ (\{1_{X^*}\} \wedge \{\beta\}) \in \{X^* \wedge S^q,\ S^0\}_{-n} \cong \{X^*,\ S^0\}_{q-n}$ *jest izomorfizmem*
$$D^u\colon \{S^0,\ X\}_q \cong \{X^*,\ S^0\}_{q-n}.$$

Remark 5. Note that $\{X,\ S^0\}_{-n} = \pi^n_S(X)$ and $\{S^0,\ X\}_n = \pi^S_n(X)$ are the stable (co-)homotopy groups and the duality homomorphsims D_u, D^u are slant-product with u and u composed with interchange of the factors. In these terms the above definition is equivalent to the following: A stable homotopy class $u \in \{X^* \wedge X, S^n,\}$ is the (right) n-duality if the homomorphisms

$$- \backslash u^*(\iota_n) : \tilde{\pi}^S_q(X^*) \longrightarrow \tilde{\pi}^{n-q}_S(X) \quad \text{and} \quad - \backslash u^*(\iota_n) : \tilde{\pi}^S_q(X) \longrightarrow \tilde{\pi}^{n-q}_S(X^*) \tag{4.1}$$

are isomorphism.

Zad. 4.1.1. Udowodnić, że jeżeli przekształcenie $f\colon S^n \wedge S^m \to S^{n+m}$ jest homotopijną równoważnością, to element $\{f\} \in \{S^n \wedge S^m,\ S^0\}_{-n-m}$ jest dualnością $(n+m)$-wymiarową.

Rozwiązanie. Wystarczy zauważyć, że w tym przypadku przekształcenie $D_{\{f\}}$ jest złożeniem izomorfizmów
$$\{S^q,\ S^n\} \cong \{S^q \wedge S^m,\ S^n \wedge S^m\} \xrightarrow{\cong} \{S^q \wedge S^m,\ S^{n+m}\} \cong \{S^m,\ S^{n+m-q}\},$$
gdzie drugie przekształcenie jest indukowane przez izomorfizm $\{f\}$. Identyczny argument działa dla $D^{\{f\}}$. □

[1] Opracował Wojciech Hury

Zad. 4.1.2. Udowodnić, że jeżeli element $u \in \{X^* \wedge X,\ S^0\}_{-n}$ jest dualnością n-wymiarową, to element $u' \in \{X \wedge X^*,\ S^0\}_{-n}$ odpowiadający elementowi u przy homeomorfizmie $s\colon X \wedge X^* \to X^* \wedge X$ jest także dualnością n-wymiarową.

Rozwiązanie. Dla dowolnego q rozpatrzmy homeomorfizm $s'\colon S^q \wedge X^* \to X^* \wedge S^q$ i dwa izomorfizmy

$$t\colon \{X^* \wedge S^q,\ X^* \wedge X\} \xrightarrow{\cong} \{S^q \wedge X^*,\ X \wedge X^*\}, \quad t'\colon \{S^q \wedge X^*,\ S^n\} \xrightarrow{\cong} \{X^* \wedge S^q,\ S^n\},$$

gdzie $t(f) = s^{-1} \circ f \circ s'$ oraz $t'(g) = g \circ s'$. W następującym przemiennym diagramie

$$\begin{tikzcd}
& \{S^q \wedge X^*,\ X \wedge X^*\} \arrow[r, "u' \circ -"] & \{S^q \wedge X^*,\ S^n\} \\
\{S^q,\ X\} \arrow[ur, "- \wedge X^*"] \arrow[dr, "X^* \wedge -"'] & & \\
& \{X^* \wedge S^q,\ X^* \wedge X\} \arrow[r, "u \circ -"] \arrow[uu, "\cong"', "t"] & \{X^* \wedge S^q,\ S^n\} \arrow[uu, "\cong"', "t'"]
\end{tikzcd}$$

pójście górą odpowiada przekształceniu $D_{u'}$, natomiast pójście dołem odpowiada przekształceniu D^u. Zatem $D_{u'}$ jest izomorfizmem. Analogiczny argument działa dla $D^{u'}$. $\square$

Zad. 4.1.3. Załóżmy, że $u \in \{X^* \wedge X,\ S^0\}_{-n}$ jest dualnością n-wymiarową. Udowodnić, że dla dowolnych Y, Z można określić izomorfizmy

$$D_u\colon \{Y,\ Z \wedge X^*\}_q \xrightarrow{\cong} \{Y \wedge X,\ Z\}_{q-n},$$
$$D^u\colon \{Y,\ X \wedge Z\}_q \xrightarrow{\cong} \{X^* \wedge Y,\ Z\}_{q-n},$$

przyjmując

$$D_u(\{\alpha\}) = (1_Z \wedge u) \circ (\{\alpha\} \wedge \{1_X\})$$
$$D^u(\{\beta\}) = (u \wedge 1_Z) \circ (\{1_{X^*}\} \wedge \{\beta\}).$$

Rozwiązanie. Dowód będzie przebiegał indukcyjnie po liczbie komórek. Zauważmy najpierw, że teza zadania jest prawdziwa w przypadku, gdy Y i Z są sferami (wynika to wprost z definicji dualności n-wymiarowej). Z określenia przekształceń D_u i D^u wynika, że są one naturalne ze względu na zmienne Y, Z. W kroku indukcyjnym wykorzystamy długi ciąg dokładny grup stabilnych przekształceń dla korozwłóknienia. Załóżmy, że tezę twierdzenia udowodniliśmy dla wszystkich kompleksów Y o liczbie komórek nie przekraczającej k oraz wszystkich kompleksów Z o liczbie komórek nie przekraczającej k'. Weźmy teraz dowolny kompleks Y' o liczbie komórek $k+1$, powstały przez doklejenie komórki p-wymiarowej do pewnego kompleksu Y i dowolny kompleks Z o liczbie komórek nie przekraczającej k'. Pokażemy, że D_u jest izomorfizmem dla Y' oraz Z. Przypomnijmy, że jeśli $A \hookrightarrow W$ jest korozwłóknieniem, to dla dowolnego V mamy długie ciągi dokładne

$$\cdots \to \{W/A,\ V\}_q \to \{W,\ V\}_q \to \{A,\ V\}_q \to \{W/A,\ V\}_{q-1} \to \ldots$$

oraz

$$\cdots \to \{V,\ A\}_q \to \{V,\ W\}_q \to \{V,\ W/A\}_q \to \{V,\ A\}_{q-1} \to \ldots$$

W naszym przypadku $Y \hookrightarrow Y'$ jest oczywiście korozwłóknieniem a $Y'/Y \cong S^p$. Mamy zatem następujący przemienny diagram (przemienność poszczególnych kwadratów wynika z funktorialności D_u)

$$\begin{tikzcd}
\{Y,\ Z \wedge X^*\}_{q+1} \arrow[r] \arrow[d, "\cong"', "D_u"] & \{S^p,\ Z \wedge X^*\}_q \arrow[r] \arrow[d, "\cong"', "D_u"] & \{Y',\ Z \wedge X^*\}_{q-1} \arrow[r] \arrow[d, "D_u"] & {} \\
\{Y \wedge X,\ Z\}_{q+1-n} \arrow[r] & \{S^p \wedge X,\ Z\}_{q-n} \arrow[r] & \{Y' \wedge X,\ Z\}_{q-n} \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \{Y,\ Z \wedge X^*\}_{q-1} \arrow[r] \arrow[d, "\cong"', "D_u"] & \{S^p,\ Z \wedge X^*\}_{q-1} \arrow[d, "\cong"', "D_u"] \\
{} \arrow[r] & \{Y \wedge X,\ Z\}_{q-n} \arrow[r] & \{S^p \wedge X,\ Z\}_{q-1-n},
\end{tikzcd}$$

gdzie dolny ciąg dokładny odpowiada korozwłóknieniu $Y \wedge X \hookrightarrow Y' \wedge X$. W analogiczny sposób dowodzi się tezy dla D^u oraz dla kompleksu Z' powstałego przez doklejenie jednej komórki do pewnego kompleksu Z. □

Dla danych dualności n-wymiarowych $u \in \{X^* \wedge X,\ S^0\}_{-n}$ oraz $v \in \{Y^* \wedge Y,\ S^0\}_{-n}$ możemy tak określić izomorfizm

$$D(u,v)\colon \{X,\ Y\}_q \xrightarrow{\cong} \{Y^*,\ X^*\},$$

by następujący diagram był przemienny

$$\begin{tikzcd}
\{X,\ Y\}_q \arrow[rr, "D(u,v)"] \arrow[dr, "\cong", "D^v"'] & & \{Y^*,\ X^*\}_q \arrow[dl, "\cong"', "D_u"] \\
& \{Y^* \wedge X,\ S^0\}_{q-n}. &
\end{tikzcd}$$

Zad. 4.1.1. Prove that for every multiplicative spectrum an n-duality $u \in \{X^* \wedge X,\ S^0\}_{-n}$ induces isomorphisms $u^*(\iota_n)\colon \tilde{\mathbb{E}}_q(X) \to \tilde{\mathbb{E}}^{n-q}(X^*)$.

Zad. 4.1.4. Udowodnić, że $D(v', u') = (D(u,v))^{-1}\colon \{Y^*,\ X^*\}_q \xrightarrow{\cong} \{X,\ Y\}_q$.

Rozwiązanie. Teza zadania wynika z przemienności następującego diagramu

$$\begin{tikzcd}
& \{Y^* \wedge X,\ S^0\}_{q-n} \arrow[dd, "\cong"] & \\
\{Y^*,\ X^*\}_q \arrow[ur, "D_u"] \arrow[dr, "D^{u'}"'] & & \{X,\ Y\}_q \arrow[ul, "D_v"'] \arrow[dl, "D_{v'}"] \\
& \{X \wedge Y^*,\ S^0\}_{q-n}, &
\end{tikzcd}$$

gdzie pionowa strzałka jest indukowana przez homeomorfizm $\mathbb{R}\colon X \wedge Y^* \to Y^* \wedge X$. □

Zad. 4.1.5. Niech elementy $u \in \{X^* \wedge X,\ S^0\}_{-n}$, $v \in \{Y^* \wedge Y,\ S^0\}_{-n}$, $w \in \{Z^* \wedge Z,\ S^0\}_{-n}$ będą dualnościami n-wymiarowymi i niech $\{\alpha\} \in \{X,\ Y\}_p$ oraz $\{\beta\} \in \{Y,\ Z\}_q$. Udowodnić, że w zbiorze $\{Z^*,\ X^*\}_{p+q}$ zachodzi równość

$$D(u,w)(\{\beta\} \circ \{\alpha\}) = (D(u,v)\{\alpha\}) \circ (D(v,w)\{\beta\}).$$

Rozwiązanie. Oznaczmy $\{\alpha'\} = D(u,v)\{\alpha\}$ oraz $\{\beta'\} = D(v,w)\{\beta\}$. Z definicji przekształceń $D(-,-)$ wynikają następujące tożsamości

$$v \circ (\{1_{Y^*}\} \wedge \{\alpha\}) = u \circ (\{\alpha'\} \wedge \{1_X\}), \tag{4.2}$$
$$w \circ (\{1_{Z^*}\} \wedge \{\beta\}) = v \circ (\{\beta'\} \wedge \{1_Y\}). \tag{4.3}$$

Zauważmy również, że aby dowieść tezy, należy pokazać równość

$$w \circ (\{1_{Z^*}\} \wedge (\{\beta\} \circ \{\alpha\})) = u \circ ((\{\alpha'\} \circ \{\beta'\}) \wedge \{1_X\}).$$

Sprawdźmy, że tak jest:

$$\begin{aligned}
w \circ (\{1_{Z^*}\} \wedge (\{\beta\} \circ \{\alpha\})) &= w \circ (\{1_{Z^*}\} \wedge \{\beta\}) \circ (\{1_{Z^*}\} \wedge \{\alpha\}) \\
&= v \circ (\{\beta'\} \wedge \{1_Y\}) \circ (\{1_{Z^*}\} \wedge \{\alpha\}) && \text{(z (4.3))} \\
&= v \circ (\{\beta'\} \wedge \{\alpha\}) \\
&= v \circ (\{1_{Y^*}\} \wedge \{\alpha\}) \circ (\{\beta'\} \wedge \{1_X\}) \\
&= u \circ (\{\alpha'\} \wedge \{1_X\}) \circ (\{\beta'\} \wedge \{1_X\}) && \text{(z (4.2))} \\
&= u \circ ((\{\alpha'\} \circ \{\beta'\}) \wedge \{1_X\}).
\end{aligned}$$

□

Niech $f\colon X^* \wedge X \to S^n$ oraz $g\colon Y^* \wedge Y \to S^n$ będą takimi przekształceniami, że $\{f\}$ i $\{g\}$ są n-wymiarowymi dualnościami i niech $\alpha\colon X \to Y$, $\beta\colon Y^* \to X^*$ spełniają

$$f \circ (\beta \wedge 1_X) \simeq g \circ (1_{Y^*} \wedge \alpha)\colon Y^* \wedge X \to S^n \tag{4.4}$$

(skąd wynika, że $D(\{f\},\{g\})\{\alpha\} = \{\beta\}$). Niech $C(\alpha)$, $C(\beta)$ będą odpowiednio stożkami przekształceń α i β. Rozpatrzmy ciągi kodokładne

$$X \xrightarrow{\alpha} Y \xrightarrow{i} C(\alpha) \xrightarrow{k} \Sigma X \xrightarrow{\Sigma\alpha} \Sigma Y, \tag{4.5}$$
$$Y^* \xrightarrow{\beta} X^* \xrightarrow{i'} C(\beta) \xrightarrow{k'} \Sigma Y^* \xrightarrow{\Sigma\alpha} \Sigma X^*. \tag{4.6}$$

Zad. 4.1.6. Udowodnić, że istnieje takie przekształcenie $h\colon C(\beta) \wedge C(\alpha) \to S^{n+1}$, że następujące diagramy są przemienne z dokładnością do homotopii

$$\begin{tikzcd}
X^* \wedge C(\alpha) \arrow[r, "1\wedge k"] \arrow[d, "i'\wedge 1"'] & X^* \wedge \Sigma X \arrow[r, Leftrightarrow] & \Sigma(X^* \wedge X) \arrow[d, "\Sigma f"] \\
C(\beta) \wedge C(\alpha) \arrow[rr, "h"] & & S^{n+1}
\end{tikzcd} \tag{4.7}$$

$$\begin{tikzcd}
C(\alpha) \wedge Y \arrow[r, "k'\wedge 1"] \arrow[d, "1\wedge i"'] & \Sigma Y \wedge Y \arrow[r, Leftrightarrow] & \Sigma(Y^* \wedge Y) \arrow[d, "\Sigma g"] \\
C(\beta) \wedge C(\alpha) \arrow[rr, "h"] & & S^{n+1}
\end{tikzcd} \tag{4.8}$$

Wywnioskować stąd, że element $\{h\} \in \{C(\beta) \wedge C(\alpha),\ S^0\}_{-n-1}$ jest (n+1)-wymiarową dualnością.

Rozwiązanie. Niech $H_t\colon Y^*\wedge X \to S^n$ będzie homotopią od $H_0 = f\circ(\beta\wedge 1_X)$ do $H_1 = g\circ(1_{Y^*}\wedge\alpha)$. Przekształcenie h określamy następująco

$$\begin{aligned} h(x^*, y) &= *, \\ h((y^*, t), y) &= \Sigma g(y^*, y, t), \\ h(x^*, (x, t)) &= \Sigma f(x^*, x, t), \\ h((y^*, t_1), (x, t_2)) &= \Sigma H_{\frac{t_1}{t_1+t_2}}(y^*, x, \max(t_1, t_2)). \end{aligned}$$

Pozostawiamy czytelnikowi sprawdzenie, że przekształcenie h jest ciągłe (diagramy 4.7 i 4.8 są przy tak określonym h przemienne punktowo a nie tylko z dokładnością do homotopii).

Pokażemy teraz, że $D_{\{h\}}$ jest izomorfizmem. Rozpatrzmy następujący diagram

$$\begin{tikzcd}
\{S^q,\ Y^*\} \arrow[r, "D_{\{g\}}", "\cong"'] \arrow[d, "\beta_*"'] & \{S^q\wedge\Sigma Y,\ S^{n+1}\} \arrow[r, "\cong"'] \arrow[d, "(1_{S^q}\wedge\Sigma\alpha)^*"'] & \{\Sigma Y,\ S^{n+1-q}\} \arrow[d, "\Sigma\alpha^*"'] \\
\{S^q,\ X^*\} \arrow[r, "D_{\{f\}}", "\cong"'] \arrow[d, "i'_*"'] & \{S^q\wedge\Sigma X,\ S^{n+1}\} \arrow[r, "\cong"'] \arrow[d, "(1_{S^q}\wedge k)^*"'] & \{\Sigma X,\ S^{n+1-q}\} \arrow[d, "k^*"'] \\
\{S^q,\ C(\beta)\} \arrow[r, "D_{\{h\}}"] \arrow[d, "k'_*"'] & \{S^q\wedge C(\beta),\ S^{n+1}\} \arrow[r, "\cong"'] \arrow[d, "(1_{S^q}\wedge i)^*"'] & \{C(\beta),\ S^{n+1-q}\} \arrow[d, "i^*"'] \\
\{S^q,\ \Sigma Y^*\} \arrow[r, "D_{\{g\}}", "\cong"'] \arrow[d, "\Sigma\beta_*"'] & \{S^q\wedge Y,\ S^{n+1}\} \arrow[r, "\cong"'] \arrow[d, "(1_{S^q}\wedge\alpha)^*"'] & \{Y,\ S^{n+1-q}\} \arrow[d, "\alpha^*"'] \\
\{S^q,\ \Sigma X^*\} \arrow[r, "D_{\{f\}}", "\cong"'] & \{S^q\wedge X,\ S^{n+1}\} \arrow[r, "\cong"'] & \{X,\ S^{n+1-q}\},
\end{tikzcd}$$

w którym kolumny są dokładne – lewa powstaje przez przyłożenie funktora $\{S^q,\ -\}$ do ciągu (4.6) a prawa przez przyłożenie funktora $\{-,\ S^{n+1-q}\}$ do ciągu (4.5) – oraz kwadraty po prawej stronie są przemienne. Aby pokazać, że $D_{\{h\}}$ jest izomorfizmem wystarczy pokazać przemienność kwadratów w lewej części diagramu. Pokażemy przemienność dwóch górnych kwadratów – dla dolnych argumenty są analogiczne. Niech $\gamma\in\{S^q,\ Y^*\}$.

$$\begin{aligned} (1_{S^q}\wedge\Sigma\alpha)^*(D_{\{g\}}(\gamma)) &= \Sigma g\circ(\gamma\wedge 1_{\Sigma Y})\circ(1_{S^q}\wedge\Sigma\alpha) \\ &= \Sigma(g\circ(\gamma\wedge 1_Y)\circ(1_{S^q}\wedge\alpha)) \\ &= \Sigma(g\circ(1_{Y^*}\wedge\alpha)\circ(\gamma\wedge 1_X)) \\ &= \Sigma(f\circ(\beta\wedge 1_X)\circ(\gamma\wedge 1_X)) && \text{(z (4.4))} \\ &= \Sigma(f\circ((\beta\circ\gamma)\wedge 1_X)) \\ &= \Sigma f\circ((\beta\circ\gamma)\wedge 1_{\Sigma X}) \\ &= D_{\{f\}}(\beta_*(\gamma)). \end{aligned}$$

Dla drugiego kwadratu przyjmijmy $\eta \in \{S^q,\ X^*\}$. Mamy

$$\begin{aligned}(1_{S^q} \wedge k)^*(D_{\{f\}}(\eta)) &= \Sigma f \circ (\eta \wedge 1_{\Sigma X}) \circ (1_{S^q} \wedge k) \\ &= \Sigma f \circ (1_{X^*} \wedge k) \circ (\eta \wedge 1_{C(\alpha)}) \\ &= h \circ (i' \wedge 1_{C(\alpha)}) \circ (\eta \wedge 1_{C(\alpha)}) && \text{(z (4.7))} \\ &= h \circ ((i' \circ \eta) \wedge 1_{C(\alpha)}) \\ &= D_{\{h\}}(i'_*(\eta)).\end{aligned}$$

Dowód, że $D^{\{h\}}$ jest izomorfizmem przebiega analogicznie. □

Zad. 4.1.7. Niech X będzie dowolnym CW-kompleksem. Wykazać, że wówczas można znaleźć liczbę całkowitą n, dla której istnieje przestrzeń X^* oraz n-wymiarowa dualność $u \in \{X^* \wedge X,\ S^0\}_{-n}$.

Rozwiązanie. Będziemy postępować indukcyjnie. Zauważmy, że z zadania 4.1.1 dostajemy tezę dla sfer. Załóżmy teraz, że kompleks Y powstaje przez doklejenie komórki $(m+1)$-wymiarowej (przekształceniem $\alpha\colon S^m \to X$) do kompleksu X, dla którego mamy dualność $u \in \{X^* \wedge X,\ S^0\}_{-n}$. Rozpatrzmy dualność n-wymiarową $v \in \{S^{n-m} \wedge S^m,\ S^0\}_{-n}$ i element $\{\beta\} = D(u,v)\{\alpha\} \in \{X^*,\ S^{n-m}\}$. Oczywiście $Y \cong C(\alpha)$ i wystarczy teraz skorzystać z zadania 4.1.6, żeby dostać dualność $n+1$-wymiarową

$$\{h\} \in \{C(\beta) \wedge Y,\ S^0\}_{n+1}.$$

□

4.2 Homological Duality and SW-duality

Let h^* and h_* be respectively cohomology and homology theories assciated to a multiplicative spectrum. However we'll never use explicit definition, we need various properties of products.

Definition 4.2.1. *An h^*-orientation of an n-dimensional topological manifold M is a class $U_M \in h^n(M \times M, M \times M \setminus \Delta)$ such that for each $x \in M$ its restriction $U_M|x \in h^n(M, M \setminus \{x\}) \simeq \tilde{h}^n(S^n)$ equals $\pm\sigma^n(1)$. A maniofld which has an h^*-orientation is called h^*-orientable.*

Theorem 4.2.1. *For any h^*-orientable n-dimensional topological manifold M and pair of closed subsets $B \subset A \subset M$ the slant product with $U_M|(A,B) \times (M \setminus B, M \setminus A)$ defines a duality isomorphism*

$$h_q(M \setminus B, M \setminus A) \xrightarrow{\simeq} \bar{h}^{n-q}(A,B)$$

where $\bar{h}^*(A,B) := \operatorname{colim}_{(U,V)\supset(A,B)} h^*(U,V)$ and $(U,V) \supset (A,B)$ is a poset of open neighborhoods.

Proposition 4.2.1. *If M is a smooth manifold then there is a bijective correspondence between its h^*-orientations and h^*-Thom classes of it is tangent bundle.*

Proof. The normal bundle of the diagonal $\Delta \subset M \times M$ is naturally isomorphic to the tangent bundle of M. Conclusion follows easily from the tubular neighborhood theorem 1.2.1. □

Proposition 4.2.2. *If E', E'' are vector bundles over same base space and $E := E' \oplus E''$. Assume that two of three bundles has an h^*-Thom class. Then the third one also has an h^*-Thom class such that $U_E = U_{E'}U_{E''}$.*

Proof. ADD □

Corollary 4.2.1. *Stably parallelizable manifolds, in particular spheres, are h^*-orientable in any cohomology theory.*

Proof. A manifold M^n is stably parallelizable if there exists trivial bundles such that $\theta^{n+k} = TM \oplus \theta^k$. Since trivial bundles are orientable in any cohomology theory, conclusion follows from Prop. 4.2.2. For any sphere $\theta^{n+1} = TS^n \oplus \theta^1$. $\square$

Zad. 4.2.1. Define explicitly a π_S^*-orientation of sphere in $\pi_S^n(S^n \times S^n, S^n \times S^n \setminus \Delta)$ i.e. a suitable stable map $(S^n \times S^n, S^n \times S^n \setminus \Delta) \to (S^n, S^n \setminus \{1\})$.

Corollary 4.2.2 (Alexander Duality)**.** *For any cohomology theory h^* and a closed subset $A \subset S^{n+1}$ there is an isomorphism*

$$h_{q+1}(S^{n+1}, S^{n+1} \setminus A) \xrightarrow{\simeq} \bar{h}^{n-q}(A)$$

which defines an isomorphism:

$$\tilde{h}_q(S^{n+1} \setminus A) \xrightarrow{\simeq} \bar{\tilde{h}}^{n-q}(A)$$

Proof. The first isomorphism follows directly from Thm. 4.2.1 and Prop. 4.2.1. The second from decompositions: $h_{q+1}(S^{n+1}, S^{n+1} \setminus A) \simeq \tilde{h}_q(S^{n+1} \setminus A) \oplus \tilde{h}_{q+1}(S^{n+1})$ and $\bar{h}^{n-q}(A) \simeq \bar{\tilde{h}}^{n-q}(A) \oplus \tilde{h}^{n-q}(S^0)$. Moreover $\tilde{h}_{q+1}(S^{n+1}) \simeq \tilde{h}_{q-n}(S^0) = h^{n-q}(S^0)$ and the duality map preserves the decomposition. $\square$

In order to relate the Alexander duality to the Spanier-Whitehead duality we need to look closer at the duality homomorphisms. They are defined by the slant products for which the following diagram commutes in an appropriate sense. (We denote $S := S^{n+1}$ for short). Assume that $A^* \subset S \setminus A$ is a compact polyhedron (or more generally an NDR) such that inclusion is a homotopy equivalence.

$$\begin{tikzcd}
h^{n+1}(S \times S, S \times S \setminus \Delta) \otimes h_{q+1}(S, A^*) \arrow[r] & h^{n+1}(S \times A, A \times A^*) \otimes h_{q+1}(S, A^*) \arrow[r, "/"] \arrow[d, bend left, "\partial"] & h^{n-q}(A) \\
h^n(S \times S \setminus \Delta) \otimes h_{q+1}(S, A^*) \arrow[u, "id\otimes\delta"] \arrow[r] & h^n(A \times A^*) \otimes h_q(A^*) \arrow[r, "/"] \arrow[u, bend left, "\delta"] \arrow[d, bend left] & h^{n-q}(A) \arrow[u, "="'] \\
& \tilde{h}^n(A \wedge A^*) \otimes \tilde{h}_q(A^*) \arrow[r, "/"] \arrow[u, bend left] & \tilde{h}^{n-q}(A) \arrow[u]
\end{tikzcd} \tag{4.9}$$

where ∂, δ denote boundary homomorphism in the relevant long exact sequences. The inclusion $S \setminus A \subset S$ is not in general a cofibration, thus we understand $\tilde{h}^n(A \wedge (S \setminus A)) = \tilde{h}^n(S \times A, S \times \{a_0\} \cup \{1\} \times (S \setminus A))$.

Proposition 4.2.3. *If $u \in \tilde{h}^n(A \wedge (A^*))$ is an element such that $\delta(u_A) = U_S|(S \times A, A \times A^*))$ then the homomorphism $u_A/-\colon \tilde{h}_q(A^*) \to \tilde{h}^{n-q}(A)$ is an isomorphism.*

Proof. Conclusion follows from the above diagram and the duality theorem. $\square$

To construct the element u_A choose a point $pt \in S \setminus (A \cup A^*)$ and identify the space $S \setminus \{pt\} = \mathbb{R}^{n+1}$. Consider the map $u\colon A \times A^* \to S^n$ given by the formula $u(x,y) := \frac{x-y}{||x-y||}$.

Proposition 4.2.4. *If the sets A, A^* are connected then restriction $u|A \vee A^*$ is null-homotopic thus u defines a map $u\colon A \wedge A^* \to S^n$*

Proof. The Alexander duality in singular cohomology give us an isomorphism $\tilde{H}_q(A) \simeq \tilde{h}^{n-q}(A^*)$ and the other way around. Hence $h^q(A) = h^q(A^*) = 0$ for $q \geq n$ and $[A, S^n] = [A^*, S^n] = 0$. $\square$

Proposition 4.2.5. *For any cohomology theory h^* the equality $\delta u^*(\iota_n) = U_S|(S \times A, A \times A^*))$ where U_S is an orientation of S.*

Proof. Conclusion follows from the following commutative diagram. (We denote $j \colon S \setminus \{pt\} \subset S$ and as usual we identify $S \setminus \{pt\}$ with $\mathbb{R}^{n+1}$) :

$$\begin{tikzcd}
h^{n+1}(S \times S, S \times S \setminus \Delta) \arrow[d, "(j\times j)^*"] \arrow[dr] & \\
h^{n+1}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \Delta) \arrow[r] & h^{n+1}(A \times S, A \times A^*) \\
\tilde{h}^n(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \Delta) \arrow[u, "\delta", "\simeq"'] \arrow[r] & \tilde{h}^n(A \times A^*) \arrow[u, "\delta"] \\
\tilde{h}^n(S^n) \arrow[u, "u^*", "\simeq"'] \arrow[ur, "u^*"] &
\end{tikzcd} \tag{4.10}$$

The key observation is that the element $(j \times j)^* U_S$ is an h^*-orientation of $\mathbb{R}^n$ and the map $u \colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \Delta \to S^n$ is a homotopy equivalence. $\square$

Corollary 4.2.3. *For any pair A, A^* of closed, connected disjoint subpolyhedra such that inclusion $A^* \subset S^{n+1} \setminus A$ is a homotopy equivalence the map $u \colon A \wedge A^* \to S^n$ is a SW-duality map.*

Proof. Lat proposition and the Alexander duality imply that $u^*(\iota_n)/- \colon \tilde{h}_q(A^*) \xrightarrow{\simeq} \tilde{h}^{n-q}(A)$ is an isomorphism. We have to argue for the symmetric statement i.e. that $u^*(\iota_n)/- \colon \tilde{h}_q(A) \xrightarrow{\simeq} \tilde{h}^{n-q}(A^*)$ is an isomorphism. It is enough to show that the inclusion $A \subset S \setminus A^*$ is a h^*-homology isomorphism, what follows from the Alexander duality. After suspension it will become a homotopy equivalence (suspension of a connected space is simply connected). $\square$

Chapter 5

The Pontriagin - Thom theorem

5.1 The construction

Consider the following situation. Let $\eta = (p\colon E \to Y)$ be a smooth vector bundle and let $i\colon X \to E$ be an embedding such that the composition $\pi = f : X \to Y$ is a proper map.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "i"] \arrow[r, "f"] & Y \end{tikzcd} \tag{5.1}$$

The Thom construction associates with this embedding a homotopy class of maps $Y^\eta \to X^{\nu(i)}$ where Y^η and $X^{\nu(i)}$ denote the Thom spaces of the appropriate bundles.

We will formulate a description of the Thom space in the form convenient for our purposes. Let $\eta = \{p : E \to Y\}$ be a vector bundle over a manifold Y and $Y^\eta = E \cup \{*\}$. We define topology on Y^η by taking as a basis of neighborhoods of $\{*\}$ sets $Y^\eta \backslash A$ where $A \subset E$ is closed and $p|_A : A \to Y$ is proper. The Thom space of a bundle over paracompact space is paracompact.

Zad. 5.1.1. Check that if a base space of a vector bundle is locally compact then the above definition of topology in the Thom space coincides with the usual one.

To describe the Thom construction on embedding $i\colon X \to E$ let us fix a Riemannian metric on the manifold E. This gives us a tubular neighborhood map h defined by the diagram

$$\begin{tikzcd} \nu(i) \simeq TX^\perp \arrow[r, "h"] \arrow[d, "\text{compression}"] & Y^\eta \\ Z \arrow[r, "\exp"] & E \arrow[u, "\subset"'] \end{tikzcd} \tag{5.2}$$

Denote $U := h(\nu(i))$ and assume that closure of U in E is a closed subset of the Thom space X^η - it is possible since tubular nbhd can be chosen arbitrary small, and $i(V) \subset E$ is closed in X^η.

The Riemannian metric on E defines through isomorphism $TX^\perp \simeq \nu(i)$ a metric g on $\nu(i)$. We denote by $D_g\nu(i)$ and $S_g\nu(i)$ the disc bundle and the sphere bundle with respect to this metric. Let

$\mathring{D}_g\nu(i) := D_g\nu(i) - S_g\nu(i)$. We define a quotient map i^* as composition:

$$\begin{tikzcd}
Y^\eta \arrow[rr, "i^*"] \arrow[d] & & X^{\nu(i)} \\
Y^\eta/(Y^\eta \setminus h(\mathring{D}_g\nu(i))) & & D_g\nu(i)/S_g\nu(i) \arrow[ll, "\simeq"'] \arrow[u, "\simeq"]
\end{tikzcd} \tag{5.3}$$

Properness of i is needed to show that the bottom map in the diagram 5.3, which is obviously bijective, is a closed map.

The homotopy class of the map $i^* \colon Y^\eta \to X^{\nu(i)}$ is independent of the choice of tubular neighborhood with respect to a given metric. It is independent also of the choice of a particular metric on E, as any two are homotopic through the Riemannian metrics. Note however that it is essential that we restrict ourselves to the tubular neighborhoods determined by some metric.

Proposition 5.1.1. *Assume that embeddings $i_0, i_1 \colon X \to E$ are isotopic through an isotopy $H \colon X \times \mathbb{R} \to E$. The isotopy defines a homotopy class of isomorphisms of the normal bundles $\nu(i_0) \simeq \nu(i_1)$. If $\widetilde{H} \colon X \times \mathbb{R} \to E \times \mathbb{R}$ is an embedding determined by H then the composition*

$$X \times \mathbb{R} \xrightarrow{\widetilde{H}} E \times \mathbb{R} \xrightarrow{p \times \mathrm{id}} Y \times \mathbb{R}$$

is a proper embedding and it defines a homotopy commutative diagram of the Thom spaces

$$\begin{tikzcd}
& Y^\eta \arrow[dl] \arrow[dr] & \\
X^{\nu(i_0)} \arrow[rr, "\simeq"] & & X^{\nu(i_1)}
\end{tikzcd} \tag{5.4}$$

Proof. Take the product metric on $E \times E$ and apply the Thom construction to $\widetilde{H}$ on each level. This gives the required homotopy. $\square$

Proposition 5.1.2. *If $i_0, i_1 \colon X \to E$ are embeddings which are stably isotopic (i.e. are isotopic after composing with an inclusion $E \subset E \oplus \theta^k$) then the diagram 5.4 commutes in the stable homotopy category.*

Proof. In view of the previous proposition this one follows from Whitney embedding theorem. The fact that the obtained isotopy is proper is guaranteed by the compactness of X. $\square$

Let us consider the following generalization of the situation described above. As before, consider embedding $i : X \to E$ such that the composition

$$\begin{tikzcd}
& E \arrow[d, "p"] & E' \arrow[dl, "p'"] \\
X \arrow[ur, "i"] \arrow[r, "f"] & Y &
\end{tikzcd} \tag{5.5}$$

is proper and suppose $\alpha = (p \colon E' \to Y)$ is another vector bundle.

Theorem 5.1.1. *The Thom construction defines a canonical homotopy class of map $Y^{\eta\oplus\alpha} \to X^{\nu(i)\oplus f^*\alpha}$ which makes the following diagram commutative:*

$$\begin{tikzcd}
Y^{\eta\oplus\alpha} \arrow[rr] \arrow[d, "\Delta"] & & X^{\nu(i)\oplus f^*\alpha} \arrow[d, "\Delta"] \\
Y^{\eta} \wedge Y^{\alpha} \arrow[dr] & & X^{\nu(i)} \wedge X^{f^*\alpha} \arrow[dl, "id\wedge f"] \\
& X^{\nu(i)} \wedge Y^{\alpha} &
\end{tikzcd} \tag{5.6}$$

Proof. Without loss of generality we can assume that $p'\colon E' \to Y$ is a smooth bundle. Let $p''\colon E'' \to Y$ be the Whitney sum $\alpha \oplus \eta$. With $i : X \to E$ and the vector bundle α we associate an embedding $i''\colon X \to E''$ $i''(x) := (i(x), f(x))$ The Thom construction applied to i'' gives us a map $X^{\nu(i')} \to Y^{\alpha\oplus\eta}$.

There is a canonical isomorphism of $\nu(i')$ with $\nu(i) \oplus f^*\alpha$. To establish this recall first that $TE'' = p'^*(TY \oplus (\alpha \oplus \eta))$.

$$\begin{aligned}
&\nu(i'') = i''^* p''^* (TY \oplus (\alpha \oplus \eta))/_{TX} = (i''^* p''^* (TY \oplus \eta) \oplus i''^* p''^* \alpha)/_{TX} \\
&= i^* p^* (TY \oplus \eta) \oplus f^*\alpha/_{TX} = i^* p^* (TY \oplus \eta)/_{TX} \oplus f^*\alpha = \nu(i) \oplus f^*\alpha.
\end{aligned} \tag{5.7}$$

Thus we obtain the desired map $X^{\nu(i)\oplus f^*\alpha} \to Y^{\eta\oplus\alpha}$. The commutativity of the diagram 5.1.1 follows immediately from definition of the Thom construction. □

5.2 Maps between the Thom spaces

Let $\xi = (p'\colon E' \to B)$ be a n-dimensional vector bundle and $\eta = (p\colon E \to Y)$ be a smooth k-dimensional vector bundle over an l-dimensional manifold Y. We shall consider submanifolds $V \subset E$ of codimension n such that the restriction $p|_V\colon V \to Y$ is a proper map. A ξ-structure on such a submanifold is a map $\nu(i) \to \xi$, where $i : V \hookrightarrow E$ is the inclusion and a submanifold with a ξ-structure is called a ξ-submanifold of η.

Definition 5.2.1. *Two ξ-submanifolds of the vector bundle η*

$$\begin{tikzcd}
& E \arrow[d, "p"] \\
V_r \arrow[ur, "i_r"] \arrow[r, "f_r"] & Y
\end{tikzcd} \qquad \varphi_r\colon \nu(i_r) \to \xi \qquad r = 0, 1 \tag{5.8}$$

are bordant if there exists a ξ-submanifold $j : W \hookrightarrow E \times \mathbb{R}$ of codimension n, such that $F := (p \times \mathrm{id}) \circ j : W \to Y \times \mathbb{R}$ is a proper map i.e. the following diagram commutes:

$$\begin{tikzcd}
& E \times \mathbb{R} \arrow[d, "p\times id"] \\
W \arrow[ur, "j"] \arrow[r, "F"] & Y \times \mathbb{R}
\end{tikzcd} \qquad \varphi\colon \nu(j) \to \xi \tag{5.9}$$

and the the following compatibility conditions are satisfied:

1. *W is transversal to $E \times \{0,1\}$ and $W \cap (E \times \{r\}) = V_k$ for $r = 0,1$.*

2. *ξ-structure $\varphi\colon \nu(j) \to \xi$ corresponds under the identification $\nu(j)|_{W\cap(E\times\{r\})} = \nu(i_r)$ to $\varphi_r\colon \nu(i_r) \to \xi$.*

Bordism is an equivalence relation in the set of ξ-submanifolds of η. The set of equivalence classes of this relation is denoted by $L(\eta;\xi)$.

Remark 5.2.1.

1. If Y is compact then this definition of cobordism coincides with the definition of bordism of compact submanifolds of a manifold E (see Bröcker, tom Dieck [4]).

2. Let $i\colon V \hookrightarrow E$ be a ξ-submanifold as in the Def. 5.2.1, and suppose ξ-structures $\varphi_r\colon \nu(i) \to \xi$, $r = 0,1$ are homotopic as vector bundle maps. Then $(i\colon V \to E, \quad \varphi_r\colon \nu(i) \to \xi)$, $r = 0,1$ are bordant.

3. It is possible to give the analogous definition and prove similar theorems if $E \to Y$ is a locally trivial smooth map.

For each ξ-submanifold of η the map obtained from the Thom construction composed with the map induced on the Thom spaces by a ξ-structure gives a homotopy class of pointed maps $Y^\eta \to B^\xi$.

Proposition 5.2.1. *Bordant ξ-submanifolds define homotopic pointed maps $Y^\eta \to B^\xi$.*

Thus we obtain a map

$$P\colon L(\eta,\xi) \longrightarrow [Y^\eta, B^\xi]_*.$$

The following Pontriagin–Thom style theorem holds.

Theorem 5.2.1. *For any smooth vector bundle η over a manifold Y and a vector bundle ξ over a CW-complex B the map $P\colon L(\eta,\xi) \to [Y^\eta, B^\xi]_*$ is bijective.*

Proof. Our proof follows Bröcker, tom Dieck [4] Thm. 3.1 (only step 1. needed a small modification). First we will prove this theorem in the case when ξ is a smooth bundle, showing that P is injective and surjective. We show that P is surjective. Let $f\colon Y^\eta \to B^\xi$ be a pointed map. We need to find a proper ξ-submanifold

$$\begin{tikzcd} & E \arrow[d, "p"] \\ V \arrow[ur, "i"] \arrow[r, "g"'] & Y \end{tikzcd} \tag{5.10}$$

$\nu(i) \xrightarrow{\varphi} \xi$ such that $P([f,i,\varphi]) = [f]$. To construct $[f,i,\varphi]$ we deform it in several steps.

Step 1. There is a map $f_1 \in [f]$ which is differentiable on $A = f_1^{-1}(E') = f^{-1}(E')$ and is transversal to the zero section $B \hookrightarrow B^\xi$.

Proof of step 1. On the space E' introduce a metric d for which compact sets are exactly the closed sets bounded with respect to d (it is enough to embedd E' as a closed subset of the Euclidean space). There is a homotopy $H\colon A \times I \to E'$ with $H_0 = f|_A$, H_1 differentiable and transversal to the zero section and moreover for every $a \in A, \quad t \in I, \quad d(H_t(a), f(a)) \le 1$.

Prolong H to $\widetilde{H} : Y^\eta \times I \to B^\xi$ putting $\widetilde{H}(x,t) = \infty$ for $x \notin A$. To demonstrate the continuity of $\widetilde{H}$ it remains to show that for $U \ni \infty$ being an open neighborhood of ∞ in B^ξ, $\quad h^{-1}(U)$ is open in $Y^\eta \times I$. Let $D = B^\xi \setminus U$. We prove that $h^{-1}(D)$ is a complement of an open neighborhood of $\infty \times I$ in $Y^\eta \times I$. Indeed $h^{-1}(D)$ is closed in E: this is the case since $\widetilde{H}^{-1}(D) = h^{-1}(D) \subset f^{-1}\{x \mid d(x, D) \leq 1\} \times I \subset A \times I$ is closed as a subset of $A \times I$. It is also closed as a subset of $E^\eta \times I$ since $f^{-1}\{x \mid d(x, D) \leq 1\} \subset E^\eta$ is closed. We set $f_1 := \widetilde{H}_1$.

Step 2. Let $N = f_1^{-1}(B)$. It is a submanifold of E and continuity of f_1 implies that $p| \colon N \to Y$ is a proper map. Let U be a tubular neighborhood of N in E such that its complement is a nbhd of ∞. There is a map $f_2 : Y^\eta \to B^\xi$, homotopic to f_1 such that

1. f_2 is transversal to the zero section, $f_2^{-1}(B) = N$.
2. $f_2(x) = \infty$ for $x \notin U$.

Proof of step 2. Let $V \subseteq U \subset A$ be a tubular neighborhood and $\overline{V} \subseteq U$ be a closed subset of Y^η. Let $s \colon Y^\eta \to [0,1]$ be a smooth function such that $s^{-1}(0) = \overline{V}, \quad s^{-1}[0,1) = U$. Let

$$H(x,t) = \begin{cases} \frac{1}{1-ts(x)} \cdot f_1(x) & (x \in A \text{ and } t < 1) \text{ or } (x \in U \text{ and } t = 1) \\ \infty & \text{elsewhere} \end{cases}$$

and put $f_2 = H(-, 1)$.

Step 3. There is a map $f_3 : Y^\eta \to B^\xi$, homotopic to f_2 such that:

1. $U = f_3^{-1}(E')$.
2. the composition $\nu(N) \to U \xrightarrow{f_3} E'$ is a vector bundle map, denoted by φ.

Proof of step 3. We construct f_3 as differential of the tubular nbhd $\nu(N) \to E$ in direction of fibers. [cf. Bröcker-tom Dieck loc.cit.]

It is clear from the construction that $P[N \subset E, \phi] = [f_3] = [f]$. To prove that the map P is injective we apply the same procedure to produce a bordism from a homotopy $H \colon Y^\eta \times I \to X^\xi$.

Remark 5.2.2. The inverse map to $P \colon L(\eta, \xi) \to [Y^\eta, B^\xi]$ can also be constructed more directly. We start from a map $f_1 : Y^\eta \to B^\xi$ which is differentiable on $f_1^{-1}(E)$ and transversal to $B \hookrightarrow B^\xi$. We define $Q'[f_1] := [f_1^{-1}(B), Df_1]$ where $Df_1 : \nu(f_1^{-1}(B)) \to \nu(B) \simeq \xi$.

So far we assumed that ξ is a smooth bundle. Now let ξ be a vector bundle over an arbitrary CW-complex. Suppose first that B is a finite dimensional, locally finite countable complex. Every such complex can be imbedded as a deformation retract of some neighborhood in Euclidean space $U \subset \mathbb{R}^N$. Let $r \colon U \to B$ be the retraction. We have an induced bundle:

$$\begin{tikzcd} r^*E' \arrow[r] \arrow[d] & E' \arrow[d] \\ U \arrow[r] & B \end{tikzcd} \tag{5.11}$$

We may assume that $r^*E' \to U$ is a smooth bundle. On the other hand it is clear that $L(\eta, \xi) = L(\eta, r^*\xi)$ and $[Y^\eta, B^\xi] = [Y^\eta, U^{r^*\xi}]$, and thus $L(\eta, \xi) = [Y^\eta, B^\xi]$ in case B is a finite dimensional, locally finite countable complex. Suppose now that B is an arbitrary CW- complex. A version of cellular approximation theorem implies that $L(\eta, \xi) = \operatorname{colim}_C L(\eta, \xi|_C)$ where C are finite dimensional, locally finite countable subcomplexex of B (since every structure $\nu(i) \to \xi$, where $V \hookrightarrow E$ is the submanifold considered, factors through $\xi|_C$). From the approximation theorem we have also $[Y^\eta, B^\xi] = \operatorname{colim}_C [Y^\eta, C^\xi]$ and the equality holds for arbitrary simplicial complex B. $\square$

Zad. 5.2.1. Suppose $\eta = \eta' \oplus \theta^1$. Then $Y^\eta \simeq \Sigma Y^{\eta'}$ thus the set $[Y^\eta, B^\xi] = [\Sigma Y^{\eta'}, B^\xi]$ has a group structure. Describe in geometric terms the corresponding group structure in the set $L(\eta, \xi)$.

Theorem 5.2.1 can be formulated in a relative form.

Definition 5.2.2. *A pair (Y, A) is a relative n-dimensional manifold iff A is a closed subset of Y and $Y \backslash A$ is a n-dimensional manifold.*

Assume that $\eta = (E \xrightarrow{p} Y)$ is a vector bundle over Y such that its restriction $\eta|_{Y\backslash A}$ over $Y \setminus A : E(\eta|_{Y\backslash A}) \xrightarrow{p} Y \setminus A$ is a smooth bundle.

Definition 5.2.3. *A proper submanifold of E over a relative manifold (Y, A) is a submanifold of $E(\eta|_{Y\backslash A})$ such that the projection on the base is proper as a map into Y.*

Let $i : W \hookrightarrow E(\eta|_{Y\backslash A})$ a proper submanifold. As in the absolute case a ξ-structure on W is a bundle map $\nu(i) \to \xi$ where ξ is a vector bundle over CW complex B and $\dim \xi = \operatorname{codim} W$.

Definition 5.2.4. *Two ξ-submanifolds of E over a relative manifold (Y, A) are bordant over (Y, A) iff they are bordant as submanifolds of $E(\eta|_{Y\backslash A})$ and in addition the submanifold of $E(\eta|_{Y\backslash A}) \times \mathbb{R}$ establishing the cobordism is a proper ξ-submanifold over a relative manifold $(Y \times \mathbb{R}, A \times \mathbb{R})$.*

We denote by $L(\eta, \eta|_A; \xi)$ the equivalence classes of the above relation. The following theorem holds:

Theorem 5.2.2. *For any vector bundle η over a relative manifold (Y, A) and an arbitrary vector bundle ξ over a CW-complex B the Pontriagin–Thom construction establishes bijection between*

$$L(\eta, \eta|_A; \xi) \to [Y^\eta / A^{\eta|_A}, B^\xi].$$

Zad. 5.2.2. Prove Thm. 5.2.2.

5.3 Stabilization. Bordism and Cobordism Groups.

Assume that we are given a vector bundle spectrum $\xi = (\xi_k, \varepsilon_k)$, $\dim \xi = 0$. Let Y be a fixed manifold. We will consider $(\dim Y - n)$-dimensional ξ_k-submanifolds of the successive product bundles θ_Y^{k-n} over Y (k varies) whose projection on Y is a proper map (see chapter 4.). The natural inclusion $\mathbb{R}^{k-n} \hookrightarrow \mathbb{R}^{k-n+1}$ on the first coordinates induces a map

$$\varkappa_k : L(\theta_Y^{k-n}, \xi_k) \to L(\theta_Y^{k-n+1}, \xi_{k+1}).$$

If $V \subset Y \times \mathbb{R}^{k-n}$ is a ξ_k-submanifold we can define ξ_{k+1}-structure on $V \subset Y \times \mathbb{R}^{k-n+1}$ to be $\nu_{k+1}(V) \simeq \nu_k(V) \oplus \theta^1 \to \xi_k \oplus \theta^1 \xrightarrow{\varepsilon_k} \xi_{k+1}$. The following diagram commutes:

$$\begin{tikzcd} L(\theta_Y^{k-n}, \xi_k) \arrow[r] \arrow[d] & {[S^{k-n}Y^+, B_k^{\xi_k}]} \arrow[d] \\ L(\theta_Y^{k-n+1}, \xi_{k+1}) \arrow[r] & {[S^{k-n+1}Y^+, B_{k+1}^{\xi_{k+1}}]} \end{tikzcd} \tag{5.12}$$

Passing to the limit on both sides we obtain:

$$\operatorname{colim}_k [S^{k-n}Y^+, B_k^{\xi_k}] = \operatorname{colim}_k L(\theta_Y^{k-n}, \xi_k)$$

If Y has the homotopy type of finite CW complex then $\operatorname{colim}_k[S^{k-n}Y^+, B_k^{\xi_k}]$ is the n^{th} cohomology group of Y with coefficients in the spectrum $\{B_k^{\xi_k}, \varepsilon_k\}$. We denote this theory by $B^*(-;\xi)$. We have also a similar formula for homology groups.

$$B_n(Y,\xi) = \operatorname{colim}_k[S^{k+n}, Y^+ \wedge B_k^{\xi_k}] = \operatorname{colim}_k[S^{k+n}, (Y \times B_k)^{\theta_Y^0 \times \xi_k}] = \operatorname{colim}_k L(\theta_{pt}^{k+n}, \theta_Y^0 \times \xi), \quad (5.13)$$

Notice that all submanifolds in $L(\theta_{pt}^{k+n}, \theta_Y^0 \times \xi)$ are compact since their projection on the one-point space must be proper.

Remark 5.3.1. If Y is a finite CW complex then there exists an open manifold of the homotopy type of Y (e.g. an open neighborhood of Y in $\mathbb{R}^m$ for suitable m).

The Whitney embedding and isotopy theorems 1.3.1 enable us to get rid of particular embeddings and thus to interpret $\operatorname{colim}_k L(\theta_Y^{k-n}, \xi_k)$ in terms of the abstract manifolds.

Let $f : V \to Y$ be a smooth map and $\nu^S(f)$ a stable normal bundle of f. We define dimension of a smooth map $f\colon V \to Y$ as $\dim(f) := \dim Y - \dim V$. We will introduce an equivalence relation in the set of n-dimensional proper maps $f\colon V \to Y$ endowed with ξ-orientations (V varies).

Definition 5.3.1. *Let $\xi = (\xi_k, \varepsilon_k)$ be a vector bundle spectrum,* $\dim \xi = 0$*. Two n-dimensional ξ-oriented proper maps $(f_r\colon V_r \to Y, \alpha_r\colon \nu^S(f_r) \to \xi)$, $r = 0, 1$ are bordant if there exists a ξ-oriented n-dimensional ξ-oriented proper map endowed $f\colon W \to Y \times \mathbb{R}$, $\quad \alpha\colon \nu^S(g) \to \xi$ such that:*

1. *$f\colon W \to Y \times \mathbb{R}$ is transversal to $Y \times \{0,1\}$ and $V_r = f^{-1}(Y \times \{r\})$*
2. *For $r = 0, 1$ the restriction of the ξ-structure $\alpha\colon \nu^S(f) \to \xi$ to $\nu^S(f)|_{f^{-1}(Y \times \{r\})} = \nu^S(f_r)$ coincides with α_r.*

Proposition 5.3.1. *The relation defined in Def. 5.3.1 is an equivalence relation; set of its classes is denoted by $\mathcal{B}^n(Y,\xi)$.*

There exists a natural map:

$$\Phi : \operatorname{colim}_k L(\theta_Y^{k-n}, \xi_k) \longrightarrow \mathcal{B}^n(Y,\xi)$$

which with every proper ξ_k-submanifold $i\colon V \subset Y \times \mathbb{R}^{k-n}$ associates the manifold V and a proper map $\pi|_V : V \to Y$ where $\pi : Y \times \mathbb{R}^{k-n} \to Y$ is the projection. The map of bundle spectra $\nu^S(i) \to \xi$ is induced in the obvious way by $\nu(i) \to \xi_k$. It is clear that this map is well defined.

Theorem 5.3.1. *The map Φ is a bijection.*

Proposition 5.3.2. *Let $i_0, i_1 : V \to Y \times \mathbb{R}^{k-n}$ be embeddings lifting the same proper map f i.e. $pi_0 = pi_1$ and let $\nu(i_0) \to \xi_k$, $\quad \nu(i_1) \to \xi_k$ be the ξ_k-structures. If there exists an isotopy for which the ξ_k-structures correspond under the isomorphism $\nu(i_0(V)) \to \nu(i_1(V))$ defined by the isotopy then the ξ_k-submanifolds $(i_0(V), \nu(i_0) \to \xi_k)$ and $(i_1(V), \nu(i_1) \to \xi_k)$ are bordant.*

Proof. If $H\colon V \times \mathbb{R} \to Y \times \mathbb{R}^{k-n} \times \mathbb{R}$ is the isotopy then $H(Y \times \mathbb{R}) \hookrightarrow Y \times \mathbb{R}^{k-n} \times \mathbb{R}$ is the submanifold establishing the required bordism, where the ξ_k-structure on $\nu(H(V \times \mathbb{R}))$ arises from the composition

$$\nu(H(V \times \mathbb{R})) \to \nu(H(V \times \{0\})) \to \nu(i_0) \to \xi_k.$$

□

Remark 5.3.2. If the condition in the proposition 5.3.2 is fulfilled then the bundle maps $\nu(i_0) \to \xi_k$ and $\nu(i_1) \to \xi_k$ determine the same map of bundle spectra $\nu^S(f) \to \xi$. Also, if they determine the same map of bundle spectra then conclusion of Prop. 5.3.2 is true for large k and thus $(i_0(V), \nu(i_0) \to \xi_k)$ and $(i_1(V), \nu(i_1) \to \xi_k)$ represent the same element in $\operatorname{colim}_k L(\theta_Y^{k-n}, \xi_k)$.

Proof of Theorem 5.2.1. It is clear that Φ is surjective. We will prove that Φ is injective. Suppose that images of two ξ_k-submanifolds $i_r\colon : V_r \hookrightarrow Y \times \mathbb{R}^{k-n}$ equipped with ξ-structures $\alpha_r : \nu(V_r) \to \xi_k$, are bordant in the sense of Def. 5.3.1. Thus there exists a manifold W and a proper ξ-oriented map $g\colon W \to Y \times \mathbb{R}$ satisfying conditions 1) - 3) of Def. 5.3.1. Let $j\colon W \to \mathbb{R}^{k-n}$ be an embedding (we can assume k is large) and consider the diagonal map:

$$\begin{tikzcd} & Y \times \mathbb{R}^{k-n} \times \mathbb{R} \arrow[d, "\pi \times id"] \\ W \arrow[ur, "\tilde{g} := (g,j)"] \arrow[r, "g"] & Y \times \mathbb{R} \end{tikzcd} \tag{5.14}$$

with the ξ_k-structure $\beta\colon \nu(\widetilde{g}) \to \xi_k$ inducing the ξ-orientation of g. According to Def. 5.2.1, the ξ-submanifold $\widetilde{g}(W)$ establishes bordism in $L(\theta_Y^{k-n}, \xi_k)$ between $U_0 := \widetilde{g}(g^{-1}(Y \times \{0\}))$, $\beta|_{U_0} : \nu(\widetilde{g}(W))|_{U_0} \to \xi_k$ and $U_1 := \widetilde{g}(g^{-1}(Y \times \{1\}))$, $\quad \beta|_{U_1} : \nu(\widetilde{g}(W))|_{U_1} \to \xi_k$.

Consider embeddings $i_0 : V_0 \hookrightarrow Y \times \mathbb{R}^{k-n}$ and $V_0 \xrightarrow{\varphi_0} g^{-1}(Y \times \{0\}) \xrightarrow{\widetilde{g}} Y \times \mathbb{R}^{k-n}$. From condition 2) we have $\pi i_0 = \pi \widetilde{g} \varphi_0$ and from condition 3) of Def. 5.3.1 we see that ξ_k-structures on $\nu(i_0(V))$ and $\nu(\widetilde{g}\varphi_0 V) = \varphi_0^* \nu(\widetilde{g}(g^{-1}(Y \times \{0\})))$ define the same ξ-orientation $\nu(\pi i_0) \to \xi$. Analougously for V_1 and U_1. Thus injectivity of Φ follows from Prop. 5.3.2. $\square$

We will give now another definition of the cobordism relation, similar to the well-known definition of geometric bordism.

Definition 5.3.2. *Two n-dimensional ξ-oriented proper maps $(f_r : V_i \to Y, \quad \alpha_r\colon \nu^S(f_r) \to \xi)$, $r = 0, 1$ are bordant iff there exists a manifold with boundary $(W, \partial W)$ and a ξ-oriented proper map $f\colon W \to Y$ such that:*

1. *there is a diffeomorphism $\partial W = V_0 \sqcup V_1$ such that $f|_{\partial W} = f_0 \sqcup f_1$,*
2. *the ξ-orientation of f agrees with those of f_r, $\quad i = 0, 1$ in the sense that the following diagram for f_0 and the analogous for f_1 commutes:*

$$\begin{tikzcd} \nu^S(f_r) \arrow[r, "\delta_r"] \arrow[d, "\alpha_r"] & (\nu^S(f)|_{V_r} \oplus \theta^1)[+1] \arrow[d, "\alpha \oplus id"] \\ \xi & \xi \oplus \theta^1[+1] \arrow[l, "\varepsilon."] \end{tikzcd} \tag{5.15}$$

where the isomorphism δ_r in the top row is defined by the inner normal field for V_0 and the outer normal field for V_1.

Theorem 5.3.2. *Definitions 5.3.1 and 5.3.2 are equivalent.*

Proof. It is clear that two maps bordant in the sense of Def. 5.3.1 are also bordant in the sense of Def. 5.3.2.

Suppose that two ξ-oriented maps $(f_r\colon V_r \to Y, \alpha_r\colon \nu(f_r) \to \xi)$, $r = 0, 1$ are equivalent in the sense of Def. 5.3.2 and $g : W \to Y, \beta : \nu(g) \to \xi$ is a bordism between them We will show that they are bordant in the sense of Def. 5.3.1

Step 1. Let $f\colon W \to \mathbb{R}$ be a Morse function on $(W, \partial W)$. We define a map $h\colon W \to Y \times \mathbb{R}$, $h(w) := (g(w), f(w))$. It is a proper map such that

1. $h(W) \subseteq Y \times [0,1]$,
2. $h^{-1}(Y \times \{i\}) = \varphi(V_i), \quad i = 0, 1$,
3. $h \pitchfork Y \times \{i\}, \quad i = 0, 1$,
4. $h|_{\varphi(V_i)} = g|_{\varphi(V_i)}, \quad i = 0, 1$.

We can glue smoothly $\varphi(V_0) \times (-\infty, 0]$ and $\varphi(V_1) \times [1, +\infty)$ to W and prolong h to get a manifold without boundary $\widetilde{W}$ and $\widetilde{h} : \widetilde{W} \to Y \times \mathbb{R}$ putting $\widetilde{h}(x,t) = (h(x), t)$ for $x \in \varphi(V_0 \cup V_1), \quad t \leq 0$ or $t \geq 1$ (see the figure below).

Next we define the ξ-orientation of $\widetilde{h} \colon \widetilde{W} \to Y \times \mathbb{R}$. The map $\beta \colon \nu(g) \to \xi$ determines the ξ-orientation $\nu(h) \to \xi$ because $\nu^S(h) \simeq \nu^S(g)$ (any embedding of W into $Y \times \mathbb{R} \times \mathbb{R}^{k-n}$ "over h" is an embedding into $Y \times \mathbb{R}^{k-n+1}$ "over g"). The latter ξ-orientation can be prolonged in a natural way to the ξ-orientation $\gamma \colon \nu^S(\widetilde{h}) \to \xi$.

We claim that $(\widetilde{h} \colon \widetilde{W} \to Y \times \mathbb{R}, \gamma \colon \nu^S(\widetilde{h}) \to \xi)$ is a bordism between $(f_r \colon V_r \to Y, \alpha_r \colon \nu^S(f_r) \to \xi)$ $r = 0, 1$ in the sense of Def. 5.3.1.

Step 2. We see that $(\widetilde{h} \colon \widetilde{W} \to Y \times \mathbb{R}, \gamma \colon \nu^S(\widetilde{h}) \to \xi)$ sets up an equivalence in the sense of Def. 5.3.1 between $(V_r \xrightarrow{i} Y, \nu(f_r) \xrightarrow{\beta_r} \xi)$ for $r = 0, 1$ where β_r is the ξ-orientation: $\nu^S(f_r) \to \nu^S(\widetilde{h})|_{V_r} \xrightarrow{\gamma} \xi$.

Step 3. It remains to show that the ξ-orientations $\alpha_r \colon \nu^S(f_r) \to \xi$ and $\beta_r \colon \nu^S(f_0) \to \xi$ $r = 0, 1$ are the same. Consider an embedding $j \colon \widetilde{W} \to (Y \times \mathbb{R}) \times \mathbb{R}^{k-n}$ lifting $\tilde{h}$ with $\widetilde{\gamma} \colon \nu(j) \to \xi_k$ being the representative of $\gamma : \nu^S(\widetilde{h}) \to \xi$ as on the diagram:

$$\begin{tikzcd} & (Y \times \mathbb{R}) \times \mathbb{R}^{k-n} \arrow[d] \\ \widetilde{W} \arrow[ur, "j"] \arrow[r, "\tilde{h}"'] & Y \times \mathbb{R} \end{tikzcd} \tag{5.16}$$

The manifold V_0 is then imbedded into $Y \times \{0\} \times \mathbb{R}^{k-n}, \quad j(V_0) \hookrightarrow Y \times \{0\} \times \mathbb{R}^{k-n}$ where $\nu_k(j(V_0))$ is the normal bundle. Then $\widetilde{\gamma}|_{V_0} \colon \nu_k(j(V_0)) = \nu(\widetilde{W})|_{V_0} \to \xi_k$ represents $\beta_0 \colon \nu^S(f_0) \to \xi$ and let $\widetilde{\alpha}_0 \colon \nu_k(j(V_0)) \to \xi_k$ represents $\alpha_0 \colon \nu^S(f_0) \to \xi$. When we consider $j(V_0)$ as a submanifold of $Y \times \mathbb{R}^{k-n+1}$ with the normal bundle $\nu_{k+1}(j(V_0))$ the above structures are respectively:

$$\nu_{k+1}(j(V_0)) \xrightarrow{\varkappa, \simeq} \nu_k(j(V_0)) \oplus \theta^1 \xrightarrow{\widetilde{\gamma_1} \oplus \mathrm{id}} \xi_k \oplus \theta^1 \xrightarrow{\varepsilon} \xi_{k+1}$$

$$\nu_{k+1}(j(V_0)) \xrightarrow{\varkappa, \simeq} \nu_k(j(V_0)) \oplus \theta^1 \xrightarrow{\widetilde{\alpha}_0 \oplus \mathrm{id}} \xi_k \oplus \theta^1 \xrightarrow{\varepsilon_k} \xi_{k+1}$$

where $\varkappa$ denotes the canonical isomorphism determined by the orientation of the Euclidean space. The isomorphism

$$\delta_0 \colon \nu^S(V_0) \simeq (\nu^S(W) \oplus \theta^1)[+1]$$

proves that condition Definition 5.3.2 2 is fulfilled. $\square$

5.4 Geometric description of homology and cohomology defined by Thom spectra

When Y is a manifold then we have a geometric interpretation of cohomology groups $B^*(Y, \xi)$ defined by the Thom spectrum $\mathrm{Th}(\xi)$.

$$\mathcal{B}^n(Y; \xi) \simeq \operatorname{colim}_k [S^{k-n} Y^+, B_k^{\xi_k}] = \tilde{B}^n(Y^+, \xi)$$

Recall that elements of $\mathcal{B}^n(Y;\xi)$ are bordism classes of n-dimensional ξ-oriented proper maps $V \xrightarrow{f} Y$ (Def. 5.3.1). The bijection is established by composition

$$\mathcal{B}^n(Y;\xi) \xrightarrow{\Phi^{-1}} \operatorname{colim}_k L(\theta_Y^{k-n}, \xi_k) \xrightarrow{\simeq} \operatorname{colim}_k [S^{k-n}Y^+, B_k^{\xi_k}] = \tilde{B}^n(Y^+;\xi)$$

which will be also denoted by P and called the Pontriagin–Thom isomorphism.

If X is an arbitrary CW-complex similarly we obtain a geometric description of homology groups defined by the Thom spectrum $\mathrm{Th}(\xi)$.

$$B_n(X;\xi) = \operatorname{colim}_k [S^{k+n}, X^0 \wedge B_k^{\xi_k}] = \mathcal{B}^{-n}(\mathrm{pt}, \xi \times \theta_X^0) =: \mathcal{B}_n(\mathrm{pt}, \xi \times \theta_X^0)$$

Elements of $\mathcal{B}^{-n}(\mathrm{pt}, \xi \times \theta_X^0)$ are represented by proper maps into a point $M \to pt$ i.e. compact manifolds M and ξ-structure is which is described by a map $f\colon M \to X$ and a morphism $\nu^S(M) \to \xi$.

Similarly we can give a geometric interpretation of the relative groups. Let (Y, A) be a relative manifold. We define the relative groups $\mathcal{B}^n(Y, A;\xi)$:

Definition 5.4.1. *Let $(f_r\colon V_r \to Y\backslash A, \nu^S(f_r) \to \xi)$, $\quad r = 0,1$ be proper ξ-oriented maps into (Y, A) (i.e. the composition $f_i : V_i \to Y\backslash A \hookrightarrow Y$ is proper for $i = 0,1$). We say that they are equivalent (bordant) if there is a proper ξ-oriented map $(g : W \to Y\backslash A, \nu(g) \to \xi)$ into (Y, A) which establishes a bordism between them in $Y\backslash A$. The set of equivalence classes of this relation we denote $\mathcal{B}^n(Y, A;\xi)$.*

Theorem 5.4.1. *$\mathcal{B}^*(Y, A;\xi)$ is isomorphic to $B^*(Y, A;\xi)$.*

Proof. It follows from Thm. 5.2.2. □

Next we describe a geometric interpretation of the relative bordism groups [Conner, Floyd [5]]. Let (X, A) be an arbitrary topological pair. Relative singular n-dimensional manifold is a map $(V, \partial V) \to (X, A)$ where n-dimensional compact manifold V is ξ-oriented. Such manifold bounds if there exists an $(n+1)$-dimensional compact manifold with boundary $(W, \partial W)$ such that $V \subset \partial W$ and a map $F\colon (W, \partial W) \to (X, A)$ such that $F|V = f$ and $F(\partial W \setminus V) \subset A$. Two relative singular manifolds are bordant if their disjoint sum with inverse ξ-structure on one of them, bounds. The set of equivalence classes of compact n-dimensional ξ-oriented singular manifolds we denote $\mathcal{B}_n(X, A;\xi)$.

Zad. 5.4.1. Check directy that the functors $\mathcal{B}_n(X, A;\xi)$ satisfy the Eilenberg-Steenrod axioms.

Zad. 5.4.2. Prove directly the MV-exact sequence, without refereeing to the relative groups. For any two open subsets $U_1 \cup U_2 = X$ there exists a long exact sequence

$$\ldots \to \mathcal{B}_k(U_1 \cap U_2;\xi) \xrightarrow{i_{1*} \oplus i_{2*}} \mathcal{B}_k(U_1;\xi) \oplus \mathcal{B}_k(U_2;\xi) \xrightarrow{j_{1*} - j_{2*}} \mathcal{B}_k(X;\xi) \xrightarrow{\partial} \mathcal{B}_{k-1}(U_1 \cap U_2;\xi) \to \ldots$$

where ∂ is defined as follows Let $f\colon M \to X$ be a singular manifold and $V_i := \phi^{-1}(U_i)$. There exists submanifolds with boundary $M_i \subset M$ such that $M = M_1 \cup M_2$ and $M_i \subset V_i$, $M_1 \cap M_2 = \partial M_1 = \partial M_2$. We define $\partial[f] := [f|\partial M_1]$.

Theorem 5.4.2. *The Pontriagin-Thom construction defines an isomorphism $\mathcal{B}_n(X, A;\xi) \simeq B_n(X, A;\xi)$.*

5.5 Universal properties of the Thom spectra

5.5.1 ξ-oriented functors

Definition 5.5.1. *Let ξ be a 0-dimensional vector bundle spectrum. A ξ-orientation of a multiplicative cohomology theory $\tilde{h}^*$ is a choice of Thom classes $U_k \in \tilde{h}^k(\mathrm{Th}(\xi_k))$ such that $\varepsilon_k^*(U_{k+1}) = \sigma(U_k)$, and if the spectrum ξ is multiplicative $U_{k+l}|\xi_k \oplus \xi_l = U_k U_l$.*

For any bundle η a ξ-structure $\eta \xrightarrow{\bar{f},f} \xi$ defines a Thom class U_η. Indeed, if $\eta \oplus \theta^k \xrightarrow{\bar{f},f} \xi_{n+k}$ represents a ξ-structure on η then $\sigma^{-k}\bar{f}^*(U_{n+k}) \in \tilde{h}^n(\mathrm{Th}(\eta))$ is a Thom class of η.

If theory $\tilde{h}^*$ is defined just on some spaces (finite dimensional CW-complexes, manifolds etc.) and the bases of bundles ξ_k are not necessary in the category, then ξ-orientation of $\tilde{h}^*$ is an assignment in natural way of the $\tilde{h}^*$-Thom class to each ξ-structure. As we'll see there in general there are many choices of ξ-orientations.

We need to consider also functors slightly more general then cohomology theories; namely homotopy invariant functors with values in pointed sets, groups, or rings for which the suspension isomorphism holds but there is no exact sequence i.e. functors those satisfying conditions 1) and 3) of Defn. 2.1.2. We will call such functors *cohomology-like* functors (CLF - for short). If $\tilde{F}\colon \mathcal{T}_* \to \mathcal{M}onoid^*$ is a CLF whose values are monoids then we assume that the suspension isomorphism is given by multiplication with the element $\sigma(1) \in \tilde{F}(S^1)$ where $1 \in \tilde{F}(S^0) = F(pt)$ and call such functors *multiplictaive cohomology-like* functors (MCLF - for short). They may also have an additive structure i.e.values in the category of rings.

An important example of MCLF will be functors of the form $h^*(-)\otimes_{h^*} R$, where R is a graded h^*-module. The tensor product preserves isomorphisms, but not necessary exactness thus $h^*(-)\otimes_{h^*} R$ doesn't have to be a cohomology theory.

For any MCLF we can define an ξ-orientation as a choice of classes $U_k \in \tilde{F}^k(B_k^{\xi_k})$ such that $\varepsilon_k^*(U_{k+1}) = \sigma(U_k)$, $U_{k+l}|\xi_k \oplus \xi_l = U_k U_l$ and $U_0 = 1$. We'll show that the cobordism theory $\mathcal{B}^*(-;\xi)$ is naturally ξ-oriented and it is universal among the ξ-oriented cohomology theories.

Proposition 5.5.1. *Let ξ be a 0-dimensional multiplicative vector bundle spectrum. For any graded MCLF $\tilde{F}\colon \mathcal{T}_* \to \mathcal{M}onoid^*$ there is a bijection between set of ξ-orientations in $\tilde{F}$ and set of natural multiplicative transformations functors $\tilde{B}^*(-;\xi) \to \tilde{F}$.*

Proof. Consider $\tilde{B}^*(-;\xi)$ as a functor to the category of monoids, where operation is defined by the multiplicative structure on $\tilde{B}^*(-;\xi)$. We recall that the canonical Thom class

$$U_k^{\mathrm{can}} \in \tilde{B}^k(B_k^{\xi_k};\xi) = \mathrm{colim}_r[S^{r-k}B_k^{\xi_k}, B_r^{\xi_r}]$$

is defined simply as the class of the identity map $B_k^{\xi_k} = B_k^{\xi_k}$. If $\Phi\colon \tilde{B}^* \to \tilde{F}$ is a natural transformation, then elements $\Phi(U_k^{\mathrm{can}}) \in F^k(B_k^{\xi_k})$ define a ξ-orientation in F. Conversely, any choice of the ξ-orientation $U = \{U_k \in F^k(B_k^{\xi_k})\}$ defines a multiplicative natural transformation $\Phi_U\colon \tilde{B}^* \to \tilde{F}$. $\square$

5.5.2 The transfer maps

Remark 6. Note that the Thom space of a smooth bundle α is a relative manifold $(\mathrm{Th}(\alpha), \infty)$. In geometric interpretation of the cobordism groups the Thom class is represented by the zero-section $s_0\colon B_k \to \mathrm{Th}(\xi_k)$, if ξ_k is smooth.

Note that if F^* is ξ-oriented CLF then ξ-oriented proper maps between manifolds admit a transfer map (*Gysin map, Umkehr Homomorphismus*). For a proper ξ-oriented smooth map $f\colon V \to Y$ of dimension $d = \dim Y - \dim V$ consider a diagram:

$$\begin{tikzcd}
 & Y \times \mathbb{R}^k \arrow[d, "\pi"] \\
V \arrow[ur, "\tilde{g})"] \arrow[r, "g"] & Y
\end{tikzcd} \tag{5.17}$$

and ξ-structure $\nu(\tilde{g}) \to \xi$. It leads to the PT-map $Y^{\theta^k} = S^k Y^+ \to W^{\nu(\tilde{g})}$. We define the transfer map $f_\natural \colon h^*(W) \to h^{*-d}(Y)$ as a composition:

$$\begin{tikzcd}
F^*(W) \arrow[r, dotted, "f_\natural"] \arrow[d, "\cup \text{with Thom class}"] & F^{*+d}(Y) \arrow[d, "\simeq"] \\
F^{*+(d+k)}(W^{\nu(\tilde{g})}) \arrow[r] & F^{*+(d+k)}(Y^{\theta^k})
\end{tikzcd} \tag{5.18}$$

Note that the transfer map shifts dimensions of cohomology by dimension of the map. If $F^* = h^*$ is a cohomology theory then similarly one can define a transfer map in the corresponding homology theory h_*.
The transfer maps satisfy similar properties to the induced maps:

1. If $V_0 \xrightarrow{f} V_1 \xrightarrow{g} Y$ are two ξ-oriented proper maps then its composition gf is naturally ξ-oriented and $(gf)_\natural = g_\natural f_\natural$.

2. Assume that

$$\begin{tikzcd}
V_0 \times_Y V_1 \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"] & V_0 \arrow[d, "f"] \\
V_1 \arrow[r, "g"] & Y
\end{tikzcd} \tag{5.19}$$

   is a pull-back square of manifolds, where g is transversal to f, f is ξ-oriented and $\tilde{g}$ carries induced ξ-structure. Then the following diagram commutes:

$$\begin{tikzcd}
h(V_0 \times_Y V_1) \arrow[d, "\bar{f}_\natural"] & V_0 \arrow[l, "\bar{g}^*"'] \arrow[d, "f_\natural"] \\
h(V_1) & h(Y) \arrow[l, "g^*"']
\end{tikzcd} \tag{5.20}$$

Zad. 5.5.1. Let F be a contravariant homotopy functor to the category of sets defined on smooth maps and which admits transfer map for ξ-oriented proper maps. Then for any element $a \in F(pt)$ there is a unique natural transformation $\Phi \colon \mathcal{B}^* \to F$, commuting with the transfer maps, and such that $\Phi(id) = a$ where $id \in \mathcal{B}^0(pt)$.

Hint. For any proper ξ-oriented map $V \xrightarrow{f} Y$ we define $\Phi([f]) := F_\natural(f) F(p_V)(a)$. We have to check that definition depends only on the cobordism class of f.

Chapter 6

Formal Group Laws

6.1 Definitions

Let R^* be a graded ring such that $R^q = 0$ for $q > 0$. We will consider graded ring of formal power series $R^*[[x, y]]$ in two variables x, y in gradation 1 (if $\mathrm{char}(R^*) = 2$) or in even gradation if it is odd. Series of gradation n are of the form

$$\sum_{i,j\geq 0} a_{ij} x^i y^j$$

where $\deg(a_{ij}) = n - (i + j)$. Note that $R^*[[x, y]] = \mathrm{inv.}\lim_{n,m} R^*[x, y]/(x^n, y^m)$ where inverse limit is taken in the category of graded groups.

Definicja 6.1.1. A commutative *formal group law (FGL)* over R^* is a series $F \in R^*[[x, y]]$ satisfying the following conditions:

1. **(Neutral Element)** $F(x, 0) = x$, $F(0, y) = y$.
2. **(Associativity)** $F(F(x, y), z) = F(x, F(y, z))$,
3. **(Commutativity)** $F(x, y) = F(y, x)$.

Usually one writes FGL's in the form

$$F(x, y) = x + y + \sum_{i,j>0} a_{ij} x^i y^j.$$

Commutativity means that $a_{ij} = a_{ji}$.

Proposition 6.1.1. *Let F be a FGL over R^*. The set of power series of gradation 1 $R^*[[x]]_1$, equipped with operation defined for every $\phi, \psi \in R^*[[x]]_1$ by the formula:*

$$(\phi *_F \psi)(x) := F(\phi(x), \psi(x))$$

is an abelian group.

Proof. Defining properties of FGL imply that neutral element is the zero series, and operation $*$ is associative and commutative. An easy calculation shows that it has an inverse element. For that it is enough to find a series $\phi(x)$ such that $F(x, \phi(x)) = 0$. □

The set of all FGL over R^*, denoted $FGL(R^*)$ is the set of objects of a category.

Definition 6.1.1. *A morphism of FGL over R^*, $F \to G$ is a series $\alpha(x) = \sum_{i>0} a_i x^i$ of gradation 1 (i.e. $\deg a_i = i-1$) such that $\alpha(F(x,y)) = G(\alpha(x), \alpha(y))$. Set of morphisms $F \to G$ we denote $\mathrm{Hom}_{R^*}(F,G)$. Composition is defined by composition of series. Identity is given by the series $\iota(x) = x$.*

A morphism $\alpha(x)$ is an isomorphism if and only if $a_1 \in R^0$ is invertible. The inverse morphism to α is a series α^{-1} such that $(\alpha \circ \alpha^{-1})(x) = \alpha(\alpha^{-1}(x)) = x$.

Proposition 6.1.2. *The group of invertible series acts on the set $FGL(R)$ by the formula*

$$\alpha_* F(x,y) := \alpha F(\alpha^{-1}(x), \alpha^{-1}(y)).$$

Any ring homomorphism $h\colon R \to S$ induces a functor $h_*\colon FGL(R) \to FGL(S)$ which which assigns to every FGL over R^*, $F(x,y) = x + y + \sum_{i,j>0} a_{ij} x^i y^j$ the series $\alpha_* F(x,y) := x + y + \sum_{i,j>0} \alpha(a_{ij}) x^i y^j$, which obviously is a FGL over S. Similarly on morphisms it is defined via application of h to the coefficients of the series. Moreover for every isomorphism $\alpha \in FGL(R)$ the following diagram commutes:

$$\begin{tikzcd} FGL(R) \arrow[r, "h_*"] \arrow[d, "\alpha_*"] & FGL(S) \arrow[d, "h_*(\alpha)"] \\ FGL(R) \arrow[r, "h_*"] & FGL(S) \end{tikzcd} \tag{6.1}$$

Example 5. For every ring R^* we define

1. *an additive group* $F_+(x,y) = x + y$,

2. *a multiplicative group* $F_{m,a}(x,y) = x + y + axy$, defined for any $a \in R^{-1}$

Assume that the ring R^* is a $\mathbb{Q}$ -algebra and a is invertible. Then additive group F_+ and multiplicative group $F_{m,a}$ are isomorphic. The isomorphism $\alpha\colon F_+ \to F_{m,a}$ is given by the formal series $\alpha(x) := a^{-1}(e^{ax} - 1)$. Indeed:

$$a^{-1}(e^{a(x+y)} - 1) = a^{-1}(e^{ax} - 1) + a^{-1}(e^{ay} - 1) + a(a^{-1}(e^{ax} - 1)a^{-1}(e^{ay} - 1))$$

The inverse morphism is the series $\lambda(x) := a^{-1} \log(1 + ax)$.

Definition 6.1.2. *Logarithm of FGL F is an series λ which defines an isomorphism $\lambda\colon F \to F_+$.*

In general, logarithm is not unique, and not every formal group admits a logarithm.

6.2 Universal Formal Group Law

Consider a functor $GF\colon Rings \to Set$ from the category of graded rings to the category of sets (even small categories) which assigns to every graded ring R^* the set (or small category) $FGL(R^*)$ of formal groups of gradation 1 over R^* and to any ring homomorphism $\alpha\colon R \to S$ a "push-forward" map.

Proposition 6.2.1. *The functor $GF\colon Rings^* \to Set$ is representable i.e. there exists a FGL (L, F_L) such that the map $\mathrm{Hom}(L,S) \ni h \mapsto h_* F_L \in FGL(R)$ is bijective.*

Proof. It is easy to construct the representing FGL, but much more difficult to understand its structure. We consider the graded polynomial ring over R^* generated by symbols u_{ij} such that $\dim u_{ij} = 1 - i - j$ and define $L := R^*[u_{ij}]/I$ where I is the ideal generated by equations on coefficients resulting from conditions 1)-3) in Defn. 6.1.1. The FGL over L is defined as $F_u(x, y) := x + y + \sum_{i,j>0} u_{ij}x^iy^j$ and for any FGL over R^* $F_u(x, y) := x + y + \sum_{i,j>0} u_{ij}x^iy^j$ we define the corresponding homomorphism by $\phi_(u_{ij}) := a_{ij}$. □

Action of the group of invertible power series over R described in Prop.6.1.2 induces via the bijection described in Prop. 6.2.1 an action on $\mathrm{Hom}(L, R)$. If $h\colon L \to R$ is a homomorphism defining a FGL $F_h \in FGL(R)$ then $\alpha * h\colon L \to R$ is a homomorphism which induces the FGL $\alpha_* F_h$.

Formal groups which are important for our topological applications satisfy an additional condition: $F(x, x) = 0$. Since $\mathrm{char}\, R^* = 2$ it means that $a_{ii} = 0$ for all $i > 0$. It is easy to extend Prop. 6.2.1 to such FGL. It turns out that all such that $F(x, x) = 0$ admit a logarithm i.e. they are isomorphic to the additive group.

Theorem 6.2.1 (M. Lazard)**.** *Let R^* be ring of characteristic 2, and $F(x, y)$ and FGL over R^* such that $F(x, x) = 0$. Then F jest is isomorphic to the additive group. Moreover there exists a unique logarithm $\lambda_F(x) = x + a_1x^2 + a_2x^3 + ...$ such that $a_j = 0$ if $j = 2^i - 1$ for some i.*

Remark 7. If i $a \neq 0$ then the additive FGL F_+ is not isomorphic to the multiplicative FGL $F_{m,a}$.

The universal FGL over a ring of characteristic 2, satisfying condition $F(x, x) = 0$ can be defined explicitly as follows.

Proposition 6.2.2. *Let $L := \mathbb{Z}_2[a_2, a_4, a_5, ..]$ be a graded polynomial ring such that* $\deg a_i = -i$ *and $i \neq 2^j - 1$. Let $\lambda_L(x) \in L[[x]]$ be defined as $\lambda_L(x) := x + \sum_{i>0} a_ix^{i+1}$ and*

$$F_L(x, y) := \lambda_{L*}^{-1} F_a = \lambda_L^{-1}(\lambda_L(x) + \lambda_L(y)).$$

Then (L, F_L) is a universal FGL for FGL's over $\mathbb{Z}_2$-algebras such that $F(x, x) = 0$.

Proof. Let (R^*, F) be an arbitrary FGL, satisfying assumptions of Thm 6.2.1 . Let

$$\lambda_F(x) = x + \sum_{i \neq 2^j - 1} b_ix^{i+1}$$

be the unique logarithm of F. We set $h(a_i) := b_i$, i.e. $\phi_*\lambda_L = \lambda_F$. We have:

$$h_*F_L = h_*(\lambda_L^{-1}(\lambda_L(x) + \lambda_L(y))) = h_*\lambda^{-1}(h_*\lambda_L(x) + h_*\lambda_L(y)) = \lambda_F^{-1}(\lambda_F(x) + \lambda_F(y)) = F(x, y).$$

□

In terms of the action of the group of power series over R on homomorphisms $L \to R$ the homomorphism $h = (\lambda_F)^{-1} * \epsilon$ where $\varepsilon\colon L \to \mathbb{Z}_2 \subset R$ is the augumentation homomorphism (thus corresonds to the additive group).

6.3 Existence of a logarithm[1]

Theorem 6.3.1. *Let R be a $\mathbb{Z}_2$-algebra and F — a commutative formal group law over R such that $F(x,x)=0$. Then there exist a logarithm of F i.e. a power series $\lambda = x + a_1x^2 + a_2x^3 + \ldots \in R[[x]]$ such that $\lambda(F(x,y)) = \lambda(x) + \lambda(y)$. Moreover the logarithm is unique if we add a condition that $a_j = 0$ if $j = 2^i - 1$ for some i.*

Remark 6.3.1. Such a logarithm will be called a good logarithm. The condition $F(x,x) = 0$ is of course necessary.

Notation. For $F, G \in R[[x,y]]$ we'll write $F \equiv_n G$ if $F - G$ has no monomials of degree $< n$.

Proof.

1. If F, G are formal groups over R and $F \equiv_n G$ then $F \equiv_{n+1} G + a\Gamma$ for some Γ (a homogenuous polynomial of degree n) which satisfies the following: $\Gamma(x,y) = \Gamma(y,x)$, $\Gamma(x,0) = 0 = \Gamma(0,y)$, $\Gamma(x,y) + \Gamma(x+y,z) = \Gamma(x,y+z) + \Gamma(y,z)$.

   The first and the second conditions are obvious. Let $G(x,y) = x + y + G_2(x,y)$, then

   $$F(F(x,y),z) \equiv_{n+1} G(F(x,y),z) + \Gamma(F(x,y),z) \equiv_{n+1} F(x,y) + z + G_2(F(x,y),z) + \Gamma(x+y,z) \equiv_{n+1}$$
   $$\equiv_{n+1} G(x,y) + \Gamma(x,y) + z + G_2(G(x,y),z) + \Gamma(x+y,z) \equiv_{n+1} G(G(x,y),z) + \Gamma(x,y) + \Gamma(x+y,z).$$

   Analogously $F(x, F(y,z)) \equiv_{n+1} G(x, G(y,z)) + \Gamma(x, y+z) + \Gamma(y,z)$.

2. Let $B_n(x,y) = (x+y)^n - x^n - y^n$ and

   $$C_n(x,y) = \begin{cases} B_n(x,y) & \text{if } n \text{ is not a prime power} \\ \frac{1}{p}B_n(x,y) & \text{if } n = p^k, p - \text{ prime} \end{cases}$$

   They a priori belong to $\mathbb{Z}[x,y]$, but we'll send them to $R[x,y]$ by the unique homomorphism $\mathbb{Z} \to R$. Then C_n is primitive, i.e. the highest common factor of its coeffitient is 1, hence the image of C_n in $R[x,y]$ is non-zero.

   Indeed, take p prime and assume that $n \neq p^k$. Then $p \nmid \binom{n}{p^l}$ where l is such that $p^l < n < p^{l+1}$.

   If $n = p^k$, then we want to find l such that $p^2 \nmid \binom{p^k}{l}$. It suffices to take $l = p^{k-1}$.

3. Then if a homogenuous polynomial Γ of degree n satisfies the properties listed in point 1, then $\Gamma = aC_n$ for some $a \in R$.

   Obviously aC_n is a non-zero polynomial in $R[x,y]$ satifying conditions from ooint 1. We'll that every coefficient of such polynomial Γ is determined by a single one.

   Take $\Gamma(x,y) = \sum_{i=0}^{n} a_i x^i y^{n-i}$ satysfying conditions from 1. Then $a_0 = a_n = 0$, $a_i = a_{n-i}$, and after substituting this to the last condition and comparing coefficients near $x^i y^j z^{n-i-j}$ we obtain $\binom{i+j}{j} a_{i+j} = \binom{n-i}{j} a_i$ for $i \neq 0$ and $i + j \neq n$. In particular $(i+1)a_{i+1} = \binom{n-1}{i} a_1$ and $(i+1)a_{i+1} = (n-i)a_i$. Then even coefficients a_{2k} are completely determined by a_1, from the symmetry of coefficients we obtain that for n odd all coefficients are determined by a_1.

   For $n = 2m$ (even) from the last equation we obtain that $a_{2k+1} = 0$, so $\Gamma(x,y) = \sum_{i=0}^{m} a_{2i} x^{2i} y^{2m-2i}$. Hence $\Gamma'(x,y) = \sum_{i=0}^{m} a_{2i} x^i y^{m-i}$ also satisfies the properties from point1 and has degree $\frac{n}{2}$. Decreasing n by induction we obtain the thesis.

4. Corollary: if F, G are formal groups over R and $F \equiv_n G$ then $F \equiv_{n+1} G + aC_n$ for some $a \in R$.

[1] Opracował Jakub Pawlikowski

5. If $F \equiv_{n+1} G + a_n B_n$ then there exist a power series $f_n \in R[[x]]$ s.t. $f_n \equiv_n x$ and $f_n(F(x,y)) \equiv_{n+1} G(f_n(x), f_n(y))$.

   Put $f_n(x) = x - ax^n$. Then

$$\begin{aligned} f_n(F(x,y)) =& F(x,y) - aF(x,y)^n \equiv_{n+1} F(x,y) - a(x+y)^n \equiv_{n+1} G(x,y) + a(x+y)^n - ax^n - ay^n - \\ & - a(x+y)^n = G(x,y) - ax^n - ay^n \equiv_{n+1} G(f_n(x), f_n(y)) \end{aligned}$$

6. If F is as in the theorem, then there exist a good logarithm of F.

   We will construct a sequence of invertible power series $f_n \in R[[x]]$ such that $f_n \equiv_n x$ such that an associated sequence $g_n = f_n \circ \ldots \circ f_1$ is convergent to a good logarithm.

   Suppose that we already have g_{n-1} such that $H := g_{n-1}(F(g_{n-1}^{-1}(x), g_{n-1}^{-1}(y))) \equiv_n x + y$. It is enough to construct $f_n \equiv_n x$ s.t $f_n(H(f_n^{-1}(x), f_n^{-1}(x))) \equiv_{n+1} x + y$ and take $g_n = f_n \circ g_{n-1}$.

   From point 4 we know, that $H(x,y) \equiv_{n+1} x + y + aC_n(x,y)$ for some $a \in R$.

   - If $n \neq 2^k$, then $C_n(x,y) = B_n(x,y)$ (because $2 = 0$), so we can take $f_n(x) = x - ax^n$ from point 5.
   - If $n = 2^k$, then $0 = H(x,x) \equiv_{n+1} 2x + aC_n(x,x) \equiv_{n+1} aC_n(x,x)$. But in $\mathbb{Z}[x,y]$ we have $C_n(x,x) = \frac{1}{2}(\binom{n}{1} + \ldots \binom{n}{n-1})x^n = (2^{n-1} - 1)x^n$, so in $R[x,y]$ a has to be equal to 0.

   Therefore we have built a logarithm $\lambda = \ldots \circ f_k \circ \ldots \circ f_1$, which has no coefficients near x^{2^k}.

7. The good logarithm is unique.

   Let $\lambda, \lambda' \colon F \to F_a$ be two good logarithms of F. Then $\lambda' \circ \lambda^{-1}$ is an isomorphism of the additive group F_a, which has no coefficients near x^{2^k}. It is straightforward that indeed $\lambda' \circ \lambda^{-1} = id$, so $\lambda = \lambda'$.

□

6.4 Formal group law of the oriented cohomology theory

Back to topology. Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and for any n, m consider the Segre embedding of the product of projective spaces $\mu_{n,m} \colon \mathbb{F}P(n) \times \mathbb{F}P(m) \to \mathbb{F}P(k)$ where $k = mn + m + n$.

Zad. 6.4.1. Check that $\mu_{n,m}^! H_k \simeq H_n \otimes H_m$.

Let h^* be an $\mathbb{F}$-oriented cohomology theory. Then

$$h^*(\mathbb{F}P(n)) = h^*[x_n^{\mathbb{F}}]/((x_n^{\mathbb{F}})^{n+1}) \quad \deg x_n^{\mathbb{F}} = \dim_{\mathbb{R}} \mathbb{F}, \quad h^*(\mathbb{F}P(n) \times \mathbb{F}P(m)) = h^*[x_n^{\mathbb{F}}, y_m^{\mathbb{F}}]/((x_n^{\mathbb{F}})^{n+1}, (y_m^{\mathbb{F}})^{m+1}).$$

Let $h^*(\mathbb{F}P(k)) = h^*[z_k^{\mathbb{F}}]/((z_k^{\mathbb{F}})^{k+1})$. Then

$$\mu^*(z_{nm}^{\mathbb{F}}) \in h^*[x_n^{\mathbb{F}}, y_m^{\mathbb{F}}]/((x_n^{\mathbb{F}})^{n+1}, (y_m^{\mathbb{F}})^{m+1})$$

is a restricted power series in two variables which defines a formal series $F_{h^*}(x,y) \in h^*[[x,y]]$. Properties of the Segre embedding imply that $F_{h^*}(x,y)$ is a FGL.

Chapter 7

Unoriented bordism and cobordism

Our aim is to calculate the (co-)bordism ring of nonoriented manifolds. Similarly one can calculate (co-)borism ring of stably almost complex manifolds.

7.1 Definitions

We briefly summarize geometric description of the homology and cohomology theory associated to the spectrum of classifying bundles over grassmannians i.e. determined by the pre-spectrum $\{BO_n^{\gamma_n}\}$. For any space X we have:

$$N_n(X) := \mathcal{B}_n(X;\gamma_*) = \{M \xrightarrow{f} X \mid M \text{ compact manifold, } \dim M = n\}/\sim_{\text{compact bordism}}$$

and the induced homomorphism by any map $X \xrightarrow{g} Y$ is given by composition $g_*([f]) = [gf]$. Cohomology (cobordism) groups are geometrically defined only for smooth manifold Y:

$$N^n(Y) := \mathcal{B}^n(Y;\gamma_*) = \{V \xrightarrow{f} Y \mid f - \text{smooth, proper map, } n = \dim Y - \dim V\}/\sim_{\text{proper bordism}}$$

and the induced homomorphism by a smooth map $X \xrightarrow{g} Y$ is given by the transversal pullback:

$$\begin{tikzcd} V \times_Y X \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"] & V \arrow[d, "f"] \\ X \arrow[r, "g"] & Y \end{tikzcd} \tag{7.1}$$

Note that we do not impose any γ-structure since every real vector bundle has unique γ-structure.

Zad. 7.1.1. Recall geometric definitions of the relative groups.

In the sets N^* and N_* addition is given by disjoint sum and multiplication by the cartesian product of maps. Note that $h^{-k}(pt) = N_k(pt)$. As usual we denote $N_* := N_*(pt)$ and $N^* := N^*(pt)$ for short. This is the ring we want to calculate. Note that nonoriented (co-)bordism functors have values in $\mathbb{Z}_2$-modules, and for any space Y, $N^*(Y)$ is a $\mathbb{Z}_2$-algebra. For any space X, the cobordism ring $N^*(X)$ is a N^*-module.

Zad. 7.1.2. Calculate the groups N_q for $q < 3$.

With the above descriptions of (co-)bordism groups the Poincare duality holds tautologically for any compact manifold X : $N^q(X) \simeq N_{n-q}(X)$.

Zad. 7.1.3. Prove that $N^*(S^n)$ is a free N^*-module with two generators: $1 = [id\colon S^n \to S^n] \in h^0(S^n)$ and $x = [1 \to S^n] \in h^n(S^n)$. For every space X and $n \geq 0$ multiplication

$$N^*(X) \otimes_{h^*} N^*(S^n) \to N^*(X \times S^n)$$

is an isomorphism and induces an isomorphism $N^*(X) \to \tilde{N}^*(X^+ \wedge S^n)$.. $N^*(S^q) \simeq h^*[x]/(x^2)$.

Zad. 7.1.4. Calculate N^*-algebra $N_*(S^{n_1} \times \ldots \times S^{n_k})$.

Zad. 7.1.5.

$$N^q(X, x_0) := \{\phi : Z \to X \setminus x_0 \,:\, \dim X - \dim Z = q \,,\, f \,\text{smooth, proper in}\, X\}/ \sim_{\text{proper bordism}} .$$

If X is a manifold then $N^*(X \sqcup pt, pt) \simeq N^*(X)$

Zad. 7.1.6. Check that $\times$ defines a bilinear multiplication

$$N^p(X, x_0) \times N^q(Y, y_0) \xrightarrow{\wedge} N^{p+q}(X \wedge Y, [x_0, y_0]),$$

such that for $Y = S^q$ it is an isomorphism

$$N^*(X, x_0) \otimes_{N^*} N^*(S^q, y_0) \simeq N^*(X \wedge S^q, [x_0, y_0]).$$

7.2 Formal group of the nonoriented cobordism

Let η be a k-dimensional real vector bundle. Recall construction of the canonical Thom class in nonoriented cobordism. In homotopical interpretation the class

$$U^{\text{can}} \in \tilde{N}^k(\text{Th}(\eta)) = \text{colim}\,[S^r\,\text{Th}(\eta), BO_{r+k}^{\gamma^{r+k}}]$$

is represented by the classifying map $\text{Th}(\eta) \to \text{Th}(\gamma_k)$. If η is a smooth bundle, then in geometric interpretation the canonical Thom class is represented by the zero-section $B \subset \text{Th}(\eta)$. Thus the Euler class $e(\eta) \in N^k(B)$ is represented by the transversal self-intersection $B \pitchfork B \subset B$.

We can give a geometric description of the N^*-algebra $N^*(\mathbb{R}P(n))$. We denote $\mathbb{P}^n := \mathbb{R}P(n)$ for short.

Proposition 7.2.1. *There are isomorphisms of N^*-algebras:*

1. $N^*[x]/(x^{n+1}) \simeq N^*(\mathbb{P}^n)$
2. $N^*[x, y]/(x^{n+1}, y^{m+1}) \simeq N^*(\mathbb{P}^n \times \mathbb{P}^m)$.

such that $x \in N^1(\mathbb{P}^n)$ (resp. y) is represented by the hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$, and $x^k = [\mathbb{P}^{n-k} \subset \mathbb{P}^n]$.

For every nonnegative integers n, m consider external tensor product of the canonical bundles $\gamma_n^1 \hat{\otimes} \gamma_m^1$ over $\mathbb{P}^n \times \mathbb{P}^m$ and its Euler class $e(\gamma_n^1 \hat{\otimes} \gamma_m^1) \in N^1(\mathbb{P}^n \times \mathbb{P}^m)$. From Prop. 7.2.1 it follows that there exist elements $\bar{a}_{ij} \in N^{1-i-j}$ such that

$$e(\gamma_n^1 \hat{\otimes} \gamma_m^1) = x + y + \sum_{0<i\leq n\,, 0<j\leq m} \bar{a}_{ij} x^i y^j$$

The coefficients stabilize i.e. $\bar{a}_{ij} = \bar{a}'_{ij}$ for $n \leq n'$, $m \leq m'$ and $i \leq n$, $j \leq m$. When $n, m \to \infty$ we obtain a power series in $N^*[[x, y]]$:

$$F_{N^*}(x, y) :== x + y + \sum_{i,j\geq 1} \bar{a}_{ij} x^i y^j$$

such that $\bar{a}_{ij} \in N^{-(i+j-1)}$. For any line bundles α, β it calculates the Euler class $e(\alpha \otimes \beta)$:

$$e(\alpha \otimes \beta) = e(\alpha) + e(\beta) + \sum_{i,j \geq 1} \bar{a}_{ij} e(\alpha)^i e(\beta)^j.$$

Proposition 7.2.2. *The power series F_{N^*} is a commutative formal group such that $F_{N^*}(x,x) = 0$.*

For every n, m let $f_{n,m} \colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$ be the Serge map $f_{n,m}(L, L') = L \otimes L'$ in coordinates given by the formula $f_{n,m}([t_0; t_1; ..; t_n], [z_0; ..; z_m]) := [t_0 z_0; ..; t_i z_j; ..; t_n z_m]$. For fixed natural numbers $0 \leq m \leq n$ the Milnor manifold $H(m,n)$ is defined as the hypersurface in $\mathbb{P}^n \times \mathbb{P}^m$ satisfying the equation $x_0 y_0 + ... + x_i z_i = 0$, where x_k and z_k are homogeneous coordinates for $\mathbb{P}^n$ i and $\mathbb{P}^m$ respectively. This equation defines a generic hyperplane intersecting the image of the Segre embedding $f_{n,m} \colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$ transversaly.

Proposition 7.2.3.

$$F_{N^*}(x,y) = \frac{x + y + \sum\limits_{m,n>0}^{\infty} [H(m,n)] x^m y^n}{\mathbb{P}(x)\mathbb{P}(y)}$$

where $\mathbb{P}(x) := \sum_i [\mathbb{P}^i] x^i$ and $H(m,n)$ are the Milnor manifolds.

Proof. We know that $f^!_{n,m} \gamma^1_r = \gamma^1_n \hat{\otimes} \gamma^1_m$. Consider the induced map on the cobordism gropus $f_{n,m} \colon N^1(\mathbb{P}^r) \to N^1(\mathbb{P}^n \times \mathbb{P}^m)$. It maps the standard generator $[\mathbb{P}^{r-1} \subset \mathbb{P}^r]$ to so-called Milnor manifold $[H(n,m) \subset \mathbb{P}^n \times \mathbb{P}^m] \in N^1(\mathbb{P}^n \times \mathbb{P}^m)$. Let $F_{N^*}(x,y) = x + y + \sum a_{i,j} x^i y^j$. We need to calculate coefficients. By definition:

$$[H(n,m) \subset \mathbb{P}^n \times \mathbb{P}^m] = \sum_{i,j \geq 1} a_{i,j} x^i y^j = \sum_{i,j \geq 1} a_{i,j} [\mathbb{P}^{n-i} \subset \mathbb{P}^n][\mathbb{P}^{n-j} \subset \mathbb{P}^m].$$

thus

$$[H(n,m)] = \sum_{i,j \geq 1} a_{i,j} x^i y^j = \sum_{i,j \geq 1} a_{i,j} [\mathbb{P}^{n-i}][\mathbb{P}^{n-j}].$$

Let u, v be new variables. By multiplying both sides with $u^n v^m$ and summing up we obtain:

$$\sum_{m,n} [H(m,n)] u^m v^n = \sum_{k,l} a_{kl} u^k v^l (\sum_{m \geq k} [\mathbb{P}^{m-k}] u^{n-k}) (\sum_{n \geq l} [\mathbb{P}^{n-l}]) v^{m-l}$$

$\square$

Chapter 8

The Thom classes and multiplicative transformations

8.1 Multiplicative transformations

Let (L, F_L) be the universal FGL described in Prop. 6.2.2 and $f_N\colon L \to N^*$ be the ring homomorphism such that $f_{N*}F_L = F_{N^*}$. It defines an L-algebra structure on the cohomology theory $N^*(-)$. For any ring homomorphism $h\colon L \to R$ (i.e. for every FGL over R) we will consider a functor from the category of pointed spaces to the category of graded abelian groups.

$$\tilde{N}^*_h(X) := \tilde{N}^*(X) \otimes_L R_h$$

(subscript h indicates that L acts on R via the ring homomorphism $h\colon L \to R$). The functor $\tilde{N}^*_h(X)$ is a cohomology-like functor (cf. Sec. 5.5.1) ; it carries both additive and multiplicative structure. Moreover its values are right R-modules.

Note that the tensor product is a direct sum in the category of L-algebras. There are canonical homomorphisms

$$j_N\colon N^*(X) \ni x \mapsto x \otimes_L 1 \in \tilde{N}^*(X) \otimes_L R_h \quad \text{and} \quad j_R\colon R \ni r \mapsto 1 \otimes_L r \in \tilde{N}^*(X) \otimes_L R_h.$$

Every power series over one of the factors can be pushed-forward via j_N or j_N to power series over $\tilde{N}^*_h$; we denote $j_{R*}\varphi =: \bar{\varphi}$ i.e. if $\varphi(x) = \sum r_i x^i$ then $\bar{\varphi}(x) = \sum (1 \otimes_L r_i)x^i$.

If γ is the spectrum of the universal real vector bundles, then the functor $\tilde{N}^*_h(X)$ has the canonical γ-orientation $U^{\mathrm{can}}_h := U^{\mathrm{can}} \otimes_L 1$. Multiplication $U^{\mathrm{can}}_h \cup -\colon N^*_h(X) \to \tilde{N}^*_h(X^\alpha)$ is an isomorphism for any vector bundle α. A multiplicative Thom class (MTC) in $\tilde{N}^*_h(-)$ is a γ-orientation satisfying a normalization axiom i.e.

Definition 8.1.1. *Multiplicative Thom class in $\tilde{N}^*_h(X)$ associates to every q-dimensional vector bundle $E \to X$ an element $u(E) \in \tilde{N}^q_h(\mathrm{Th}(E))$ such that:*

1. $\bar{f}^*(u(E)) = u(f^*E)$
2. $u(E_1 \times E_2) = u(E_1) \times u(E_2)$
3. $u(\mathbb{R}) = \sigma(1) \otimes_L 1 \in \tilde{N}^1_h(S^1)$.

Theorem 8.1.1. *For every MTC $u(-)$ in $\tilde{N}^*_h(-)$ there exists exactly one multiplicative natural transformation of the CLF*

$$\Theta_u\colon \tilde{N}^*(-) \to \tilde{N}^*_h(-).$$

such that for every bundle $E \to X$ the equality $\Theta_u(U^{can}) = u(E)$ holds. The transformation Θ_u is also additive thus it is a transformation of the ring-valued functors.

Proof. It follows immediately from Prop. 5.5.1 □

8.2 The exponential (difference) class

As we already noticed the canonical Thom class establishes a natural isomorphism of $\tilde{N}_h^*(X)$-modules $U_N^{\text{can}} \cup -: N_h^*(X) \xrightarrow{\cong} \tilde{N}_h^*(X^\alpha)$ i.e. $\tilde{N}_h^*(X^\alpha)$ is a free $\tilde{N}_h^*(X)$-module with single generator U_h^{can}. Thus for any other MTC $u(-)$ and vector bundle $E \to X$ there exist an element $v(E) \in N_N^0(X)$ such that for every vector bundle E: $u(E) = v(E) \cup U_h^{\text{can}}(E)$. Conversely, we have:

Proposition 8.2.1. *If the assignment $E \mapsto v(E) \in N_h^0(X)$ satisfies the following conditions:*

1. *(Natural) $f^*v(E) = v(f^*E)$*
2. *(Multiplicative) $v(E \times E') = v(E)v(E')$*
3. *(Normed) $v(\mathbb{R}) = 1$*

Then $u(E) := v(E) \cup U_N^{can}(E)$ is a MTC.

Proposition 8.2.2. *If v, w are two exponential classes in $N_h^0(-)$ such that $v(\gamma_n^1) = w(\gamma_n^1)$ for all $n \geq 0$ then $v = w$.*

Proof. Carefully apply method of the splitting principle. □

The assignment which satisfies the above conditions (1-3) we call an *exponential class*. Exponential classes are in bijective correspondence with MTC.

Proposition 8.2.3. *For every homomorphism $h: L \to R$ and a power series*

$$\varphi(x) = x + r_1x^2 + r_2x^3 + ... \in R[[x]]$$

of gradation 1 there exists exactly one exponential class $v_\varphi \in \tilde{N}_h^0(-)$ such that for any line bundle η over a space X,

$$v_\varphi(\eta) = \bar{\varphi}_1(e_\eta \otimes_L 1) = 1 + e_\eta \otimes r_1 + e_\eta^2 \otimes r_2 + ... \in N_h^0(B)$$

where $\varphi(x) = x\varphi_1(x)$ and $e_\eta := e(\eta) \in N^1(X)$ is the canonical Euler class.

Proof. For every n the power series φ determines an element in $\tilde{N}_h^*(\mathbb{P}^n) = \tilde{N}^* \otimes_L R[x]/(e_n^{n+1})$ thus we can define $v(E)$ for every line bundle of finite type and it obviously satisfies the above equation. We extend the definition to arbitrary vector bundles carefully applying the method of the splitting principle. □

Applying the last proposition one can assign to each ring homomorphism $h : L \to R$ and a power series $\varphi(x) = x + r_1x^2 + r_2x^3 + ... \in R[[x]]$ of gradation 1 a natural transformation

$$\Theta_\varphi : \tilde{N}^*(-) \to \tilde{N}_h^*(-)$$

in following steps:

1. we assign to a series φ an exponential class v_φ using Prop. 8.2.3

2. the exponential class v_φ defines MTC by the formula $u_\varphi(E) = v_\varphi(E) \cup (U_N^{\mathrm{can}}(E) \otimes 1)$

3. universal property of the functor N^*, MCT u_φ implies existence of the natural transformation $\Theta_\varphi : \tilde{N}^*(-) \to \tilde{N}_h^*(-)$ (Thm. 8.1.1) such that $\Theta_\varphi(U_N^{\mathrm{can}}(E)) = u_\varphi(E)$.

Proposition 8.2.4. *For any line bundle η over X let $e_\eta := e(\eta) \in N^1(X)$ be its Euler class. Then*

$$\Theta_\varphi(e_\eta) = \bar{\varphi}(e_\eta \otimes 1) = e_\eta \otimes 1 + e_\eta^2 \otimes r_1 + e_\eta^3 \otimes r_2 + \dots \tag{8.1}$$

Proof. $e_\eta = s_0^* U^{can}(\eta) \in N^1(X)$ and $u_\varphi(\eta) = v_\varphi(\eta) \cup U^{can}(\eta)$. Since Θ_φ is a natural transformation

$$\Theta_\varphi(e_\eta) = s_0^*(\Theta_\varphi(U^{can}(\eta)) = s_0^*(v_\varphi(\eta) \cup U^{can}(\eta)) = \bar{\varphi}_1(e_\eta \otimes_L 1) s_0^*(U^{can}(\eta)) \otimes 1 = \bar{\varphi}(e_\eta \otimes 1).$$

□

Example 6. For the identity series $\iota(x) = x$, $\Theta_\iota(x) = x \otimes id$. If $\varphi \in L[[x]]$ and $h = id$ then the transformation $\Theta_\varphi \colon \tilde{N}^*(-) \to \tilde{N}_{id}^*(X) = \tilde{N}^*(-)$ is defined by the series $f_{N*}\alpha$.

Recall that the group of formal series $\varphi(x) = x + r_1x^2 + r_2x^3 + ... \in R[[x]]$ of gradation 1 acts on the set of homomorphisms $\mathrm{Hom}\,(L^*, R^*)$. Indeed, each homomorphism $h \colon L \to R$ defines FGL F_h on R^*. Define the formal group $\varphi_* F_h(x, y) := \varphi F(\varphi^{-1}(x), \varphi^{-1}(y))$. The corresponding homomorphism we denote $\varphi * h \colon L \to R$. It is easy to check that it is a group action.

Lemma 8.2.1. *For any homomorphism $h \colon L \to R$ and a series $\varphi(x) = x + r_1x^2 + r_2x^3 + ... \in R[[x]]$ let $g := \varphi * h \colon L \to R$, $\bar{g}(l) := (j_1 g)(l) = 1 \otimes_L g(l)$. Then the following diagram commutes:*

$$\begin{tikzcd} L \arrow[rr, "\bar{g}"] \arrow[dr, "f_N"'] & & N^* \otimes_L R_h \\ & N^* \arrow[ur, "\Theta_\varphi"'] & \end{tikzcd} \tag{8.2}$$

Proof. To prove that two homomorphism defined on the universal FGL ring L are equal it is enough to show that they induce the same FGL i.e. $\bar{g}_* F_L = (\Theta_\varphi)_* f_{N*} F_L = (\Theta_\varphi)_* F_N$. Let $F_L(x, y) = \sum a_{ij} x^i y^j$.

Consider two natural homomorphisms: $N^* \xrightarrow{i_1} N^* \otimes_L R_h \xleftarrow{i_2} R$ and corresponding extensions of formal groups: $i_{1*} F_N$ and $i_{2*}(\varphi_* F_h)$. Note that

$$\begin{aligned} i_{1*}F_N(x,y) = (F_N \otimes_L 1)(x,y) = \sum (f_N(a_{ij}) \otimes_L 1) x^i y^j = \sum (1 \otimes_L h(a_{ij})) x^i y^j = \\ = (1 \otimes_L h)_* F_L(x,y) = (1 \otimes_L F_h)(x,y) = i_{2*} F_h(x,y) \end{aligned} \tag{8.3}$$

For any series F over R denote $\bar{F} := i_{2*}F$. By definition $(\varphi_* F_h)(x, y) := \varphi F_h(\varphi^{-1}(x), \varphi^{-1}(y))$ thus $(\varphi_* F_h)(\varphi(x), \varphi(y)) = \varphi F_h(x, y)$ and the same equality holds for extended groups:

$$\bar{\varphi} \bar{F}_h(x, y) = \overline{\varphi_* F_h}(\bar{\varphi}(x), \bar{\varphi}(y))$$

Then we have:

$$\bar{g}_* F_L(x,y) = \sum (1 \otimes g(a_{ij})) x^i y^j = 1 \otimes \sum g(a_{ij}) x^i y^j = (1 \otimes \varphi_* F_h)(x,y) = \overline{\varphi_* F_h}(x,y) \tag{8.4}$$

Thus we need to prove that $\overline{\varphi_* F_h}(x, y) = \Theta_{\varphi *} F_N(x, y)$. We calculate:

$$(\Theta_\varphi)_* F_N(x,y) = (\Theta_\varphi)_* (\sum f_N(a_{ij}) x^i y^j) = \sum \Theta_\varphi f_N(a_{ij}) x^i y^j =: G(x,y)$$

For any line bundles ξ, η we denote the Euler classes with subscript e.g. $e(\xi) := e_\xi$. By definition we have:

$$\Theta_\varphi(e_{\xi\otimes\eta}) = \bar{\varphi}(e_{\xi\otimes\eta} \otimes 1).$$

We express both sides of the equation:

$$\begin{aligned} \Theta_\varphi(e_{\xi\otimes\eta}) = \Theta_\varphi(\sum f_N(a_{ij}) e_\xi^i e_\eta^j) = \sum \Theta_\varphi f_N(a_{ij}) \Theta_\varphi e_\xi^i \Theta_\varphi e_\eta^j = \\ = \sum \Theta_\varphi f_N(a_{ij}) \bar{\varphi}(e_\xi)^i \bar{\varphi}(e_\eta)^j = G(\bar{\varphi}(e_\xi), \bar{\varphi}(e_\eta)) \end{aligned} \tag{8.5}$$

and

$$\bar{\varphi}(e_{\xi\otimes\eta} \otimes 1) = \bar{\varphi}\bar{F}_h(e_\xi \otimes 1, e_\eta \otimes 1) \tag{8.6}$$

thus

$$\overline{\varphi_* F_h}(\bar{\varphi}(e_\xi \otimes 1), \bar{\varphi}(e_\eta \otimes 1)) = \bar{\varphi}\bar{F}_h(e_\xi \otimes 1, e_\eta \otimes 1) = G(\bar{\varphi}(e_\xi \otimes 1), \bar{\varphi}(e_\eta \otimes 1)) \tag{8.7}$$

Since it is true for any line bundles we obtain that

$$\overline{\varphi_* F_h}(x, y) = (\Theta_{\varphi *} F_N(x, y).$$

Hence for every coefficient of the universal FGL we have: $\Theta_\varphi f_N(a_{ij}) = 1 \otimes_L g(a_{ij})$ (cf. 8.4). The coefficients of F_L generate the ring L, hence $\Theta_\varphi f_N = 1 \otimes_L g$. $\square$

Proposition 8.2.5. *For every series $\varphi(x) = x + r_1 x^2 + r_2 x^3 + ... \in R[[x]]$ of gradation 1, the natural transformation $\Theta_\varphi : \tilde{N}^*(-) \to \tilde{N}^*_h(-)$ extends to a natural transformation*

$$\begin{tikzcd}
\tilde{N}^*_{\varphi * h}(-) \arrow[rr, "\bar{\Theta}_\varphi"] & & \tilde{N}^*_h(-) \\
& \tilde{N}^*(-) \arrow[ul, "\Theta_x"] \arrow[ur, "\Theta_\varphi"'] &
\end{tikzcd} \tag{8.8}$$

such that $\bar{\Theta}_{\varphi_2 \circ \varphi_1} = \bar{\Theta}_{\varphi_1} \circ \bar{\Theta}_{\varphi_2}$ and $\bar{\Theta}_x = id$. The transformation $\bar{\Theta}_\varphi$ is an isomorphism.

Proof. We define a group homomorphism $\bar{\Theta}_\varphi \colon \tilde{N}^*(-) \otimes_{\mathbb{Z}} R \to \tilde{N}^*_h(-) = N^*(-) \otimes_L R_h$ by the formula $\bar{\Theta}_\varphi(x \otimes r) := \Theta_\varphi(x)(1 \otimes r)$. Let $g := \varphi * h$ for short. To prove that $\bar{\Theta}_\varphi$ factors through the tensor product $\tilde{N}^*_g(-)$ we have to show that for any element $l \in L$

$$\bar{\Theta}_\varphi(xl \otimes r) := \Theta_\varphi(x)\Theta_\varphi(f_N(l))(1 \otimes r) = \Theta_\varphi(x)(1 \otimes g(l) r) =: \bar{\Theta}_\varphi(x \otimes g(l) r).$$

Indeed, by Lemma 8.2.1 $\Theta_\varphi(f_N(l)) = 1 \otimes g(l)$ hence for every $r \in R$ the equality $\Theta_\varphi(f_N(l))(1 \otimes r) = 1 \otimes g(l) r$ holds. $\square$

Consider the augmentation homomorphism $\varepsilon \colon L \to \mathbb{Z}_2 \subset L$, $\varepsilon(1) = 1$, $\varepsilon(a_i) = 0$ and let the series $\lambda_L(x)$ be a logarithm of the universal formal group, defined in Prop. 6.2.2 . Obviously (cf. Prop. 6.2.2) $\lambda^{-1} * \varepsilon = id$. Consider the transformation homomorphism

$$\begin{tikzcd}
N^*(-) = \tilde{N}^*_{\lambda^{-1} * \varepsilon}(-) \arrow[rr, "\bar{\Theta}_{\lambda^{-1}}", "\simeq"'] & & \tilde{N}^*_\varepsilon(-) \\
& \tilde{N}^*(-) \arrow[ul, "\Theta_x = id"] \arrow[ur, "\Theta_{\lambda_L^{-1}}"'] &
\end{tikzcd} \tag{8.9}$$

Theorem 8.2.1. *The ring homomorphism $f_N \colon L \to N^*$ is injective.*

Proof. Consider commutative diagram defined in Prop. 8.2.1.

$$\begin{tikzcd} L \arrow[rr, "1\otimes_L id"] \arrow[dr, "f_N"'] & & N^* \otimes_L L_\varepsilon \\ & N^* \arrow[ur, "\simeq", "\Theta_{\lambda_L^{-1}}"'] & \end{tikzcd} \tag{8.10}$$

The top horizontal arrow is obviously injective ($N^0 = \mathbb{Z}_2$) thus also f_N is injective. ☐

For proving in the next chapter that f_N is indeed an isomorphism we need the following:

Proposition 8.2.6. *The functor $\tilde{h}^*(-) := \tilde{N}^*(-) \otimes_L \mathbb{Z}_2$ is a cohomology theory and there is an isomorphism of cohomology theories $\tilde{N}^*(-) \xrightarrow{\simeq} \tilde{h}^*(-) \otimes_{\mathbb{Z}_2} L$.*

Proof. We have to check exactness axiom. For a cofibration sequence $X \to Y \to C$ consider the induced sequence $\tilde{h}^*(C) \to \tilde{h}^*(Y) \to \tilde{h}^*(X)$ is exact. Note that according to 8.9 $\tilde{h}^*(-) \otimes_{\mathbb{Z}_2} L \simeq N^*(-)$ thus is a cohomology theory. Since L is a free $\mathbb{Z}_2$-module it follows that tensoring with L is faithfully flat, thus the sequence $\tilde{h}^*(C) \to \tilde{h}^*(Y) \to \tilde{h}^*(X)$ is exact. ☐

In the next chapter we'll prove that $h^*(-)$ is a classical cohomology theory (i.e. $h^q(pt) = 0$ for $q \neq 0$), thus the following theorem holds:

Theorem 8.2.2. *There exists a natural isomorphism of multiplicative cohomology theories*

$$\tilde{N}^*(X) \xrightarrow{\simeq} \tilde{H}^*(X; \mathbb{F}_2) \otimes_{\mathbb{F}_2} L,$$

in particular an isomorphism of rings with formal group laws: $(N^*, F_N) \simeq (L, F_L)$.

Chapter 9

Cohomological operations in cobordism

9.1 Construction of the Steenrod squares in cobordism

Theorem 9.1.1. *There exist (additive) natural transformations $R^i \colon \tilde{N}^n(X, A) \to \tilde{N}^{n+i}(X)$, $R := \sum R^i$ such that:*

1. *Naturality. If $f \colon (X, x_0) \to (Y, y_0)$ then $R \circ f^* = f^* \circ R$.*
2. *Stability. The operations R^i commute with the suspension isomorphism.*
3. *The Cartan formula. $R(a \times b) = R(a) \times R(b)$.*
4. *If $a \in \tilde{N}^n(X)$, then $R^n(a) = a \cup a$, and $R^i(a) = 0$ for $i > n$.*

The rest of the present section will be devoted to construction of the Steenrod squares. Similarly as in the classical cohomology, they can be considered as a kind of "filtration" of the squaring operation:

$$\tilde{N}^n(X/A) = N^n(X, A) \ni x \mapsto x \times x \in N^{2n}(X \times X, X \times A \cup A \times X) = \tilde{N}^{2n}(X/A \wedge X/A).$$

Note that because $\operatorname{char} N^* = 2$, the squaring map is an additive homomorphism. The "filtration" comes from action of the cyclic group $\mathbb{Z}_2$ on $X \times X$ by permuting factors. We will consider pointed spaces and equivariant cohomology of pointed $\mathbb{Z}_2$-spaces.

For an arbitrary pointed space (X, x_0) consider its smash-square $(X \wedge X, \hat{x}_0)$ - to shorten notation we'll denote its base-point $[x_0, x_0] =: \hat{x}_0$. The cyclic group $\mathbb{Z}_2$ acts on $(X \wedge X, x_0)$ by permutation of coordinates. For a universal principal $\mathbb{Z}_2$-bundle $E\mathbb{Z}_2 \to B\mathbb{Z}_2$ consider an associated fiber bundle pair $E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (X \wedge X, x_0) \to B\mathbb{Z}_2$.The diagonal map $\Delta \colon (X, x_0) \subset (X \wedge X, x_0)$ is precisely inclusion of the $\mathbb{Z}_2$-fixed-point set and it induces an inclusion of the trivial bundle:

$$\begin{tikzcd}
(X, x_0) \arrow[rr, "\Delta"] \arrow[d] & & (X \wedge X, \hat{x}_0) \arrow[d] \\
B\mathbb{Z}_2 \times (X, x_0) \arrow[rr, "\Delta_{\mathbb{Z}_2}"] \arrow[dr] & & E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (X \wedge X, \hat{x}_0) \arrow[dl] \\
& B\mathbb{Z}_2 &
\end{tikzcd} \tag{9.1}$$

The top square defines the following diagram of cohomology groups in which we'll be looking for the natural lifting of the squaring map (dotted arrow), and show that it is unique:

Proposition 9.1.1. *There exists unique natural transformation P^e for which the diagram:*

$$\begin{tikzcd} & \tilde{N}^{2n}_{\mathbb{Z}_2}(X \wedge X) \arrow[r, "\Delta^*_{\mathbb{Z}_2}"] \arrow[d, "j^*"] & \tilde{N}^{2n}_{\mathbb{Z}_2}(X) \arrow[d, "j^*"] \\ \tilde{N}^n(X) \arrow[ur, dotted, "P^e_X"] \arrow[r, "x \times x"] & \tilde{N}^{2n}(X \wedge X) \arrow[r, "\Delta^*"] & \tilde{N}^{2n}(X) \end{tikzcd} \tag{9.2}$$

commutes.

Proof. In order to construct the natural transformation P^e we'll refer to the universal properties of the unoriented cobordism. We need to prove that $\tilde{N}^{2n}_{\mathbb{Z}_2}(X \wedge X)$ is a MCT in which all bundles are naturally oriented. Indeed, for any vector bundle $E \to X$, the bundle $E \times E \to X \times X$ is a $\mathbb{Z}_2$-vector bundle and

$$\mathrm{Th}(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (E \times E)) \simeq (E\mathbb{Z}_2)^+ \wedge_{\mathbb{Z}_2} \mathrm{Th}(E \times E) \simeq (E\mathbb{Z}_2)^+ \wedge_{\mathbb{Z}_2} (\mathrm{Th}(E) \wedge \mathrm{Th}(E)).$$

The $2n$-dimensional vector bundle $E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (E \times E) \to E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (X \times X)$ has the Thom class $U^{can}_{E\mathbb{Z}_2 \times_{\mathbb{Z}_2}(E\times E)} \in \tilde{N}^{2n}_{\mathbb{Z}_2}(\mathrm{Th}(E) \wedge \mathrm{Th}(E))$. Moreover its restriction to $E \times E \to X \times X$ is equal to $(U^{can}_E)^2 \in \tilde{N}^{2n}(\mathrm{Th}(E) \wedge \mathrm{Th}(E)))$.

Thus there exists unique natural transformation $P^e_X \colon \tilde{N}^n(X) \to \tilde{N}^{2n}_{\mathbb{Z}_2}(X \wedge X)$ such that for every vector bundle $P^e_{\mathrm{Th}(E)}(U^{can}_E) = U^{can}_{E\mathbb{Z}_2 \times_{\mathbb{Z}_2}(E\times E)}$. □

We're ready to define the Steenrod squaring operations. Consider composition of maps in diagram 9.2 with the Künneth isomorphism and denote it P^*, and it i–th component by P_i.

$$\begin{tikzcd} \tilde{N}^n(X) \arrow[r, dotted, "P^*"] \arrow[d, "P^e"] & (\tilde{N}^*(X)[t])^{2n} \arrow[r, "\simeq"] & \bigoplus\limits_{i=0}^{\infty} \tilde{N}^{2n-i}(X) \\ \tilde{N}^{2n}_{\mathbb{Z}_2}(X \wedge X) \arrow[r, "\Delta^*_{\mathbb{Z}_2}"] & \tilde{N}^{2n}((B\mathbb{Z}_2)^+ \wedge X) \arrow[u, "\simeq"'] & \end{tikzcd} \tag{9.3}$$

where $\deg t = 1$. Thus $P^*(x) = \sum\limits_{i=0}^{\infty} P_i(x)t^i$ where $P_i(x) \in N^{2n-i}(X, x_0)$. Note that in the diagram 9.3 two arrows $\Delta^*_{\mathbb{Z}_2}$ and Künneth isomorphism $\simeq$ are additive and multiplicative. The lifting P^e is only a transformation of the pointed set valued functors, preserving multiplication. In the next section we'll prove that in fact it preserves addition and cross product.

Definition 9.1.1. *We define operations $R^i \colon N^n(X, x_0) \to N^{n+i}(X, x_0)$ by the formula*

$$R^i(x) := P_{n-i}(x) \in \tilde{N}^{2n-(n-i)}(X) = \tilde{N}^{n+i}(X)$$

From the definition it follows immediately that $R^i(x) = 0$ for $i > \deg(x)$ and $R^n(x) = x^2$ for $n = \deg(x)$.

9.2 Checking axioms

We'll prove that natural transformations of Set-valued functors $R^i\colon \tilde{N}^n(X) \to \tilde{N}^{n+i}(X)$, defined in Def. 9.1.1 satisfy the following properties (in different order then in Thm. 9.1.1) :

1. If $a \in N^n(X, A)$, then $R^i(a) = 0$ for $i > n$, $R^n(a) = a \cup a$,
2. $R^k(a \times b) = \sum_{i+j=k} R^i(a) \times R^j(b)$,
3. Operations R^k commute with the suspension isomorphism, thus they are additive.

Proposition 9.2.1. *If $a \in \tilde{N}^n(X)$, then $R^i(a) = 0$ for $i > n$ and $R^n(a) = a \cup a$*

Proof. It follows immediately from the definition. □

Proposition 9.2.2 (The Cartan formula)**.** $R^k(a \times b) = \sum_{i+j=k} R^i(a) \times R^j(b)$

Proof. The Cartan formula is equivalent to the claim that transformation

$$P^e_X\colon \tilde{N}^n(X) \to N^{2n}(E\mathbb{Z}_2^+ \wedge_{\mathbb{Z}_2} X)$$

is multiplicative (cf. remark before Defn. 9.1.1). i.e. that the following diagram of natural transformations commutes:

$$\begin{tikzcd}
\tilde{N}^n(X) \times \tilde{N}^m(Y) \arrow[r, "\times"] \arrow[d, "P^e_X \times P^e_Y"] & \tilde{N}^{n+m}(X \wedge Y) \arrow[d, "P^e_{X\wedge Y}"] \\
\tilde{N}^{2n}_{\mathbb{Z}_2}(X \wedge X) \times \tilde{N}^{2m}_{\mathbb{Z}_2}(Y \wedge Y) \arrow[r, "\times"] & \tilde{N}^{2(n+m)}_{\mathbb{Z}_2}(X \wedge Y)^2)
\end{tikzcd} \tag{9.4}$$

□

Corollary 9.2.1. *The operations R^i are stable i.e. it commutes with the suspension isomorphism:*

$$\begin{tikzcd}
\tilde{N}^n(X) \arrow[r, "R^i"] \arrow[d, "\sigma", "\simeq"'] & \tilde{N}^{n+i}(X) \arrow[d, "\sigma", "\simeq"'] \\
\tilde{N}^{n+1}(\Sigma X) \arrow[r, "R^i"] & \tilde{N}^{n+i+1}(\Sigma X)
\end{tikzcd} \tag{9.5}$$

Proof. The suspension isomorphism σ is given by cross product with $\iota_1 \in \tilde{N}^1(S^1)$: $\sigma(x) = \iota_1 \times x$. It is easy to check that $R^k(\iota_1) = \begin{cases} 0 & \text{if} \quad k \neq 0 \\ \iota_1 & \text{if} \quad k \neq 0 \end{cases}$. Thus

$$R^i\sigma(x) = R^i(\iota_1 \times x) = R^0(\iota_1) \times R^i(x) = \iota_1 \times R^i(x) = \sigma R^i(x).$$

□

Proposition 9.2.3. *The operation R^i is additive i.e. $R^i(x_1 + x_2) = R^i(x_1) + R^i(x_2)$.*

Proof. It follows easily from Cor. 9.2.1.

□

Proposition 9.2.4. *For any line bundle η, $\Delta^*_{\mathbb{Z}_2} P^e(e_\eta) = p_2^!(e_\eta) e_{\gamma\hat{\otimes}\eta}$.*

9.3 F_N is the universal FGL

Consider again the augumentation homomorphism $\varepsilon\colon L \to \mathbb{Z}_2$. In view of Prop. 8.2.6 in order to prove that $f_N\colon L \to N^*$ is an isomorphism we need to show that the functor $\tilde{N}^*_\varepsilon(-) := N^*(-) \otimes_L \mathbb{Z}_2$ is the classical cohomology theory i.e.

$$\tilde{N}^q_\varepsilon(S^0) = \begin{cases} \mathbb{Z}_2 & \text{dla } q = 0; \\ 0 & \text{dla } q \neq 0. \end{cases}$$

For technical reasons we'll also consider compositions of ε with inclusions of the base ring $\mathbb{Z}_2$ into the graded polynomial rings $\mathbb{Z}_2[s]$ and $\mathbb{Z}_2[s, s^{-1}]$ where $\deg s = -1$. Note that

$$N^*(-) \otimes_L \mathbb{Z}_2[s] = \tilde{N}^*_\varepsilon(-)[s].$$

Consider the series $\varphi(x) := x + sx^2 \in \mathbb{Z}_2[s][[x]]$ and the associated natural transformation

$$\Theta_\varphi : \tilde{N}^*(X) \to \tilde{N}^*_\varepsilon(X)[s].$$

Proposition 9.3.1. *There exists a natural transformation $\bar{\Theta}_\varphi$ such that the following diagram commutes:*

$$\begin{tikzcd} & \tilde{N}^*_\varepsilon(-) & \\ \tilde{N}^*_\varepsilon(-) \arrow[ur, "id"] \arrow[rr, "\bar{\Theta}_\varphi"] & & \tilde{N}^*_\varepsilon(-)[s] \arrow[ul, "s=0"'] \\ & \tilde{N}^*(-) \arrow[ul, "\Theta_x"] \arrow[ur, "\Theta_\varphi"'] & \end{tikzcd} \tag{9.6}$$

Proof. The conclusion follows from Prop. 8.2.5 which implies existence of $\bar{\Theta}_\varphi : \tilde{N}^*_{\varphi * \bar{\varepsilon}}(X) \to \tilde{N}^*(X)[s]$. Since the series $\varphi(x) - x = sx^2$ contains only even power of x, thus it is an automorphism of the additive group, hence $\phi * \bar{\varepsilon} = \bar{\varepsilon}$ and $\bar{\Theta}_\varphi\colon \tilde{N}^*_\varepsilon(X)[s] \to \tilde{N}^*_{\bar{\varepsilon}}(X)[s]$. It is easy to see that its restriction to $N^*_\varepsilon(X) \subset N^*_\varepsilon(X)[s]$ makes the upper triangle commute. □

The next lemma describes the crucial relation between the transformation $\bar{\Theta}_\varphi$ and operations R^k defined in the previous section.

Proposition 9.3.2. *For any $x \in \tilde{N}^n(X)$ the following equation holds in the ring $\tilde{N}^*(X) \otimes_L \mathbb{F}_2[s, s^{-1}]$:*

$$\Theta_\varphi(x) = \sum_{i=0}^{\infty} R^{n-i}(x) \otimes s^{n-i}$$

Proof. First we prove that the right hand-side formula defines a stable, multiplicative natural transformation $\Theta'_\varphi\colon \tilde{N}^*(X) \to \tilde{N}^*_\varepsilon(X)[s, s^{-1}]$. Indeed:

$$\Theta'_\varphi(z_1)\Theta'_\varphi(z_2) = \sum_{i,j=0}^{\infty} R^{n_1-i}(z_1)R^{n_2-j}(z_2) \otimes_L s^{n_1+n_2-(i+j)} = \sum_{k=0}^{\infty} \sum_{i+j=k} R^{n_1-i}(z_1)R^{n_2-j}(z_2) \otimes_L s^{n_1+n_2-k} =$$

$$= \sum_{k=0}^{\infty} R^{n+m-k}(z_1 z_2) \otimes_L s^{n_1+n_2-k} = \Theta'_\varphi(z_1 z_2)$$

Stability follows immediately from the stability of the operations R^i. To show that two multiplicative, stable operations are equal it is enough to show that on the Euler classes of line bundles. Let η be a line bundle and $e_\eta \in \tilde{N}^1(X)$. We denote $t := e_\gamma$; thus we have:

$$P^*(e_\eta) = \sum_{j=0}^{\infty} R^{1-j}(e_\eta) t^j = e_\eta e_{\gamma\otimes\eta} = e_\eta(e_\gamma + e_\eta + \sum_{i,j\geq 1} \bar{a}_{ij} e_\gamma^i e_\eta^j) =$$

$$= e_\eta^2 + (e_\eta + \sum_{j=1}^{\infty} \bar{a}_{i1} e_\eta^i) t^1 + \sum_{j>1} (\sum_{i=1}^{\infty} a_{ij} e_\eta^i) t^j$$

hence

$$R^{1-j}(e_\eta) = \begin{cases} e_\eta^2 \quad \text{for} \quad j=0 \\ e_\eta + \sum_{i=1}^{\infty} \bar{a}_{i1} e_\eta^i \quad \text{for} \quad j=1 \\ \sum_{i=1}^{\infty} \bar{a}_{ij} e_\eta^i \quad \text{for} \quad j>1 \end{cases}$$

Note that $\bar{a}_{ij} \otimes_1 = 1 \otimes_L \bar{\varepsilon}(a_{ij}) = 0$ for $i, j \geq 1$. Thus

$$\Theta'_\varphi(e_\eta) = \sum_{j=0}^{\infty} R^{1-j}(e_\eta) \otimes_L t^{1-j} = e_\eta^2 \otimes_L t^1 + e_\eta = \bar{\varphi}(e_\eta \otimes 1) = \Theta_\varphi(e_\eta)$$

□

Corollary 9.3.1. *For every space X, $\tilde{N}^q_\varepsilon(X) = 0$ dla $q < 0$.*

Proof. The map $\mu\colon \tilde{N}^*(X) \to \tilde{N}^*(X)$ $\mu(x) := x \otimes_L 1$ is an epimorphism and $R^0(x) = 0$ for $\deg(x) < 0$. Thus it is enough to prove that the following diagram commutes: $\stackrel{?}{=}$

$$\begin{tikzcd} \tilde{N}^*(X) \arrow[rr, "R^0"] \arrow[dr, "\mu"'] & & \tilde{N}^*(X) \arrow[dl, "\mu"] \\ & \tilde{N}^*_\varepsilon(X) & \end{tikzcd} \tag{9.7}$$

It follows from Prop. 9.3.1 and 9.3.2 that for every $n \in \mathbb{Z}$:

$$\mu(x) \stackrel{9.3.1}{=} a_0 \circ \Theta_\varphi(x) \stackrel{9.3.2}{=} a_0(\sum_{i=0}^{\infty} R^{n-i}(x) \otimes s^{n-i}) \stackrel{*}{=} R^0(x) \otimes 1 = (\mu \circ R^0)(x).$$

Note that if $n < 0$ then $a_0(\sum_{i=0}^{\infty} R^{n-i}(x) \otimes s^{n-i}) = 0$ and also $R^0(x) = 0$, thus equality $*$ holds for every $n \in \mathbb{Z}$.

□

Corollary 9.3.2. *There exists a natural equivalence of oriented cohomology theories*

$$\tilde{N}^*(X) \xrightarrow{\simeq} \tilde{H}^*(X; \mathbb{F}_2) \otimes_{\mathbb{F}_2} L.$$

In particular $f_N\colon (L, F_L) \to (N^*, F_N)$ is an isomorphism of the formal groups, where $L = F_2[a_2, a_4, a_5, ...]$ is a graded polynomial algebra, $\deg(a_i) = -i$ where $i \neq 2^j - 1$.

9.4 Examination Problems

9.4.1 Cohomology, homology and duality

Zad. 9.4.1. Let X, X^* be pointed CW complexes. A stable homotopy class $v \in \{S^n, X^* \wedge X\}$ will be called the (left) n-duality if the slant-product homomorphisms

$$-\backslash v_*[\iota_n] : \tilde{\pi}^q_S(X^*) \longrightarrow \tilde{\pi}^S_{n-q}(X), \quad -\backslash v_*[\iota_n] : \tilde{\pi}^q_S(X) \longrightarrow \tilde{\pi}^S_{n-q}(X^*)$$

are isomorphisms. We say that X^* is n-dual to X. For left n-duality state and prove analogous facts as in Section 4.1.

Zad. 9.4.2. Prove that for an n-dual pair X, X^* there is a bijective correspondence between right and left n-dualities (Hint: Zad. 4.1.3).

Zad. 9.4.3. For any multiplicative prespectrum $\mathbb{E}$ a left n-duality $v \in \{S^n, X^* \wedge X\}$ defines isomorphisms:

$$-\backslash v_*(\iota_n) : \tilde{\mathbb{E}}^q(X^*) \longrightarrow \tilde{\mathbb{E}}_{n-q}(X) \quad \text{and} \quad -\backslash v_*(\iota_n) : \tilde{\mathbb{E}}^q(X) \longrightarrow \tilde{\mathbb{E}}_{n-q}(X^*). \tag{9.8}$$

where $\iota_n = \sigma^n(1) \in \tilde{\mathbb{E}}_0(S^0)$.

Zad. 9.4.4. Recall that for any vector bundle $E \to X$ there is a natural diagonal map $\mathrm{Th}(E) \xrightarrow{\Delta} X^+ \wedge \mathrm{Th}(E)$. Prove that if $M \subset S^n$ is a closed submanifold then composition of the Thom map and the diagonal $S^n \to \mathrm{Th}(\nu(M, S^n)) \xrightarrow{\Delta} M^+ \wedge \mathrm{Th}(\nu(M, S^n))$ is a (left) n-duality map.

9.4.2 Bordism and cobordism

Zad. 9.4.5. Oriented bordism groups of a space X can be defined as

$$\Omega_n(X) := \{M \xrightarrow{f} X \mid M \text{ oriented compact manifold}, \dim M = n\}/\sim_{\text{bordism of oriented manifolds}} .$$

Prove that oriented bordism is a homology theory defined by the vector bundle spectrum $\tilde{\gamma}_*$ of universal bundles over Grassmannians of oriented subspaces. Give a geometric description of the associated cohomology theory $\mathcal{B}^*(X; \tilde{\gamma}_*)$.

Zad. 9.4.6. Formulate and prove naturality properties of the PT-bijection $P\colon L(\eta, \xi) \longrightarrow [Y^\eta, B^\xi]_*$ (Thm. 5.2.1) with respect to pull-backs of the vector bundles.

Zad. 9.4.7. For any two pairs of vector bundles (η_i, ξ_i) describe in geometric terms the pairing $[Y^{\eta_1}, B^{\xi_1}]_* \times [Y^{\eta_2}, B^{\xi_2}]_* \xrightarrow{\wedge} [(Y_1 \times Y_2)^{\eta_1 \times \eta_2}, (B_1 \times B_2)^{\xi_1 \times \xi_2}]_*$

Zad. 9.4.8. Describe in geometric terms the duality map between cobordism and bordism groups by the Spaniet-Whitehead duality described in Zad. 9.4.4.

Zad. 9.4.9. In geometric cobordism $\mathcal{B}^*(-; \xi)$ the transfer map is defined by composition of the ξ-oriented maps. Define it in homotopical terms.

Zad. 9.4.10. Let $F\colon Top_{h*} \to Grps$ be a contravariant functor which preserves finite direct sums. Then for any space X the group operation on $F(\Sigma X)$ coicides with the induced map $F(\Sigma X) \times F(\Sigma X) \simeq F(\Sigma X \vee \Sigma X) \xrightarrow{\nu^*} F(\Sigma X)$. If $F_0, F_1\colon Top_{h*} \to Grps$ are two such functors, equipped with the suspension isomorphisms $\sigma_i\colon F_i \to F_i\Sigma$ and $\Phi\colon F_0 \to F_1$ is a natural transformation of the functors into Set_* which commutes with the suspension isomorphisms then Φ preserves the group operation i.e. $\Phi\colon F_0 \to F_1$.

9.4.3 Formal groups

Zad. 9.4.11. Let F be a FGL over R^*. The set of power series of gradation 1 $R^*[[x]]_1$, equipped with operation defined for every $\phi, \psi \in R^*[[x]]_1$ by the formula $(\phi *_F \psi)(x) := F(\phi(x), \psi(x))$ is an abelian group.

Zad. 9.4.12. The multiplicative FGL $F_m(x, y) = x + y + xy$ over a $\mathbb{F}_2$-algebra R doesn't have a logarithm.

Zad. 9.4.13. If $\alpha(x) = x + a_1x^2 + a_2x^3 + ...$ is a series over a $\mathbb{F}_2$-algebra such that $a_j = 0$ for all j is not a power of 2, then $\alpha_* F_a = F_a$. where $\alpha_* F(x, y) := \alpha F(\alpha^{-1}(x), \alpha^{-1}(y))$.

Zad. 9.4.14. Prove that if (R, F) is the universal FGL, then its coefficient generate ring R.

Zad. 9.4.15. Prove that the group of power series $\alpha(x) \in R[[x]]$, $\alpha(x) = x + \sum_{i=1}^{\infty} a_i x^{i+1}$ acts via conjugation on the set $FGL(R)$, thus on the set $\mathrm{Hom}(L, R)$, where (L, F_L) is the universal FGL.

9.4.4 Vector bundles

Zad. 9.4.16. For any vector bundle $p\colon E \to B$ equipped with action of a finite group G (G-vector bundle) and any free G-space P the induced map $\bar{p} := id \times_G p\colon P \times_G E \to P \times_G B$ is a vector bundle. The construction defines a functor from the category of G-vector bundles to the category of vector bundles.

Zad. 9.4.17. For any vector bundle $E \to X$, the bundle $E \times E \to X \times X$ is equipped with a $\mathbb{Z}_2$-vector action exchanging factors. Prove that:

$$\mathrm{Th}(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (E \times E)) \simeq (E\mathbb{Z}_2)^+ \wedge_{\mathbb{Z}_2} \mathrm{Th}(E \times E) \simeq (E\mathbb{Z}_2)^+ \wedge_{\mathbb{Z}_2} (\mathrm{Th}(E) \wedge \mathrm{Th}(E)).$$

More generally, prove that for any G-vector bundle $\mathrm{Th}(P \times_G E) \simeq P^+ \wedge_G \mathrm{Th}(E)$.

Zad. 9.4.18. Let $\eta = E \to B$ be a real line bundle equipped with the antipodal action on fibres. Then for any k the bundle $S^k \times_{\mathbb{Z}_2} E \to \mathbb{R}P^k \times B$ is isomorphic to the bundle $p_1^! \gamma_k \otimes p_2^! \eta$.

9.4.5 Operations in cobordism

Zad. 9.4.19. If v, w are two exponential classes in $N_h^0(-)$ such that $v(\gamma_n^1) = w(\gamma_n^1)$ for all $n \geq 0$ then $v = w$. (Prop. 8.2.2)

Zad. 9.4.20. For any line bundle η, $\Delta_{\mathbb{Z}_2}^* P^e(e_\eta) = p_2^!(e_\eta) e_{\gamma \hat{\otimes} \eta}$. (cf. Diagram 9.3).

Zad. 9.4.21. Prove that $\bar{\Theta}_{\varphi_2 \circ \varphi_1} = \bar{\Theta}_{\varphi_1} \circ \bar{\Theta}_{\varphi_2}$ and $\bar{\Theta}_x = id$ thus for any φ the transformation $\bar{\Theta}_\varphi$ is an isomorphism (cf. Prop 8.2.5).

Zad. 9.4.22. If $\iota_1 \in \tilde{N}^1(S^1)$ is a generator then $R^k(\iota_1) = \begin{cases} 0 & \text{if} \quad k \neq 0 \\ \iota_1 & \text{if} \quad k = 0 \end{cases}$.

Zad. 9.4.23. Calculate action of the operations R^i on the unoriented cobordism ring of the projective space $N^*(\mathbb{R}P^k)$.

Zad. 9.4.24. Prove that under isomorphism $\tilde{N}^*(X) \otimes_L (\mathbb{Z}_2)_\varepsilon \xrightarrow{\simeq} \tilde{H}^*(X; \mathbb{F}_2)$ the operations $R^i \otimes_L id$ correspond to the Steeenrod squares Sq^i.

Zad. 9.4.25. Calculate the groups N_i for $i \leq 4$ and construct their generators.

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