Notes on fibre bundles and characteristic classes

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1 Cohomology of sheaves of non-abelian groups

Cohomology of (pre-)sheaves of abelian groups is described in detail in many books on algebraic topology [Spanier 6.7-9], algebraic geometry or complex analysis. The aim of the present notes is to describe low-dimensional cohomology sets which can be defined for sheaves of not necessarily abelian groups. Interpretation of principal bundles as elements of a cohomology set gives a convenient framework to introduce characteristic classes.

1.1 Presheaves and sheaves

Let X be any topological space and $\mathcal{T}_X$ its topology i.e. set of open subsets ordered by inclusion. A presheaf of groups on X is a contravariant functor $\Gamma\colon \mathcal{T}_X \to \mathcal{G}$ from $\mathcal{T}_X$ to the category of groups. If $i\colon U \subset V$ is an inclusion and $\gamma \in \Gamma(V)$ then we denote $\Gamma(i)(\gamma) = \gamma|U$ and call it restriction of the element γ to U. Presheaves of groups on X form a category – morphisms are natural transformations of functors. It will be denoted $\mathcal{PS}h_{\mathbf{G}}(X)$. A canonical example is a constant presheaf or a presheaf of continuous functions with values in a fixed topological group G. Such presheaf is a sheaf i.e. it satisfies additional uniqueness and gluing conditions:

Definition 1.1. *A presheaf $\Gamma\colon \mathcal{T}_X \to \mathcal{G}$ is a sheaf if it satisfies the following conditions:*

1. *Given a collection $\mathcal{U}$ of open subsets of X such that $V := \bigcup_{U\in\mathcal{U}} U$ and two elements $\gamma, \gamma' \in \Gamma(V)$ such that for each $U \in \mathcal{U}$ their restrictions are equal $\gamma|U = \gamma'|U$ then $\gamma = \gamma'$;*
2. *Given a collection $\mathcal{U}$ of open subsets of X such that $V := \bigcup_{U\in\mathcal{U}} U$ and for each $U \in \mathcal{U}$ an element $\gamma_U \in \Gamma(U)$ such that for any $U', U'' \in \mathcal{U}$ restrictions are equal $\gamma_{U'}|U' \cap U'' = \gamma_{U''}|U' \cap U''$ then there exists an (unique) element $\gamma \in \Gamma(V)$ such that for each $U \in \mathcal{U}$, $\gamma|U = \gamma_U$.*

For any sheaf $\Gamma\colon \mathcal{T}_X \to \mathcal{G}$ and any point $x \in X$ we define its *stalk*: $\Gamma_x := \operatorname{colim}_{U\ni x} \Gamma(U)$, which is obviously also a group. For every open subset $U \ni x$ we have restriction homomorphism $\mathrm{res}\colon \Gamma(U) \to \Gamma_x$. Note that if for an element $\gamma \in \Gamma(U)$, we have $\gamma|x = 1$ for every $x \in U$ then $\gamma = 1$ thus $\Gamma(U) \xrightarrow{\mathrm{res}} \prod_{x\in U} \Gamma_x$ is injective.

We say that a sequence of sheaves on X: $\Gamma' \to \Gamma \to \Gamma''$ is *exact* if for every point $x \in X$ the sequence of groups $\Gamma'_x \to \Gamma_x \to \Gamma''_x$ is exact for each $x \in X$.

Proposition 1.1. *If $1 \to \Gamma' \xrightarrow{i} \Gamma \xrightarrow{p} \Gamma'' \to 1$ is a short exact sequence of sheaves on X, then for any open subset $U \subset X$ the sequence $1 \to \Gamma'(U) \to \Gamma(U) \to \Gamma''(U)$ is exact.*

Proof. For the proof consider the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \Gamma'(U) \arrow[r, "i"] \arrow[d, "\mathrm{res}"] & \Gamma(U) \arrow[r, "p"] \arrow[d, "\mathrm{res}"] & \Gamma''(U) \arrow[d, "\mathrm{res}"] & \\
1 \arrow[r] & \prod_{x\in U} \Gamma'_x \arrow[r, "i"] & \prod_{x\in U} \Gamma_x \arrow[r, "p"] & \prod_{x\in U} \Gamma''_x \arrow[r] & 1
\end{tikzcd} \tag{1}$$

in which the bottom row is exact as a product of exact sequences, and vertical arrows are injective. Diagram chasing proves that i in top row is injective.

Similarly assume that $p(\gamma) = 1$. Then $\mathrm{res}(p(\gamma)) = 1$ and therefore for every $x \in U$ there exists unique element $\gamma'_x \in \Gamma'_x$ such that $i(\gamma'_x) = \gamma_x$. Assume that $\gamma'_x = \mathrm{res}(\gamma'_{U_x})$ where $\gamma'_{U_x} \in \Gamma(U_x)$. We can choose such U_x that $i(\gamma'_{U_x}) = \gamma|U_x$.

Thus there is $U_x \ni x$ such that $i(\gamma'_x)|U_x = \gamma|U_x$. Again, since i is injective the elements $\{\gamma'_{U_x} \in \Gamma'(U_x)\}_{x\in U}$ form a compatible family. Thus there exists $\gamma' \in \Gamma(U)$ such that $\gamma'|x = \gamma'_x$. Diagram chase shows that $i(\gamma') = \gamma$. □

1.2 Cohomology pointed-sets

Theorem 1.1. *For any space X and $i = 0, 1$ there exist functors $\check{\mathrm{H}}^i(X; -): \mathcal{S}h_{\mathcal{G}}(X) \to \mathcal{S}et_*$ such that $\check{\mathrm{H}}^0(X; \Gamma) := \Gamma(X)$ and for every short exact sequence of sheaves $1 \to \Gamma' \xrightarrow{i} \Gamma \xrightarrow{p} \Gamma'' \to 1$ there exists natural transformation $\delta^0 : \check{\mathrm{H}}^0(X; \Gamma'') \to \check{\mathrm{H}}^1(X; \Gamma')$ such that the sequence*

$$1 \to \check{\mathrm{H}}^0(X;\Gamma') \xrightarrow{i_*} \check{\mathrm{H}}^0(X;\Gamma) \xrightarrow{p_*} \check{\mathrm{H}}^0(X;\Gamma'') \xrightarrow{\delta^0} \check{\mathrm{H}}^1(X;\Gamma') \xrightarrow{i_*} \check{\mathrm{H}}^1(X;\Gamma) \xrightarrow{p_*} \check{\mathrm{H}}^1(X;\Gamma'')$$

is exact.

Remark 1. If we consider sheaves of abelian groups then the sequence is a sequence of abelian groups and it extends to the long cohomology exact sequence.

We'll construct sets $\check{\mathrm{H}}^1(\mathcal{U}; \Gamma)$ for each open cover X and then pass to the limit with respect to refinements of coverings. Let $\mathcal{U} = \{U_i\}_{i\in I}$. For any sequence of indices $i_1, \ldots, i_n$ we denote $U_{i_1\ldots i_n} := U_{i_1} \cap \cdots \cap U_{i_n}$.

For every $n \geq 0$ define cochains groups and base-point (neutral elements) preserving map

$$C^n(\mathcal{U};\Gamma) := \prod_{i_0,\ldots i_n \in I} \Gamma(U_{i_0,\ldots i_n}), \quad C^0(\mathcal{U};\Gamma) \xrightarrow{\delta^0} C^1(\mathcal{U};\Gamma) \xrightarrow{\delta^1} C^2(\mathcal{U};\Gamma)$$

$$\delta^0(\{\gamma_i\}_{i\in I})_{kl} := (\gamma_k|U_{kl})(\gamma_l|U_{kl})^{-1}, \quad \delta^1(\{\gamma_{ij}\}_{i,j\in I})_{klm} := (\gamma_{kl}|U_{klm})(\gamma_{lm}|U_{klm})(\gamma_{km}|U_{klm})^{-1}.$$

Note that $\delta^1\delta^0 = \{1_{klm}\}$ is a constant map. The maps δ^i in general are not group homomorphisms; they are if the groups are abelian. Next we define sets of cocycles:

$$Z^0(\mathcal{U};\Gamma) := \ker \delta^0 = \{\{\gamma_i\}_{i\in I} \in C^0(\mathcal{U};\Gamma) \,|\, \text{for all}\, i,j\ \gamma_i|U_{ij} = \gamma_j|U_{ij}\} = \Gamma(X)$$
$$Z^1(\mathcal{U};\Gamma) := \ker \delta^1 = \{\{\gamma_{ij}\} \in C^1(\mathcal{U};\Gamma) \,|\, \text{for all}\, i,j,k\ (\gamma_{ijk}|U_{ijk})(\gamma_{jk}|U_{ijk}) = \gamma_{ik}|U_{ijk}\}$$

If values of Γ were abelian groups we would define $\check{\mathrm{H}}^1(\mathcal{U};\Gamma) := \ker \delta^1 / \operatorname{im} \delta^0$. For sheaves of not necessarily abelian groups we define:

$$\check{\mathrm{H}}^0(\mathcal{U};\Gamma) := Z^0(\mathcal{U};\Gamma) = \Gamma(X) \qquad \check{\mathrm{H}}^0(X;\Gamma) := \Gamma(X)$$

$$\check{\mathrm{H}}^1(\mathcal{U};\Gamma) := Z^1(\mathcal{U};\Gamma)/\sim$$

where $\{\gamma_{ij}\} \sim \{\gamma'_{ij}\}$ if there exists a family $\{\alpha_i \in \Gamma(U_i)\}_{i\in I}$ such that $\gamma_{ij} = (\alpha_i|U_{ij})\gamma'_{ij}(\alpha_j|U_{ij})^{-1}$.

Remark 2. The equivalence relation on $Z^1(\mathcal{U};\Gamma)$ can be interpreted as quotient of an action of the group $C^0(\mathcal{U};\Gamma)$ on $Z^1(\mathcal{U};\Gamma)$.

To define the set $\check{\mathrm{H}}^1(X;\Gamma)$ as a colimit over refined coverings we need to introduce a refinement function. If $\mathcal{U}$, $\mathcal{V}$ are two coverings then $\lambda\colon \mathcal{V} \to \mathcal{U}$ is a refinement function if $V \subset \lambda(V)$ for all $V \in \mathcal{V}$. Any refinement function defines a map $\lambda^*\colon C^*(\mathcal{U};\Gamma) \to C^*(\mathcal{V};\Gamma)$ which commutes with maps δ^*. In particular, $\lambda^*(\{\gamma_{ij}\})_{kl} := \gamma_{\lambda(k)\lambda(l)}|V_{kl}$. We need to prove that any two refinement functions $\lambda, \mu\colon \mathcal{V} \to \mathcal{U}$ define the same map $\lambda^* = \mu^*\colon \check{\mathrm{H}}^i(\mathcal{U};\Gamma) \to \check{\mathrm{H}}^i(\mathcal{V};\Gamma)$. For $i = 0$ both maps are simply identities. For $i = 1$ and any $\{\gamma_{ij}\} \in Z^1(\mathcal{U};\Gamma)$ we have:

$$\lambda^*(\{\gamma_{ij}\})_{kl} := \gamma_{\lambda(k)\lambda(l)} = \gamma_{\lambda(k)\mu(k)}\gamma_{\mu(k)\mu(l)}\gamma^{-1}_{\lambda(l)\mu(l)}$$

thus $\lambda^* = \mu^*$. (Note that $\gamma_{\lambda(k)\mu(k)} \in \Gamma(V_k)$.) Finally, we define

$$\check{\mathrm{H}}^1(X;\Gamma) := \operatorname{colim}_{\mathcal{U}} \check{\mathrm{H}}^1(\mathcal{U};\Gamma)$$

Any natural transformation $\Phi\colon \Gamma \to \Gamma'$ of sheaves over the space X defines a homomorphism $\Phi_*\colon \check{\mathrm{H}}^i(X;\Gamma) \to \check{\mathrm{H}}^i(X;\Gamma')$.

Let $1 \to \Gamma' \xrightarrow{i} \Gamma \xrightarrow{p} \Gamma'' \to 1$ be an exact sequence of sheaves. In order to define the connecting homomorphism δ^0, consider the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & C^0(\mathcal{U},\Gamma') \arrow[r, "i"] \arrow[d, "\delta^0"] & C^0(\mathcal{U},\Gamma) \arrow[r, "p"] \arrow[d, "\delta^0"] & C^0(\mathcal{U},\Gamma'') = \Gamma''(X) \arrow[d, "\delta^0"] \\
1 \arrow[r] & C^1(\mathcal{U},\Gamma') \arrow[r, "i"] \arrow[d, "\delta^1"] & C^1(\mathcal{U},\Gamma) \arrow[r, "p"] \arrow[d, "\delta^1"] & C^1(\mathcal{U},\Gamma'') \arrow[d, "\delta^1"] \\
1 \arrow[r] & C^2(\mathcal{U},\Gamma') \arrow[r, "i"] & C^2(\mathcal{U},\Gamma) \arrow[r, "p"] & C^2(\mathcal{U},\Gamma'')
\end{tikzcd} \tag{2}$$

where rows are exact sequences of groups (cf. Prop.1.1).

For every $\gamma'' \in \check{\mathrm{H}}^0(X;\Gamma'') = \Gamma(X)$ we define $\delta^0(\gamma'') \in \check{\mathrm{H}}^1(\mathcal{U};\Gamma')$, where the covering $\mathcal{U}$ depends on the choice of γ''. From exactness of the sequence of sheaves $1 \to \Gamma' \xrightarrow{i} \Gamma \xrightarrow{p} \Gamma'' \to 1$ it follows that there is a covering $\mathcal{U} := \{U_i\}_{i\in I}$ and family of elements $\gamma_i \in \Gamma(U_i)$ such that $p(\gamma_i) = \gamma''|U_i$. Consider an element $\delta^0(\{\gamma_i\} \in C^1(\mathcal{U};\Gamma)$ for which obviously $p(\delta^0(\{\gamma_i\}) = 1$. Exactness of the middle sequence implies that there is an element $\gamma'_{ij} \in C^1(\mathcal{U};\Gamma')$ such that $i(\{\gamma'_{ij}\}) = \delta^0(\{\gamma_i\})$. We set $\delta^0(\gamma'') := \{\gamma'_{ij}\}$. It is again straightforward to see that $\delta^0(\gamma'') \in \ker\delta^1$. It remains to prove that when we pass to the colimit set $\check{\mathrm{H}}^1(X;\Gamma'')$ the element $\delta^1(\gamma'')$ does not depend on choices we made and the sequence in Theorem 1.1 is exact.

Remark 3. If Γ' is a sheaf of abelian groups, then the connecting homomorphism $\delta^1\colon \check{\mathrm{H}}^1(X;\Gamma'') \to \check{\mathrm{H}}^2(X;\Gamma')$ can be defined similarly.

1.3 Functoriality of $\check{\mathrm{H}}^1(X;\Gamma)$ with respect to the space X

Any map $f\colon X\to Y$ induces a functor between categories of sheaves $f_\#\colon \mathcal{S}h_X\to\mathcal{S}h_Y$ defined as $f_\#\Gamma(V):=\Gamma(f^{-1}(V))$ (in particular $f_\#\Gamma(Y)=\Gamma(X)$) and a map of cohomology: $f^*\colon \check{\mathrm{H}}^1(Y;f_\#\Gamma)\to\check{\mathrm{H}}^1(X;\Gamma)$. Indeed, for any open covering $\mathcal{V}$ of Y we have the tautological bijection: $C^*(\mathcal{V};f_\#\Gamma)\simeq C^*(f^{-1}(\mathcal{V});\Gamma)$ thus $\check{\mathrm{H}}^*(\mathcal{V};f_\#\Gamma)\simeq\check{\mathrm{H}}^*(f^{-1}(\mathcal{V});\Gamma)$. After passing to limits withe respect to coverings of Y we obtain an isomorphism $\check{\mathrm{H}}^*(Y;\Gamma)\simeq\operatorname{colim}_{\mathcal{V}}\check{\mathrm{H}}^*(f^{-1}(\mathcal{V});\Gamma)$. Composition with the natural map $\operatorname{colim}_{\mathcal{V}}\check{\mathrm{H}}^*(f^{-1}(\mathcal{V});\Gamma)\to\operatorname{colim}_{\mathcal{U}}\check{\mathrm{H}}^1(\mathcal{U};\Gamma)=\check{\mathrm{H}}^*(X;\Gamma)$ is the requested map $f^*\colon\check{\mathrm{H}}^1(Y;f_\#\Gamma)\to\check{\mathrm{H}}^1(X;\Gamma)$.

Remark 4. The functor $f_\#$ has a left adjoint functor pull-back functor $f^!\colon\mathcal{S}h_Y\to\mathcal{S}h_X$.

2 Cohomology and G-bundles

Let G be a topological group and X a topological space. Denote $\mathcal{P}_G(X)$ the set of isomorphism classes of the principal G-bundles over X. Trivial bundle is the base-point. $\mathcal{P}_G(-)$ is obviously an contravariant functor from the category of spaces to sets with base-points. It is also functorial with respect to group homomorphisms. A homomorphism of topological groups $\phi\colon G\to K$, defines a map $\phi_*\colon\mathcal{P}_G(X)\to\mathcal{P}_K(X)$ if we assign to every G-bundle $P\to X$ a K-bundle $P\times_\phi K\to X$. Thus we have a functor of two variables $\mathcal{P}\colon(\mathcal{T}op)^{op}\times\mathcal{G}r\to\mathcal{S}et_*$. We'll interpret the functor in terms of sheaf cohomology.

For any space X and topological group G let $\Gamma^X_G\colon\mathcal{T}_X\to\mathcal{G}$ be the sheaf of continuous functions from X to G i.e. $\Gamma^X_G(U):=\mathrm{Map}\,(U,G)$. Obviously any group homomorphism $\varphi\colon G\to K$ induces a morphism of sheaves $\varphi_*\colon\Gamma^X_G\to\Gamma^X_K$.

For any map $f\colon X\to Y$ we have a morphism of sheaves $\Gamma^Y_G\to f_\#\Gamma^X_G$, thus $\check{\mathrm{H}}^1(-;\Gamma_?)\colon(\mathcal{T})^{op}\times\mathcal{G}\to\mathcal{S}_*$ is also a functor of two variables.

Theorem 2.1. *There is a natural equivalence of functors $\mathcal{P}_G(-)\simeq\check{\mathrm{H}}^1(-;\Gamma_G)$.*

Proof. Proof consists of construction of mutually inverse maps.

For every principal bundle $P\to X$ we choose a local trivialization consisting of an open covering $\mathcal{U}=\{U_i\}_{i\in I}$ and homeomorphisms $\{U_i\times G\xrightarrow{h_i}p^{-1}(U_i)\}$ which define collection of glueing functions $\{g_{ij}\colon U_{ij}\to G\}_{i,j\in I}\in Z^1(\mathcal{U};\Gamma^X_G)$ by the formula: $h_ih_j^{-1}(u,g)=(u,g_{ij}(u)g)$. It is easy to show that its image in $\check{\mathrm{H}}^1(X;\Gamma^X_G)$ doesn't depend on choice of local trivialization.

Conversely, for every cocycle $\{g_{ij}\colon U_{ij}\to G\}_{i,j\in I}\in Z^1(\mathcal{U};\Gamma^X_G)$ we define a principal bundle $P_{\{g_{ij}\}}\to X$ gluing together trivial bundles over sets $U_i\in\mathcal{U}$ via functions g_{ij}. More precisely: $P_{\{g_{ij}\}}:=\coprod_{i\in I}U_i\times G/\sim$ where $(u_j,g_j)\sim(u_i,g_i)$ iff $u_j=u_i=u$ and $g_j=g_{ij}(u)g_i$. Again, one has to check that isomorphism class of the bundle does not depend on choice of cocycle form its equivalence class.

The two maps described above are clearly inverse maps. It is easy to show that they define a natural transformation of functors (it is enough to check that one of them is natural). ☐

Zad. 1. Check that the maps are well defined.

Examples

Proposition 2.1. *If $1\xrightarrow{i}K\to G\xrightarrow{p}Q\to 1$ is an exact sequence of Lie groups then for every space X the sequence of sheaves $1\to\Gamma^X_K\to\Gamma^X_G\to\Gamma^X_Q\to 1$ is exact.*

Proof. We have to prove that the sequence of stalks $1 \to (\Gamma_K^X)_x \xrightarrow{i_x} (\Gamma_G^X)_x \xrightarrow{p_x} (\Gamma_Q^X)_x \to 1$ is exact for each $x \in X$. Exactness at the first and middle term is obvious. It remains to show that p_x is surjective. It follows easily form the fact that projection $K \to G \xrightarrow{p} Q$ is locally trivial i.e. admits local sections. □

Example 1. Consider exact sequence $1 \to GL^+(n,\mathbb{R}) \to GL(n,\mathbb{R}) \to \mathbb{Z}_2 \to 1$ and for any space X the associated cohomology exact sequence:

$$\check{\mathrm{H}}^1(X;\Gamma_{GL^+(n,\mathbb{R})}) \to \check{\mathrm{H}}^1(X;\Gamma_{GL(n,\mathbb{R})}) \to \check{\mathrm{H}}^1(X;\Gamma_{\mathbb{Z}_2}) \simeq H^1(X;\mathbb{Z}_2)$$

i.e. in terms of the principal bundles

$$\mathcal{P}_{GL^+(n,\mathbb{R})}(X) \to \mathcal{P}_{GL(n,\mathbb{R})}(X) \xrightarrow{w_1} \check{\mathrm{H}}^1(X;\Gamma_{\mathbb{Z}_2}) \simeq \check{\mathrm{H}}^1(X;\mathbb{Z}_2)$$

The map w_1 is the first Stiefel-Whitney (SW) class of the vector bundle associated with given principal $GL(n,\mathbb{R})$-bundle. Vector bundle is orientable if and only if its first SW class vanishes.

Zad. 2. Calculate first SW class of the Whitney sum of two vector bundles and of the tensor product of two 1-dimensional bundles.

Example 2. Consider the exact sequence of groups $0 \to \mathbb{Z}_2 \to Spin_n \to SO_n \to 0$ (universal covering of the special orthogonal group SO_n). We have an exact sequence:

$$\check{\mathrm{H}}^1(X;\Gamma_{Spin_n}) \to \check{\mathrm{H}}^1(X;\Gamma_{SO_n}) \xrightarrow{\delta^1} \check{\mathrm{H}}^2(X;\Gamma_{\mathbb{Z}_2}) \simeq \check{\mathrm{H}}^2(X;\mathbb{Z}_2).$$

i.e. in terms of the principal bundles

$$\mathcal{P}_{Spin_n}(X) \to \mathcal{P}_{SO_n}(X) \xrightarrow{w_2} \check{\mathrm{H}}^1(X;\Gamma_{\mathbb{Z}_2}) \simeq \check{\mathrm{H}}^1(X;\mathbb{Z}_2)$$

Homomorphism w_2 is the second Stiefel-Whitney class. It follows that an oriented vector bundle admits a *Spin*–structure iff its second Stiefel-Whitney class $w_2 = 0$.

Zad. 3. Prove that for two oriented vector bundles E_1, E_2, $w_2(E_1 \oplus E_2) = w_2(E_1) + w_2(E_2)$.

Example 3. The exact sequence of abelian groups $0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to 1$ leads to a long exact sequence (groups are abelian); in particular we have the connecting homomorphism:

$$\delta^1 \colon \check{\mathrm{H}}^1(X;\Gamma_{\mathbb{C}^*}) \to \check{\mathrm{H}}^2(X;\Gamma_{\mathbb{Z}}) \simeq \check{\mathrm{H}}^2(X;\mathbb{Z})$$

which is the first Chern class of the 1-dimensional complex vector bundles. Note that the homomorphism is an isomorphism [not obvious].

3 Classifying spaces for principal G-bundles

We begin with an important homotopy property of the locally trivial maps

Theorem 3.1 (W. Hurewicz)**.** *Every locally trivial map over a paracompact space is a fibration (i.e. has the homotopy covering property).*

For proof cf. (Spanier, 1966), (Dold, ???), (tom Dieck,??).

Corollary 3.1. *For every paracompact space B any principal G-bundle $P \xrightarrow{q} B \times I$ is isomorphic to the bundle $P_0 \times I \xrightarrow{q_0 \times id} B \times I$ where $P_0 := q^{-1}(B \times 0)$ and $q_0 := q|P_0$.*

To prove the corollary we need a lemma concerning morphisms of principal bundles.

Lemma 3.1. *Let $P \xrightarrow{q} B$ be a principal G-bundle and F any right G-space. There is a natural bijection between set of sections of the associated bundle $P \times_G F \xrightarrow{q_F} B$ and G-equivariant maps $f\colon P \to F$ i.e. such that $f(pg) = g^{-1}f(p)$.*

Let $P_i \xrightarrow{q_i} B$ $i = 1, 2$ be two principal G-bundles and $\mathrm{Hom}_B(P_1, P_2)$ denote the set of all G-morphisms $P_1 \to P_2$ over B. We will construct a locally trivial bundle $P_{12} \xrightarrow{q} B$ such that there is a bijection between set of its sections and $\mathrm{Hom}_B(P_1, P_2)$. To construct the associated bundle $P_1 \times_G P_2 \xrightarrow{q_{12}} B$ we consider P_2 with the right G-action $gp_2 := p_2g^{-1}$. Section of the bundle correspond to all equivariant maps $P_1 \to P_2$, not necessary over B. We define a subbundle of q_{12} and prove that its sections correspond to maps over B.

Proposition 3.1. *Restriction of the map q_{12} to a subspace*

$$P_{12} := \{[p_1, p_2] \in P_1 \times_G P_2 \,|\, q_1(p_1) = q_2(p_2)\}$$

is locally trivial and its sections are in bijective correspondence with G-maps $P_1 \to P_2$ over B.

Proof. The last assertion follows immediately from construction in Lemma 3.1. We prove that the map is locally trivial. For that choose a set $U \subset B$ such that both G-bundles are trivial over U and consider the bundle $(U\times G)\times_G P_2 = U\times P_2 \xrightarrow{\pi} U$ and its subset $P_{12}|U = \{(u, p_2) \in U \times P_2 \,|\, q_2(p_2) = u\}$. Since q_2 is trivial over U there is a section $s\colon U \to P_2$. The map $h\colon U \times G \to P_{12}|U$ defined as $h(u, g) := (u, s(u)g)$ is the required homeomorphism over U. □

Zad. 4. If $P_1 = P_2 = P$ then the bundle $P_{12} \to B$ is isomorphic to the associated bundle $P \times_G Ad(G) \to B$ where $Ad(G)$ denotes the group G with G-action by inner automorphisms (adjoint action).

Proof of Corollary 3.1 . Note first that for any fibration over a cylinder $E \to X \times I$ any section defined on $X \times 0$ extends to the cylinder, in particular for any two principal G-bundles $P_i \to X \times I$ any morphism $P_1|X \times 0 \to P_2|X \times 0$ over $X \times 0$ extends to a morphism $P_1 \to P_2$ over $X \times I$. For bundles $P \xrightarrow{q} B \times I$ and $P_0 \times I \xrightarrow{q_0\times id} B \times I$ there is an obvious (identity) morphism over $B \times 0$, thus there is a morphism over $B \times I$. Every morphism of principal G-bundles over fixed space is an isomorphism. □

Corollary 3.2. *Let $P \to B$ be a principal G-bundle. Any two homotopic maps $f_0, f_1\colon X \to B$ induce isomorphic principal G-bundles $f_0^!P \simeq f_1^!P$.*

For any topological group G and pointed space (X, x_0) define $\mathcal{P}_G(X, x_0)$ as a set of isomorphism classes of principal G-bundles over X with fixed path component of fibre over x_0 i.e. an element of $\mathcal{P}_G(X, x_0)$ is an isomorphism class of pairs (P, c) where $P \to X$ is a principal bundle and $c \subset p^{-1}(x_0)$ is a path-component. Note that a choice of a a point in such component defines a G-homeomorphism $G \to p^{-1}(x_0)$.

Theorem 3.2. *For every $n > 1$ the clutching function construction defines a natural bijection*

$$\pi_{n-1}(G, e) = [S^{n-1}, G]_* \to \mathcal{P}_G(S^n, 1).$$

Sketch of the proof. As usual we define mutually inverse maps in both directions and check that they are well-defined. To every function $\gamma\colon S^{n-1} \to G$ we assign the bundle $D^n \times G \coprod D^n \times G/\sim \to S^n$ where $(x, g) \sim (x, \gamma(x)g)$ if $x \in S^{n-1}$. Note that since $\gamma(1) = e$ we have fixed identification $p^{-1}(1) \simeq G$. Conversely to every isomorphism class of G-bundles we assign clutching function γ by choosing trivialisations on both hemispheres which coincide with given trivialisation over the base point. □

Remark 5. Note that if $n > 1$ then $[S^{n-1}, G]_* = [S^{n-1}, G_e]_* = [S^{n-1}, G_e]$ where G_e denotes the path-component of identity $e \in G$. Thus $\mathcal{P}_G(S^n, 1) = \mathcal{P}_{G_e}(S^n)$.

A different construction of the clutching function: for a given G-bundle over a sphere consider its pullback:

$$\begin{tikzcd} D^n \times G \arrow[r, "\tilde{q}"] \arrow[d, "p_1"] & E \arrow[d, "p"] \\ D^n \arrow[r, "q"] & S^n \end{tikzcd} \tag{3}$$

where $q\colon D^n \to D^n/S^{n-1} \simeq S^n$. The map $\tilde{q}\colon D^n \times G \to E$ restricts to a G–equivariant map $\tilde{q}\colon S^{n-1} \times G \to p^{-1}(1)$ which defines a map $\hat{q}\colon S^{n-1} \to \mathrm{Map}_G(G, p^{-1}(1)) \simeq G$. We describe the last identification in detail. It is the composition of maps $\mathrm{Map}_G(G, p^{-1}(1)) \xrightarrow{ev_e} p^{-1}(1) \xleftarrow{e_0-} G$. We define the map $\gamma\colon S^{n-1} \to G$ as composition of $\hat{q}$ with the above identification. The the map $\tilde{q}\colon D^n \times G \to E$ defines a G–homeomorphism of the quotient space $(D^n \times G)/\sim \simeq E$ where $(x_1, g_1) \sim (x_2, g_2)$ if and only if $\tilde{q}(x_1, g_1) = \tilde{q}(x_2, g_2)$. It is easy to check that this is the case when $(x_1, g_1) = (x_2, g_2)$ or $x_1, x_2 \in S^{n-1}$ and $\gamma(x_1)g_1 = \gamma(x_2)g_2$.

Proposition 3.2. *For every G-bundle $E \to S^n$ the above two construction of the clutching function lead to homotopic maps.*

Moreover from definition of the boundary map the following proposition follows:

Proposition 3.3. *For any G-bundle $(E, e_0) \to (B, b_0)$ and a map $(S^k, 1) \xrightarrow{f} (B, b_0)$ homotopy class of the clutching function of the induced bundle $f^!E \to S^k$ equals to $\partial([f]) \in \pi_{i-1}(G, e)$ where $\partial\colon \pi_k(B, b_0) \to \pi_{k-1}(p^{-1}(b_0), e_0) \simeq \pi_{k-1}(G, e)$ is the boundary homomorphism in the homotopy exact sequence of the fibration $(E, e_0) \to (B, b_0)$.*

Let $E_nG \to B_nG$ be a principal G–bundle such that $\pi_i(E_nG) = 0$ for $k < n$. Then the boundary homomorphism in the long exact sequence of the fibration $\pi_i(B_nG, b_0) \xrightarrow{\partial} \pi_{i-1}(G, e)$ is an isomorphism for $k \leq n-1$ and an epimorphism for $k = n$.

Pick-up a base point $b_0 \in B_nG$ and fix a component of $p^{-1}(b_0)$. We define a natural transformation: $\Phi_X\colon [X, B_nG]_* \to \mathcal{P}_G(X, x_0)$ which sends homotopy class $(X, x_0) \xrightarrow{f} (B_nG, b_0)$ to the isomorphism class of the induced bundle $f^!E_nG \xrightarrow{p_f} X$ with the corresponding component of the fiber $p_f^{-1}(x_0)$.

Theorem 3.3. *For any topological group G and $n > 0$ the transformation $\Phi_X\colon [X, B_nG]_* \to \mathcal{P}_G(X, x_0)$ is bijective for every connected CW-complex of dimension $< n$ and surjective for every connected CW-complex of dimension n.*

Proof. Theorem follows from the fact that Φ_X is a natural transformation of the half-exact functors cf. Topologia Algebraiczna I - Pomocnik studenta. Rozdz.5 and 6.5 or E. H. Spanier *Algebraic Topology* Sec. 7.7 , Thm. 7.7.14. □

Zad. 5. Check that $\mathcal{P}_G(-)$ is a half-exact functor.

Corollary 3.3. *For any topological group G and $n > 0$ the transformation $\Phi_X\colon [X, B_nG] \to \mathcal{P}_G(X)$ is bijective for every connected CW-complex of dimension $< n$ and surjective for every connected CW-complex of dimension n.*

Proof. Note that $[X, B_nG] \simeq [X, B_nG]_*/\pi_1(B_nG, b_0) \simeq \mathcal{P}_G(X, x_0)/\pi_0(G, e) \simeq \mathcal{P}_G(X)$. $\square$

If $n = \infty$ then we denote $EG \to BG$ and the above assertion hold for any CW-complex.

Zad. 6. Prove Thm. 3.3 directly, by induction on the dimension of CW-complex.

4 Three perspectives on principal G-bundles and vector bundles

Let us note that a principal G-bundle, up to isomorphism, admits three interpretations:

1. **(Global)** A G-space P and projection $P \to X$;
2. **(Local)** An element in the cohomology $\check{\mathrm{H}}^1(X; \Gamma_G)$, where Γ_G is the sheaf of G-valued functions on X;
3. **(Homotopical)** A homotopy class of maps $f_P \colon X \to BG$, where BG is the classifying space of the group G.

Since isomorphism classes of n-dimensional $\mathbb{F}$-vector bundles (where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) are in bijective correspondence with isomorphism classes of the principal $GL_n(\mathbb{F})$-bundles we have similar three perspectives on vector bundles:

1. **(Global)** A locally trivial bundle $E \to B$ of vector spaces;
2. **(Local)** An element in the cohomology $g_P \in \check{\mathrm{H}}^1(X; \Gamma_{GL_n(\mathbb{F})})$, where Γ_{GL_n} is the sheaf of GL_n-valued functions on X;
3. **(Homotopical)** A homotopy class of maps $f_P \colon X \to BGL_n$, where $BGL_n(\mathbb{F})$ is the classifying space of the group $GL_n(\mathbb{F})$ (e.g. Grassmannian $G_n(\mathbb{F}^\infty)$).

All construction which we perform on bundles can be interpreted in such terms. For example:

Proposition 4.1. *Let $H \subset G$ be a subgroup of a topological group G. For a principal G-bundle $P \to X$ the following conditions are equivalent:*

1. *$P \to X$ admits reduction of the structural group to $H \subset G$ i.e. there exists a principal H-bundle $P' \to X$ such that $P' \times_H G \simeq P$;*
2. *$g_P \in \operatorname{im}\{\check{\mathrm{H}}^1(X; \Gamma_H) \to \check{\mathrm{H}}^1(X; \Gamma_G)\}$;*
3. *The map f_P factors up to homotopy via the map induced by inclusion $j \colon H \subset G$ i.e. $X \xrightarrow{f_H} BH \xrightarrow{Bj} BG$.*

Zad. 7. Give three interpretations of the following properties of vector bundles:

1. Every vector bundle admits a metric;
2. A bundle is orientable;
3. A vector bundle admits k sections, linearly independent in each point;
4. A bundle splits into the Whitney sum of two (or more) subbundles.

Other examples most welcome!

5 Real, oriented, complex and quaternion vector bundles

Let $Vect_{\mathbb{F}}^n$ denote the category of n-dimensional vector spaces over $\mathbb{F}$ and $Vect_{\mathbb{F}}$ the category of all finite-dimensional $\mathbb{F}$-vector spaces. Note that a complex vector space can be considered as a real vector space equipped with an automorphism $J\colon \mathbf{V} \to \mathbf{V}$ such that $J^2 = -1$; respectively the quaternion vector space can be considered as a real vector space with a linear action of the 8-element quaternion group i.e. three complex structures satisfying anticommutativity relations. There are obvious forgetful functors:

$$\begin{tikzcd} Vect_{\mathbb{H}} \arrow[rr, "-_{\mathbb{C}}"] \arrow[dr, "-_{\mathbb{R}}"'] & & Vect_{\mathbb{C}} \arrow[dl, "-_{\mathbb{R}}"] \\ & Vect_{\mathbb{R}} & \end{tikzcd} \tag{4}$$

Moreover $Vect_{\mathbb{H}}^n \to Vect_{\mathbb{C}}^{2n}$ and $Vect_{\mathbb{C}}^n \to Vect_{\mathbb{R}}^{2n}$.

For any pair of our fields $\mathbb{F}_1 \subset \mathbb{F}_2$ there exists an extension of scalars functor $-^{\mathbb{F}_2}\colon Vect_{\mathbb{F}_1} \to Vect_{\mathbb{F}_2}$, defined as $\mathbf{V}^{\mathbb{F}_2} := \mathbb{F}_2 \otimes_{\mathbb{F}_1} \mathbf{V}$ which is called complexification or quaternionization respectively. For our fields we have a commutative diagram:

$$\begin{tikzcd} Vect_{\mathbb{H}} & & Vect_{\mathbb{C}} \arrow[ll, "\_^{\mathbb{H}}"'] \\ & Vect_{\mathbb{R}} \arrow[ul, "\_^{\mathbb{R}}"] \arrow[ur, "\_^{\mathbb{C}}"'] & \end{tikzcd} \tag{5}$$

Proposition 5.1. *For any pair of our fields $\mathbb{F}_1 \subset \mathbb{F}_2$ the corresponding forgetful functor and extension of scalars functor are adjoint. Moreover the natural bijection $\mathrm{Hom}_{\mathbb{F}_2}(\mathbf{V}^{\mathbb{F}_2}, \mathbf{W}) \simeq \mathrm{Hom}_{\mathbb{F}_1}(\mathbf{V}, \mathbf{W}_{\mathbb{F}_1})$ is an isomorphism of $\mathbb{F}_1$-vector spaces.*

We need to identify compositions of forgetful and extension functors. For it we define one more functor on complex (resp. quaternion) vector spaces $\bar{-}\colon Vect_{\mathbb{F}} \to Vect_{\mathbb{F}}$ which assigns to each vector space the same abelian group, but with conjugate $\mathbb{F}$-action. We have an important natural isomorphism $\overline{(\mathbf{V}^{\mathbb{F}_2})} \simeq \mathbf{V}^{\mathbb{F}_2}$.

Proposition 5.2. *For any $\mathbb{R}$-vector space $\mathbf{V}$ there is a natural isomorphism $(\mathbf{V}^{\mathbb{C}})_{\mathbb{R}} \simeq \mathbf{V} \oplus \mathbf{V}$. For any $\mathbb{C}$-vector space $\mathbf{W}$ there is a natural isomorphism $(\mathbf{W}_{\mathbb{R}})^{\mathbb{C}} \simeq \mathbf{W} \oplus \overline{\mathbf{W}}$.*

Zad. 8. Prove Prop. 5.2. Formulate and prove a similar proposition for forgetful and quaternionization functors.

We'll consider one more structure on the real vector spaces: an orientation. An oriented real vector space in a pair $(\mathbf{V}, or_{\mathbf{V}})$ where $or_{\mathbf{V}}$ is an orientation i.e. an equivalence class of bases. We denote $Vect_{\mathbb{R}}^{or}$ the category of real oriented vector spaces; morphisms are just isomorphisms preserving orientation. For every complex vector space $\mathbf{W}$ its realification $\mathbf{W}_{\mathbb{R}}$ has a natural orientation defined in the following way: for any basis $w_1, \dots w_n \in \mathbf{W}$ (over $\mathbb{C}$) we define vectors $w_1, iw_1, \dots w_n, iw_n \in (\mathbf{W})_{\mathbb{R}}$ which for a basis over $\mathbb{R}$ and its equivalence class doesn't depend on choice of $w_1, \dots w_n \in \mathbf{W}$. We denote such orientation $or_{\mathbf{W}_{\mathbb{R}}}^{\mathbb{C}}$.

Zad. 9. Prove the last assertion.

Since any isomorphism of complex vector spaces maps any basis to a basis, the forgetful functor $Vect_{\mathbb{C}} \to Vect_{\mathbb{R}}$ restricted to category of isomorphisms factors: $Vect_{\mathbb{C}}^{iso} \to Vect_{\mathbb{R}}^{or} \to Vect_{\mathbb{R}}^{iso}$. Consider the following diagram

$$\begin{tikzcd}
Vect_{\mathbb{C}} \arrow[rr, "(-1)^{\frac{\dim(\dim -1)}{2}}"] \arrow[dr, bend left, "-_{\mathbb{R}}"] & & Vect_{\mathbb{R}}^{or} \arrow[dl] \\
& Vect_{\mathbb{R}} \arrow[ul, bend left, "-^{\mathbb{C}}"] &
\end{tikzcd} \tag{6}$$

in which the horizontal arrow associates to each n-dimensional complex vector space $\mathbf{W}$ the oriented real vector space $(\mathbf{W}_{\mathbb{R}}, (-1)^{\frac{n(n-1)}{2}}[w_1, iw_1, \ldots, w_n, iw_n]$ and multiplication by ± 1 means changing or preserving orientation coming from the complex structure.

Proposition 5.3. *For every oriented real vector space $(\mathbf{V}, or_{\mathbf{V}})$ there is a natural (with respect to orientation preserving isomorphisms) isomorphism $((\mathbf{V}^{\mathbb{C}})_{\mathbb{R}}, or^{\mathbb{C}}_{(\mathbf{V}^{\mathbb{C}})_{\mathbb{R}}}) \simeq (\mathbf{V} \oplus \mathbf{V}, (-1)^{\frac{n(n-1)}{2}}(or_{\mathbf{V}} \oplus or_{\mathbf{V}})$.*

Proof. ZAD. □

Now we consider relation between complex projective space $\mathbb{P}(\mathbf{W})$ and the real projective space $\mathbb{P}(\mathbf{W}_{\mathbb{R}})$. There is a map $p\colon \mathbb{P}(\mathbf{W}_{\mathbb{R}}) \to \mathbb{P}(\mathbf{W})$ which assigns to every real line $L \subset \mathbf{W}$ a complex line $L^{\mathbb{C}} \subset \mathbf{W}$ generated by L.

Proposition 5.4. *The map $\pi\colon \mathbb{P}(\mathbf{W}_{\mathbb{R}}) \to \mathbb{P}(\mathbf{W})$ is a S^1-principal bundle.*

Proof. Since $\mathbf{W}$ is a complex vector space, the group of unit complex numbers acts by $\mathbb{R}$-linear automorphisms on $\mathbf{W}_{\mathbb{R}}$ and obviously its orbit space is homeomorphic to $\mathbb{P}(\mathbf{W})$. □

Zad. 10. Let $H_{\mathbf{W}} \to \mathbb{P}(\mathbf{W})$ be the complex Hopf bundle. Prove that realification of its pull-back $(\pi^! H_{\mathbf{W}})_{\mathbb{R}}$ is isomorphic to the Whitney sum of the two copies of real Hopf bundles $H_{\mathbf{W}_{\mathbb{R}}} \oplus H_{\mathbf{W}_{\mathbb{R}}}$.

Theorem 5.1. *Propositions 5.1, 5.2, 5.3 and 5.4 remain true for vector bundles.*

Proof. Every natural transformation (equivalence) between continuous functors defined between categories of vector spaces extends to a natural transformation (equivalence) between categories of corresponding vector bundles. (cf. [1] §1.2.) □

Zad. 11. Give three interpretations (Sec. 4) of the following properties of vector bundles:

- A real bundle is a realification of a complex bundle;
- A complex bundle is a complexification of a real bundle;
- A real vector bundle is oriented.

6 Vector bundles over spheres

The set of isomorphism classes of vector bundles $Vect_{\mathbb{F}}(X)$ over a space X is equipped with two operations: Whitney sum and tensor product, zero element - 0-dimensional vector bundle, and unit - 1-dimensional trivial vector bundle. The operations satisfy axioms of semi-ring structure (same as ring, no additive inverse elements). We describe monoids of vector bundles over spheres: $Vect_{\mathbb{F}}(S^n)$ for $n = 1, 2$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}$

Let $\mathbb{F} = \mathbb{R}$. Then $Vect_{\mathbb{R}}(S^n) = \coprod_{k=0}^{\infty} [S^n, BO_k]$.

For $n = 1$ inclusions $O_n(\mathbb{R}) \subset O_{n+1}(\mathbb{R})$ induce bijections: $[S^1, BO_1] \simeq [S^1, BO_2] \simeq \dots$. Two vector line bundles corresponding to elements in $[S^1, BO_1] \simeq \mathbb{Z}_2$ are the Möbius bundle $\gamma^1_{\mathbb{R}} \colon M \to S^1$ and trivial bundle θ^1. Hence the abelian monoid $Vect_{\mathbb{R}}(S^1)$ is generated by $\gamma^1_{\mathbb{R}}$ and θ^1. It is obvious that $\gamma^1_{\mathbb{R}} \otimes \gamma^1_{\mathbb{R}} = \theta^1$ and $\gamma^1_{\mathbb{R}} \oplus \gamma^1_{\mathbb{R}} = \theta^1 \oplus \theta^1$. Moreover if $\eta_1 \oplus \theta^1 \simeq \eta_2 \oplus \theta^1$ then $\eta_1 \simeq \eta_2$. The abelian group generated by the monoid is $KO(S^1) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$ (check multiplication).

Complex case: $n = 1$ every complex line bundle over circle is trivial. The case $n = 2$ is similar to the real case when $n = 1$. We have: $\mathbb{Z} \simeq [S^2, BU_1] \simeq [S^1, BU_2] \simeq \dots$ thus the abelian monoid $Vect_{\mathbb{R}}(S^1)$ is generated by $\gamma^1_{\mathbb{C}}$ and θ^1. Tensor powers in $[S^2, BU_1]$ correspond to addition in $\mathbb{Z}$., whereas $(\gamma^1_{\mathbb{C}})^m \oplus (\gamma^1_{\mathbb{C}})^n \simeq (\gamma^1_{\mathbb{C}})^{m+n} \oplus \theta^1_{\mathbb{C}}$. The abelian group generated by the monoid is $K(S^2) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

The case $n = 2$ and $\mathbb{F} = \mathbb{R}$ is more complicated. $[S^2, BO_2] \simeq \mathbb{Z}^+$ and $\mathbb{Z}_2 \simeq [S^2, BO_3] \simeq [S^2, BO_4] \simeq \dots$. The map $\mathbb{Z}^+ \simeq [S^2, BO_2] \to [S^2, BO_3] \simeq \mathbb{Z}_2$ is surjective. Thus the abelian monoid $Vect_{\mathbb{R}}(S^2)$ is generated by two bundles: 1-dimensional trivial bundle θ^1, 2-dimensional bundle $\gamma^1_{\mathbb{C}}|\mathbb{R}$ (realification of the Hopf bundle) satisfying relation $(\gamma^1_{\mathbb{C}})^2|_{\mathbb{R}} \oplus \theta^1_{\mathbb{R}} = \theta^3_{\mathbb{R}}$. Realification map $\coprod_{k=0}^{\infty} Vect^k_{\mathbb{C}}(S^2) \to \coprod_{k=0}^{\infty} Vect^{2k}_{\mathbb{R}}(S^2)$ is an additive homomorphism and it is surjective. Every odd dimensional bundle is a sum of a realification of a complex bundle and 1-dimensional real trivial bundle. $KO(S^2) \simeq \mathbb{Z} \oplus \mathbb{Z}_2$.

7 Orientation of vector bundles in general cohomology theories

7.1 Projectivization and the Thom space of a vector bundle

Let $E \to X$ be n-dimensional real or complex vector bundle and $P_E \to X$ the corresponding GL_n-principal bundle. The linear group acts clearly on the projective space $\mathbb{F}P(n-1)$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) thus we can define the associated bundle $P_E \times_{GL_n} \mathbb{F}P(n) \to X$ and call it projectivization of $E \to X$, denoted usually $\mathbb{P}(E) \to X$. Note that any monomorphism of vector bundles $E_1 \to E_2$ induces injection of projectivizations $\mathbb{P}(E_1) \to \mathbb{P}(E_2)$.

Definition 7.1. *The Thom space of a vector bundle $E \to X$ is the (pointed) space*

$$\mathrm{Th}(E) := \mathbb{P}(E \oplus \theta^1)/\mathbb{P}(E).$$

The Thom space can be viewed as a fiberwise compactification of the total space E. Note that for a 0-dimensional bundle $X \to X$, $\mathrm{Th}(X) = X^+$.

Proposition 7.1. *For every vector bundle $E \to X$ there is a natural embedding $j \colon E \to \mathrm{Th}(E)$ onto the subspace $\mathrm{Th}(E) \setminus \{[\mathbb{P}(E)]\}$. Moreover, closure of each fiber in $\mathrm{Th}(E)$ is its one-point compactification $\overline{j(E_b)} \simeq E^\bullet_b$. If the base X is compact then $\mathrm{Th}(E) = E^\bullet$ is one point compactification of the total space E.*

Proof. We define $j(e) := [e, 1]$ where $1 \in \theta_{p(e)}$. The listed properties of the map are obvious. $\square$

The Thom space construction can be viewed also as a generalization of suspension which is the Thom space of the trivial bundle.

Example 4. Let $X \times \mathbb{R} \to X$ be one dimensional product bundle. Then $\mathrm{Th}(X \times \mathbb{R}) \simeq \Sigma(X^+)$ is the reduced suspension of the pointed space $X^+ := X \sqcup \{+\}$. Hence $\mathrm{Th}(X \times \mathbb{R}^n) \simeq \Sigma^n(X^+)$ and $\mathrm{Th}(X \times \mathbb{C}^n) \simeq \Sigma^{2n}(X^+)$

Zad. 12. The Thom space construction is functorial with respect to monomorphisms of vector bundles over X. Moreover for a vector bundle $E \to X$ every map $f : Y \to X$ induces a map of the Thom spaces $\hat{f}\colon \mathrm{Th}(f^! E) \to \mathrm{Th}(E)$.

Zad. 13. Prove that the Thom space of $E \to X$ is homeomorphic to the following spaces:

1. $D(E)/S(E)$ where $D(E)$ (resp. $S(E)$) is the disc (resp. sphere) bundle with respect to some riemannian (if $\mathbb{F} = \mathbb{R}$) or hermitian metric (if $\mathbb{F} = \mathbb{C}$)
2. $S(E \oplus \theta^1)/s_1(B)$ where $s_1(b) = (0_b, 1)$.

Zad. 14. For every two vector bundles $E_i \to X_i$ there is a natural homeomorphism $\mathrm{Th}(E_1 \times E_2) \simeq \mathrm{Th}(E_1) \wedge \mathrm{Th}(E_2)$. In particular: $\mathrm{Th}(E \oplus \theta^n) \simeq \Sigma^n \mathrm{Th}(E)$.

Proposition 7.2. *Let $H_{\mathbf{V}} \to \mathbb{P}(\mathbf{V})$ be the canonical bundle over the projective space of a vector space $\mathbf{V}$ over the field $\mathbb{F} = \mathbb{R}, \mathbb{C}$. There is a natural homeomorphism $h_{\mathbf{V}}\colon \mathrm{Th}(H^*_{\mathbf{V}}) \xrightarrow{h_{\mathbf{V}}} \mathbb{P}(\mathbf{V} \oplus \mathbb{F})$ such that:*

1. *$h_{\mathbf{V}}$ is natural with respect to inclusions of vector spaces i.e. for every $\mathbf{V} \subset \mathbf{W}$ the diagram:*

$$\begin{tikzcd} \mathrm{Th}(H^*_{\mathbf{V}}) \arrow[r, "h_{\mathbf{V}}"] \arrow[d, "\subset"] & \mathbb{P}(\mathbf{V} \oplus \mathbb{F}) \arrow[d, "\subset"] \\ \mathrm{Th}(H^*_{\mathbf{W}}) \arrow[r, "h_{\mathbf{W}}"] & \mathbb{P}(\mathbf{W} \oplus \mathbb{F}) \end{tikzcd} \tag{7}$$

2. *the zero-section $\mathbb{P}(\mathbf{V}) \subset H^*_{\mathbf{V}} \subset \mathrm{Th}(H^*_{\mathbf{V}})$ is mapped onto the subspace $\mathbb{P}(\mathbf{V}) \subset \mathbb{P}(\mathbf{V} \oplus \mathbb{F})$ i.e. the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Th}(H^*_{\mathbf{V}}) \arrow[rr, "h_{\mathbf{V}}"] & & \mathbb{P}(\mathbf{V} \oplus \mathbb{F}) \\ & \mathbb{P}(\mathbf{V}) \arrow[ul, "s_0"] \arrow[ur, "i"'] & \end{tikzcd} \tag{8}$$

3. *for every 1-dimensional subspace $L \in \mathbb{P}(\mathbf{V})$ compactification of each fiber $(H^*_{\mathbf{V}})_L$ is embedded onto $\mathbb{P}(L \oplus \mathbb{F}) \subset \mathbb{P}(\mathbf{V} \oplus \mathbb{F})$, where $L \subset \mathbf{V}$.*

Proof. We will construct an embedding $h_{\mathbf{V}}\colon H^*_{\mathbf{V}} \to \mathbb{P}(\mathbf{V} \oplus \mathbb{F})$. To every form $\varphi\colon L \to \mathbb{F}$ we associate its graph which is a 1-dimensional subspace in $L \oplus \mathbb{F} \subset \mathbf{V} \oplus \mathbb{F}$. Note that $h_{\mathbf{V}}$ is continuous, injective and its image consists of all 1-dimensional subspaces of $\mathbf{V} \oplus \mathbb{F}$ but one: $\mathbb{F} \subset \mathbf{V} \oplus \mathbb{F}$. Thus $h_{\mathbf{V}}$ extends to one-point compactification of the space, hence a homeomorphism $\mathrm{Th}(H_{\mathbf{V}}) \xrightarrow{h_{\mathbf{V}}} \mathbb{P}(\mathbf{V} \oplus \mathbb{F})$.

The zero-section maps a line L to the vector $(L, 0)$ and $h_{\mathbf{V}}(L, 0) = L$ thus the diagram (1) commutes. Point (2) follows immediately from construction of $h_{\mathbf{V}}$. $\square$

Remark 6. Note that choice of the scalar product (resp. hermitian) product establishes an isomorphism of bundles $\bar{H}_{\mathbf{V}} \simeq H^*_{\mathbf{V}}$ (where for $\mathbb{F} = \mathbb{C}$, $\bar{H}_{\mathbf{V}}$ denotes complex conjugation). Since Thom space depends only on realification of a complex bundle we obtain a homeomorphism $\mathrm{Th}(H_{\mathbf{V}}) \simeq \mathbb{P}(\mathbf{V} \oplus \mathbb{F})$.

Zad. 15. Prove that the diagonal map $\Delta\colon \mathrm{Th}(E) \to X^+ \wedge \mathrm{Th}(E)$, $\Delta(e) := [p(e), e]$ where $p(\infty) := [+, \infty]$ is well-defined and continuous.

7.2 The Thom classes in cohomology theories

Let $h^n\colon \mathcal{T}op_2 \to \mathcal{A}b$ be a sequence of functors (defined for $n \in \mathbb{Z}$) defined on (some) pairs of topological spaces, containing all CW-pairs, which satisfies the Eilenberg-Steenrod axioms, with perhaps exception of the dimension axiom. We'll call such sequence generalized cohomology theory, or simply cohomology theory (for details cf. Hilton [3]). We define reduced cohomology on pointed spaces: $\tilde{h}^*(X) := h^*(X, x_0)$. If $A \subset X$ is a cofibration then ES-axioms imply that $\tilde{h}^*(X/A) := h^*(X, A)$. Note the following properties which follow from the axioms.

Proposition 7.3. *Let h^* be a cohomology theory. Then, under additional assumptions which may depend on the category on which cohomology theory is defined, the following assertion hold:*

1. *there exists the Meyer-Vietoris exact sequence for (homotopy) push-out diagrams;*
2. *for a family of pointed spaces $\{X_i\}_{i\in J}$, inclusions induce an isomorphism $\tilde{h}^*(\bigvee_{i\in J} X_i) \simeq \prod_{i\in J} \tilde{h}^*(X_i)$;*
3. *for two pointed spaces X_1, X_2 there is a split exact sequence*
$$0 \to \tilde{h}^*(X_1 \wedge X_2) \to h^*(X_1 \times X_2) \to h^*(X_1) \times h^*(X_2) \to 0.$$
4. *any self map of a sphere $f\colon S^n \to S^n$ of degree d induces a homomorphism $f^*\colon \tilde{h}^*(S^n) \to \tilde{h}^*(S^n)$ such that $f^*(u) = d \cdot u$ for each $u \in \tilde{h}^*(S^n)$.*

A multiplicative structure on h^* is a collection of natural transformations, defined for every $p, q \in \mathbb{Z}$
$$h^p(X, A) \otimes h^q(Y, B) \xrightarrow{\times} h^{p+q}(X \times Y, X \times B \cup A \times Y)$$
and an element which satisfy standard conditions for the cross-product in singular cohomology (cf. Spanier [7] 5.6.2-5). Moreover assume that $h^*(pt) \otimes h^*(pt) \xrightarrow{\times} h^*(pt)$ is a ring structure with unit $1 \in h^0(pt)$. The ring $h^*(pt)$ is called the coefficient ring and often denoted h^*.

For every pair (X, A) cohomology $h^*(X, A)$ is a graded module over $h^*(pt)$ and induced homomorphisms are h^*-module homomorphisms. For two subsets $A, B \subset X$ the cup-product $\cup$ is defined as usual as composition
$$h^p(X, A) \otimes h^q(X, B) \xrightarrow{\times} h^{p+q}(X \times X, X \times B \cup A \times X) \xrightarrow{\Delta^*} h^p(X, A \cup B),$$
where $\Delta\colon X \to X \times X$ is the diagonal map. If $A, B = \emptyset$, then $h^*(X) := h^*(X, \emptyset)$ is a ring with unit. Note that commutation of cross product with the boundary operator in the exact sequence of pair implies that the suspension homomorphism in the reduced cohomology $\sigma\colon \tilde{h}^q(X) \to \tilde{h}^{q+1}(\Sigma X)$ is given by the formula $\sigma(u) = \sigma(1) \times u$ where $1 \in \tilde{h}^0(S^0) = h^0(pt)$.

Proposition 7.4. *For every multiplicative cohomology theory*
$$\tilde{h}^*((\mathbb{R}^n)^\bullet) = \tilde{h}^*(S^n) = h^*(D^n, S^{n-1}) = h^*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$
is a free graded h^*-module with one generator $\iota_n := \sigma^n(1) \in \tilde{h}^n(S^n)$.

Note that any choice of orientation *or* of the n-dimensional real vector space $\mathbf{V}$ in the sense of linear algebra defines a generator of h^*-module $h^*(\mathbf{V}^\bullet)$. Formally, we pick up an isomorphism $A\colon \mathbf{V} \to \mathbb{R}^n$ which maps the selected orientation to the canonical orientation and define $[or]_{h^*} := A^*(\iota_n)$. This is well defined because the real linear group has two path components. Elements $\pm\iota_n \in h^n(\mathbf{V}^\bullet)$ we call h^*-orientations of $\mathbf{V}$. It may happen that they coincide (e.g. for $H^*(-;\mathbb{Z}_2)$).

Definition 7.2. *An h^*-orientation of an n–dimensional real vector bundle $E \to X$ is an element $U_E \in \tilde{h}^n(\mathrm{Th}(E))$ such that for inclusion of every fiber $i_x\colon p^{-1}(x) \to \mathrm{Th}(E)$ the restriction $i_x^*(U_E) \in \tilde{h}^n(p^{-1}(x)^\bullet)$ is an h^*-orientation of the vector space $p^{-1}(x)$. The element U_E is also called the Thom class of the bundle $E \to X$. A bundle which has a Thom class is called h^*-orientable.*

Zad. 16. If $U_E \in \tilde{h}^n(\mathrm{Th}(E))$ is the Thom class of the vector bundle $E \to Y$ and $f\colon X \to Y$ is a map then $\hat{f}^*(U_E) \in \tilde{h}^n(\mathrm{Th}(f^!E))$ is the Thom class of the induced bundle.

Zad. 17. Prove the following assertions concerning orientation of vector bundles:

1. Trivial bundle has the Thom class in any multiplicative cohomology theory.
2. Every vector bundle is orientable in the singular cohomology with $\mathbb{Z}_2$–coefficients (any $\mathbb{Z}_2$–algebra) and the Thom class is unique.
3. There is a bijection between geometric orientations of the vector bundle and its Thom classes in singular cohomology with $\mathbb{Z}$–coefficients (any ring of characteristic $\neq 2$).
4. If in a cohomology theory h^* every vector bundle is orientable then the ring h^* has characteristic 2 i.e. cohomology theory has values in $\mathbb{Z}_2$-vector spaces.
5. If h^* has characteristic $\neq 2$ then any h^*–orientable bundle is geometrically orientable (i.e. $H^*(-;\mathbb{Z})$-orientable.)

Zad. 18. For every two vector bundles $E_i \to X_i$ with Thom classes $U_{E_i} \in \tilde{h}^{n_i}(\mathrm{Th}(E_i))$ the element $U_{E_1} \times U_{E_2} \in \tilde{h}^{n_1+n_2}(\mathrm{Th}(E_1 \oplus E_2))$ (cf. Zad 14) is the Thom class of the Whitney sum $E_1 \oplus E_2$. Infer that orientability is a stable property i.e. if $E \oplus \theta^k$ is h^*-orientable then E is h^*-orientable.

Definition 7.3. *For an h^*-oriented n-dimensional vector bundle $E \to X$ we define its Euler class $e(E) := s_0^*(U_E) \in h^n(X)$ where $s_0\colon (X, \emptyset) \to (\mathrm{Th}(E), \infty)$ is the zero-section.*

Zad. 19. If a vector bundle has everywhere non-vanishing section then its Euler class in any cohomology theory vanishes.

Zad. 20. Let $E \to S^n$ be a real oriented n-dimensional vector bundle over sphere ($n > 1$). Prove two interpretations of its Euler class in singular cohomology:

1. Let $\gamma_E\colon S^{n-1} \to SO_n$ be gluing function of the bundle and consider its composition with projection $p\colon SO_n \to S^{n-1}$. Then $e(E) = \pm\deg(p\gamma_E) \in H^n(S^n) \simeq \mathbb{Z}$.
2. Prove that $E \to S^n$ has a section $s\colon S^n \to E$ such that $s(x) \neq 0$ for all $x \neq 1$. Consider a map locally defined by the section s: $f_s\colon (U, U \setminus \{1\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. Then $e(E) = \pm\deg(f_s) \in H^n(S^n) \simeq \mathbb{Z}$.

7.3 Complex oriented cohomology theories

A complex oriented cohomology theory is a generalized cohomology theory h^* which is multiplicative and it has a natural choice of the Thom class for every complex vector bundle (considered as a real bundle.) Before we give a precise definition we'll explain relation between real and complex vector bundles.

Obviously for every n-dimensional complex vector bundle $E \to X$ we can forget about complex structure and consider it as $2n$-dimensional real vector bundle $E_{\mathbb{R}} \to X$ (*a realification*) . Conversely, having a n-dimensional real vector bundle we can construct its *complexification* $E \otimes_{\mathbb{R}} \mathbb{C} \to X$.

Zad. 21. Consider realification and complexification as a pair of adjoint functors between categories of complex (reps. real) vector bundles over a fixed space.

For every complex vector space $\mathbf{V}$ we define a map $\mathbb{P}(\mathbf{V}_{\mathbb{R}} \oplus \mathbb{R}) \ni [\mathbf{v}, t]_{\mathbb{R}} \mapsto [\mathbf{v}, t]_{\mathbb{C}} \in \mathbb{P}(\mathbf{V} \oplus \mathbb{C})$ which maps $\mathbb{P}(\mathbf{V}_{\mathbb{R}})$ to $\mathbb{P}(\mathbf{V})$ and on the quotients

$$\mathbb{P}(\mathbf{V}_{\mathbb{R}} \oplus \mathbb{R})/\mathbb{P}(\mathbf{V}_{\mathbb{R}}) \to \mathbb{P}(\mathbf{V} \oplus \mathbb{C})/\mathbb{P}(\mathbf{V})$$

it is bijective, thus a homeomorphism. For every complex vector bundle $E \to X$ the map

$$\mathbb{P}(E \oplus \theta^1_{\mathbb{R}})/\mathbb{P}(E_{\mathbb{R}}) \to \mathbb{P}(E \oplus \theta^1_{\mathbb{C}})/\mathbb{P}(E)$$

is defined fibrewise and it is a local homeomorphism (also at ∞ !), thus a homeomorphism. Hence the Thom space can be constructed out of the complex projectivization.

Note that a complex vector space $\mathbf{V}$ has a uniquely defined h^*-orientation i.e. a generator $\iota_{\dim \mathbf{V}} \in \tilde{h}^{\dim \mathbf{V}}(\mathbf{V}^\bullet)$ which is defined by an $\mathbb{C}$-linear isomorphism $\mathbf{V} \to \mathbb{C}^n$. Any two such isomorphisms are homotopic because the complex linear group is path connected.

Definition 7.4. *We say that a multiplicative cohomology theory h^* is complex oriented if to every complex vector bundle $E \to X$ a Thom class $U_E \in \tilde{h}^*(\mathrm{Th}(E))$ is assigned in such way that:*

1. *For each $x \in X$, $i_x^*(U_E) = \iota_{\dim E}$*
2. *The classes U_E should be natural under pull-backs: if $f\colon Y \to X$, $U_{f^! E} = \hat{f}^* U_E$;*
3. *Multiplicativity: $U_{E_1 \oplus E_2} = U_{E_1} U_{E_2}$;*
4. *Normalization: If $H \to S^2$ is the Hopf bundle, then $e(H) = \sigma^2(1)$.*

Zad. 22. The singular cohomology theory is complex oriented.

Remark 7. There are many other examples of the complex oriented theories playing important role in algebraic topology e.g. complex K-theory, Morava K-theories, complex cobordism.

8 The Leray-Hirsch Theorem and the Splitting Principle

8.1 The Leray-Hirsch Theorem

Definition 8.1 (Spanier Sec. 5.7). *A* fibre-bundle pair *with a base space B consists of a* total pair *of spaces $(E, \dot{E})$, a* fiber pair *of spaces $(F, \dot{F})$ and a projection $E \xrightarrow{p} B$ such that there exists an open covering $\mathcal{V}$ of the space B and for each $V \in \mathcal{V}$ there is a homeomorphism over V of pairs $V \times (F, \dot{F}) \simeq (p^{-1}(V), p^{-1}(V) \cap \dot{E})$.*

Any vector bundle $E \to B$ leads to a fibre-bundle pair, where fiber is $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ i.e. $(E, \dot{E}) \to B$ where $\dot{E} = E \setminus s_0(B)$. When we pick-up a Riemannian metric on the vector bundle then we get a a fiber-bundle pair with fiber (D^n, S^{n-1}). Another important example is projectivization $(\mathbb{P}(E \oplus \theta^1), \mathbb{P}(E)) \to B$; here fiber-pair is $(\mathbb{P}(E_x \oplus \mathbb{F}), \mathbb{P}(E_x))$.

In fact for every locally-trivial bundle $\dot{E} \to B$ whose fiber is a sphere S^{n-1} we can construct a fiber-bundle pair $(E, \dot{E}) \to B$ with fiber (D^n, S^{n-1}). The total space E is obtained by construction a cone over each fibre $\dot{E}_x \subset \dot{E}$.

If h^* is a multiplicative cohomology theory then for any fiber-bundle pair cohomology $h^*(E, \dot{E})$ has a graded $h^*(B)$-module structure (in fact local triviality is not relevant here). The multiplication by elements of $h^*(B)$ is defined via the cup product and the induced homomorphism p^*:

$$h^p(B) \otimes h^q(E, \dot{E}) \xrightarrow{p^* \otimes id} h^p(E) \otimes h^q(E, \dot{E}) \xrightarrow{\cup} h^{p+q}(E, \dot{E})$$

Theorem 8.1 (Leray-Hirsch). *Let $(E, \dot{E}) \to B$ be a fiber-bundle pair over a CW-complex (with finite trivialization covering). Suppose $u_1, \ldots u_n \in h^*(E, \dot{E})$ are homogeneous elements such that for each point $b \in B$ the restrictions $i_b^*(u_1), \ldots i_b^*(u_n) \in h^*(F, \dot{F})$ form a free basis of $h^*(F, \dot{F})$ over h^*. Then the elements $u_1, \ldots u_n \in h^*(E, \dot{E})$ form a free basis of $h^*(E, \dot{E})$ over $h^*(B)$.*

We begin with proving a version of the Künneth formula in general cohomology, which is a special case of the above theorem.

Proposition 8.1 (Künneth formula). *For every finite CW-complex B and a pair $(F, \dot{F})$ such that $h^*(F, \dot{F})$ is a free h^*–module, the cross product homomorphism*

$$\bigoplus_{p+q=n} h^p(B) \otimes_{h^*} h^q(F, \dot{F}) \xrightarrow{\times} h^n(B \times (F, \dot{F}))$$

is an isomorphism for every $n \in \mathbb{Z}$.

Proof. Both graded groups $h^*(-) \otimes h^*(F, \dot{F})$ and $h^*(- \times (F, \dot{F}))$ can be extended to cohomology theories (i.e. relative groups and boundary homomorphisms defined). Exactness is preserved because $h^*(F, \dot{F})$ is a free h^*-module (it is enough to assume that it is flat). The cross product defines a natural transformation which is obviously isomorphism for $B = pt$. Hence it is an isomorphism for any finite CW-complex B. □

Proof of Thm. 8.1. We consider a homomorphism:

$$\bigoplus_{i=1}^{n} h^*(B) \ni (z_1, \ldots z_k) \mapsto \sum_{i=1}^{n} p^*(z_i) \cup u_i \in h^*(E, \dot{E})$$

and first prove (using the Kunneth formula) that it is an isomorphism for trivial fibre-bundle pair $B \times (F, \dot{F}) \to B$. General case follows by induction on skeleta, applying the Mayer-Vietoris exact sequence. □

Note that the Leray-Hirsch theorem asserts that if the elements $u_1, \dots u_n \in h^*(E,\dot{E})$ exist then $h^*(B)$-module $h^*(E,\dot{E})$ is isomorphic to $h^*(B\times F, B\times \dot{F})$ thus the $h^*(B)$-module structure doesn't distinguish between isomorphism classes of bundles.

8.2 The Thom isomorphism and the Gysin exact sequence

Let us note that the Thom class 7.2 can be defined for any fiber-bundle pair $(E,\dot{E})\to B$ whose fiber is either $(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\})$ or (D^n, S^{n-1}). It is an element $U_E \in h^n(E,\dot{E})$ whose restriction to each fiber $U|(E_x,\dot{E}_x)\in h^n(E_x,\dot{E}_x)$ is a generator. We obtain the following consequence of the Leray-Hirsch Theorem 8.1.

Corollary 8.1 (Thom Isomorphism). *Let $U_E \in \tilde{h}^n(\mathrm{Th}(E))$ be the Thom class of the fiber-bundle pair $(E,\dot{E})\to B$. Then multiplication by the Thom class: $\Phi_U\colon h^q(B)\to h^{q+n}(E,\dot{E})$ is an isomorphism $h^*(B)$-modules.*

For an h^*-orientable vector bundle $E\to B$ it means that cohomology of the Thom space $\tilde{h}^*(\mathrm{Th}(E))$ is a free graded $h^*(B)$-module generated by U_E. It follows immediately from Thm. 8.1 applied to the fibre-bundle pair $(\mathbb{P}(E\oplus\theta^1), \mathbb{P}(E))\to B$. .

Remark 8. The Thom isomorphism for one dimensional trivial bundle is the suspension isomorphism.

An important consequence of the Thom isomorphism is the Gysin exact sequence relating cohomology of a total space and base space of h^*-oriented sphere bundle $p\colon \dot{E}\to B$.

Theorem 8.2 (The Gysin sequence). *Let $p\colon \dot{E}\to B$ be h^*-oriented sphere bundle and $e(E)\in h^n(B)$ be its Euler class. Then the following sequence is exact:*

$$\dots \to h^{q-n}(B) \xrightarrow{e(E)\cup -} h^q(B) \xrightarrow{p^*} h^q(\dot{E}) \xrightarrow{\delta} h^{q-n+1}(B) \to \dots$$

8.3 Cohomology of projective spaces

Now we'll apply the Thom isomorphism theorem to complex oriented cohomology to calculate cohomology rings of complex projective spaces. If h^* is a complex oriented cohomology then for any complex projective space $\mathbb{P}(\mathbf{V})$ we define an element $x_{\mathbf{V}} \in h^2(\mathbb{P}(\mathbf{V}))$ as the Euler class of the tautological bundle $H_{\mathbf{V}}\to\mathbb{P}(\mathbf{V})$ i.e. $x_{\mathbf{V}} = U_{\mathbf{V}}|B$. We note two properties of the class $x_{\mathbf{V}}$ which are crucial for calculation of the cohomology ring $h^*(\mathbb{P}(\mathbf{V}))$.

Lemma 8.1. *The class is natural i.e. for any embedding of a subspace $\mathbf{V}\to\mathbf{W}$, $x_{\mathbf{W}}|_{\mathbb{P}(\mathbf{V})} = x_{\mathbf{V}}$. Moreover the class $x_{\mathbf{V}\oplus\mathbb{C}}\in\tilde{h}^2(\mathbb{P}(\mathbf{V}\oplus\mathbb{C}))$ corresponds under homeomorphism defined in Prop. 7.2 to a Thom class of the bundle $H_{\mathbf{V}}$.*

Proof. Naturality follows immediately from the definition of the complex oriented theory and the fact that any two embeddings are homotopic via embeddings. According to Prop. 7.2 3. embedding of the fibre over L into the Thom space corresponds to the embedding $\mathbb{P}(L\oplus\mathbb{C})\subset\mathbb{P}(\mathbf{V}\oplus\mathbb{C})$, thus $v_{\mathbf{V}\oplus\mathbb{C}}|\mathbb{P}(L\oplus\mathbb{C}) = x_{\mathbb{P}(L\oplus\mathbb{C})}\in\tilde{h}^2(S^2)$ is by the definition a generator. □

The projective space $\mathbb{P}(\mathbf{V})$ can be covered by $d := \dim_{\mathbb{C}}\mathbf{V}$ contractible sets (affine charts), thus $x^d_{\mathbf{V}} = 0$ and there is a homomorphism $\chi_{\mathbf{V}}\colon h^*[x_{\mathbf{V}}]/(x^d_{\mathbf{V}})\to h^*(\mathbb{P}(\mathbf{V}))$ where $h^*[x_{\mathbf{V}}]$ denotes graded polynomial ring with single generator in gradation 2.

Theorem 8.3. *For a complex vector space $\mathbf{V}$ of dimension d the map*

$$\chi_{\mathbf{V}}\colon h^*[x_{\mathbf{V}}]/(x^d_{\mathbf{V}})\to h^*(\mathbb{P}(\mathbf{V}))$$

is an isomorphism.

Proof. Using Prop. 7.2 and Cor. 8.1 we proceed by induction on dimension of the vector space $\mathbf{V}$. For $\dim \mathbf{V} = 1$ theorem is tautologically true. (For $\dim \mathbf{V} = 2$ it also obviously holds because $\mathbb{P}(\mathbb{C} \oplus \mathbb{C}) = S^2$ and its cohomology can be calculated using suspension isomorphism: $h^*[x_{\mathbb{C}^2}]/(x^2_{\mathbb{C}^2}) \xrightarrow{\simeq} h^*(S^2)$.

Suppose

$$h^*[x_{\mathbf{V}}]/(x^d_{\mathbf{V}}) \to h^*(\mathbb{P}(\mathbf{V}))$$

is an isomorphism, and we prove it for the space $\mathbf{V} \oplus \mathbb{C}$. Note that $x_{\mathbf{V}\oplus\mathbb{C}} \in h^2(\mathbb{P}(\mathbf{V}\oplus\mathbb{C}))$ corresponds under homomorphism induced by homeomorphism described in Prop. 7.2 to the Thom class of the bundle $H_{\mathbf{V}} \to \mathbb{P}(\mathbf{V})$ (apply 7.2.3.). Suppose the theorem holds for the space $\mathbf{V}$. We use the Thom isomorphism defined by the class $x_{\mathbf{V}\oplus\mathbb{C}}$ to prove it for the space $\mathbf{V} \oplus \mathbb{C}$. Consider the following commutative diagram, where the top arrow is defined via the cup-product and the diagonal map $\Delta\colon \mathrm{Th}(E) \to X^+ \wedge \mathrm{Th}(E)$ defined in Zad 15:

$$\begin{tikzcd}
h^{2k-2}(\mathbb{P}(\mathbf{V})) \otimes \tilde{h}^2(\mathbb{P}(\mathbf{V} \oplus \mathbb{C})) \arrow[r, "\Delta^*"] & \tilde{h}^{2k}(\mathbb{P}(\mathbf{V} \oplus \mathbb{C})) \\
h^{2k-2}(Th(H_{\mathbf{V}})) \otimes \tilde{h}^2(Th(H_{\mathbf{V}})) \arrow[r, "\cup"] \arrow[u, "s_0^* \otimes h_{\mathbf{V}}^*"] \arrow[d, "\simeq"', "h_{\mathbf{V}}^* \otimes h_{\mathbf{V}}^*"] & \tilde{h}^{2k}(Th(H_V)) \arrow[u, "h_{\mathbf{V}}^*"', "\simeq"] \arrow[d, "\simeq"', "h_{\mathbf{V}}^*"] \\
h^{2k-2}(\mathbb{P}(\mathbf{V} \oplus \mathbb{C})) \otimes \tilde{h}^2(\mathbb{P}(\mathbf{V} \oplus \mathbb{C})) \arrow[r, "\cup"] & \tilde{h}^{2k}(\mathbb{P}(\mathbf{V} \oplus \mathbb{C}))
\end{tikzcd} \tag{9}$$

According to the inductive assumption and the Thom isomorphism theorem the elements $\Delta(x^k_{\mathbf{V}} \otimes x_{\mathbf{V}\oplus\mathbb{C}}) \in \tilde{h}^*(\mathbb{P}(\mathbf{V} \oplus \mathbb{C}))$ are free generators over the coefficient ring h^*. To finish description of the multiplicative structure it is enough to prove that $\Delta(x^k_{\mathbf{V}} \otimes x_{\mathbf{V}\oplus\mathbb{C}}) = x^k_{\mathbf{V}\oplus\mathbb{C}}$. It follows immediately from the last diagram.

□

Remark 9. One can prove that if in a multiplicative cohomology theory the above description of the cohomology algebra of the complex projective space holds (with some choice of generators compatible with restrictions to subspaces) then the theory is complex oriented. In other words, orientations of the linear bundles permit to construct orientation for any complex vector bundle.

Remark 10. Analogous theorem holds for h^*-cohomology of real projective spaces if all real vector bundles are h^*-orientable (then $\deg x_{\mathbf{V}} = 1$.)

9 Characteristic classes

Let $E \to B$ a n-dimensional complex vector bundle and consider its projectivization $\mathbb{P}(E) \to B$. We notice that the bundle $\mathbb{P}(E) \to B$ satisfies assumptions of the Leray-Hirsch theorem in any complex oriented theory h^* . Consider the pull-back diagram:

$$\begin{tikzcd}
p^!_{\mathbb{P}} E \arrow[r, "\tilde{p}_{\mathbb{P}}"] \arrow[d, "p_1"] & E \arrow[d, "p"] \\
\mathbb{P}(E) \arrow[r, "p_{\mathbb{P}}"] & B
\end{tikzcd} \tag{10}$$

and note that the vector bundle $p_{\mathbb{P}}^! E \to \mathbb{P}(E)$ contains a 1-dimensional subbundle

$$H_E := \{(L, e) \in p_{\mathbb{P}}^! E \,|\, e \in L\}.$$

Restriction of $H_E|\mathbb{P}(E_b) = H_{E_b}$ to any fiber is the canonical bundle over $\mathbb{P}(E_b)$. Consider the Euler class of the bundle $x_E := e(H_E) \in h^2(\mathbb{P}(E))$ and for every point $b \in B$, $x_{E^*}|\mathbb{P}(E_b) \in h^2(\mathbb{P}(E_b))$ is a generator thus its powers $1, x_E|_b, \dots, (x_E|_b)^{n-1}$ generate freely $h^*(\mathbb{P}(E_b))$ over the coefficient ring h^*. Hence, according to the Leray-Hirch theorem the elements $1, x_E, \dots, x_E^{n-1}$ generate freely $h^*(\mathbb{P}(E))$ over $h^*(B)$. Consider the element $x_E^n \in h^{2n}(\mathbb{P}(E))$. There are unique elements $c_i(E) \in h^{2i}(B)$ such that:

$$x_E^n + \sum_{i=1}^{n} (-1)^i c_i(E) x_E^{n-i} = 0.$$

The elements $c_i(E) \in h^{2i}(B)$ describe completely the multiplicative structure of $h^*(\mathbb{P}(E))$. We define also $c_0(E) = 1$. The classes $c_i(E)$, $i = 0, \dots n$ are called the Chern classes of the bundle $E \to X$ in cohomology theory h^*. The element $c(E) := 1 + c_{(}E) + \dots + c_n(E) \in h^*(B)$ is called the total Chern class. We proceed towards proof of the following theorem:

Theorem 9.1. *Let h^* be the complex oriented cohomology theory. There is exactly one natural transformation of functors $c\colon Vect^{\mathbb{C}}(X) \to h^*(X)$ such that:*

1. *$c(E) = 1 + c_1(E) + \dots + c_n(E)$ where $c_i(E) \in h^{2i}(X)$ and $n = \dim_{\mathbb{C}} E$*
2. *$c(E_1 \oplus E_2) = c(E_1)c(E_2)$*
3. *For every complex vector space $\mathbf{V}$, $c_1(H_{\mathbf{V}}) = x_{\mathbf{V}} \in h^2(\mathbb{P}(\mathbf{V}))$.*

Construction of the total Chern class was described above. For proof of properties 1.-3. (in fact only 2. is not obvious) we will need *splitting principle.*

Theorem 9.2 (Splitting Principle)**.** *Let h^* be the complex oriented cohomology theory. To every complex vector bundle $E \to X$ one can associate functorially the flag bundle $p_{\mathbb{F}}\colon \mathbb{F}(E) \to X$ such that*

1. *the induced homomorphism $p_{\mathbb{F}}^*\colon h^*(X) \to h^*(\mathbb{F}(E))$ is injective, and*
2. *the pull-back bundle $p_{\mathbb{F}}^! E$ splits into direct sum of line bundles $p_{\mathbb{F}}^! E = L_1 \oplus \dots \oplus L_n$.*

Proof. Theorem follows easily when we apply splitting of single line bundle described above several times

$$\begin{tikzcd}
L_1 \oplus \dots L_n \arrow[r] \arrow[d] & \dots \arrow[r] & H_{Q_E} \oplus p_{1\mathbb{P}}^! H_E \oplus Q_{Q_E} \arrow[r] \arrow[d] & p_{\mathbb{P}}^! E = H_E \oplus Q_E \arrow[r, "\tilde{p}_{\mathbb{P}}"] \arrow[d, "p_1"] & E \arrow[d, "p"] \\
F(E) \arrow[r] & \dots \arrow[r] & \mathbb{P}(Q_E) \arrow[r, "p_{1\mathbb{P}}"] & \mathbb{P}(E) \arrow[r, "p_{\mathbb{P}}"] & B
\end{tikzcd} \tag{11}$$

The last base space denoted $\mathbb{F}(E)$ can be also constructed as the space of bundle associated to the principal bundle $P_E \to X$ whose fiber is the flag manifold $\mathbb{F}(\mathbb{C}^n)$. $\square$

Proof of Thm. 9.1. The assignment $E \mapsto c(E)$ is functorial since construction of projectvization and splitting over $\mathbb{P}(E)$ are functorial. Functoriality of the Euler class follows from definition of the complex oriented cohomology.

Properties 1 and 3 follow directly from definition. Indeed, for the Hopf bundle $H_{\mathbf{V}} \to \mathbb{P}(\mathbf{V})$ we have the defining equation: $x_{H_{\mathbf{V}}} - c_1(H_{\mathbf{V}}) = 0$ thus $c_1(H_{\mathbf{V}}) = x_{H_{\mathbf{V}}} = e(H_{\mathbf{V}})$.

For property 2. we prove it first for sums of line bundles i.e. if $E = L_1 \oplus \cdots \oplus L_n$ and $\dim L_i = 1$ for all i's, then $c(E) = \prod_{i=1}^{n} c(L_i)$. Consider the commutative diagram

$$\begin{tikzcd} H_E \arrow[r, "\tilde{p}_{\mathbb{P}}"] \arrow[d, "p_1"] & \bigoplus_{i=1}^{n} L_i \arrow[d, "p"] \\ \mathbb{P}(E) \arrow[r, "p_{\mathbb{P}}"] & B \end{tikzcd} \tag{12}$$

The space $\mathbb{P}(E) = \{[\mathbf{v}] = [\mathbf{v}_1, \ldots, \mathbf{v}_n] \,|\, \mathbf{v}_i \in (L_i)_b,\ b \in B\}$ and for each k consider an open subset $U_k := \{[\mathbf{v}_1, \ldots, \mathbf{v}_n] \,|\, \mathbf{v}_i \in (L_i)_b,\ \mathbf{v}_k \neq 0,\ b \in B\}$. For each point $[\mathbf{v}] \in U_k$ the map $\tilde{p}_{\mathbb{P}} \colon p_1^{-1}([\mathbf{v}]) \to (L_k)_b$ is an isomorphism thus $H_E|U_k \simeq p_{\mathbb{P}}^! L_k$, hence $(x_E - p_{\mathbb{P}}^* c_1(L_k))|U_k = 0$. The sets $\{U_k\}_{k=1}^n$ cover the space $\mathbb{P}(E)$ thus:

$$\prod_{k=1}^{n} (x_E - p_{\mathbb{P}}^* c_1(L_k)) = 0$$

and $c_i(E) = \sigma_i(c_1(L_1), \ldots, c_1(L_n)$ where σ_i is the i-th elementary symmetric polynomial. Now we obtain that $c(E) = \prod_{i=1}^{n} c(L_i)$. The general case follows from the splitting principle. For two bundles E_1, E_2 we find a map $p_{12} \colon Y \to B$ such that both bundles $p_{12}^* E_1, p_{12}^* E_2$ split into linear summands.

Uniqueness of the natural transformation satisfying conditions 1.-3. follows similarly from the splitting principle. For linear bundles it follows from the fact that every linear bundle is a pull-back of the canonical Hopf bundle. For sums of linear bundles it follows from conditions 2. For general bundles we apply the splitting principle Thm. 9.2. □

Zad. 23. For every complex oriented cohomology theory h^* and n-dimensional complex vector bundle E, $c_n(E) = e(E)$.

If h^* is a real oriented cohomology theory i.e. all vector bundles are canonically oriented in h^* then the same procedure leads to construction of the Stiefel-Whitney classes $w_i(E) \in h^i(B)$. More precisely:

Theorem 9.3. *Let h^* be the real oriented cohomology theory. There is exactly one natural transformation of functors $w \colon Vect^{\mathbb{R}}(X) \to h^*(X)$ such that:*

1. $w(E) = 1 + w_1(E) + \cdots + w_n(E)$ *where* $w_i(E) \in h^i(X)$ *and* $n = dim_{\mathbb{R}} E$
2. $w(E_1 \oplus E_2) = w(E_1) w(E_2)$
3. *for any real vector space* $\mathbf{V}$, $w_1(H_{\mathbf{V}}) = x_{\mathbf{V}} = e(H_{\mathbf{V}}) \in h^1(\mathbb{P}(\mathbf{V}))$.

10 Cohomology rings of Grassmannians

Theorem 10.1. *For any complex oriented additive cohomology theory h^* the Chern classes $c_i \in h^{2i}(BU_n)$ of the universal bundle over BU_n are generators of $h^*(BU_n)$ as a graded power series ring over h^* i.e. $h^*(BU_n) \simeq h^*[[c_1, \dots c_n]]$.*

Proposition 10.1. $h^*(\mathbb{P}(\mathbb{C}^\infty) \times \dots \times \mathbb{P}(\mathbb{C}^\infty)) \simeq h^*[[x_1, \dots x_n]]$ *where* $\deg(x_i) = 2$.

Proof.

$$h^*(\mathbb{P}(\mathbb{C}^\infty)) \simeq \lim_{n\to\infty} h^*(\mathbb{P}(\mathbb{C}^n)) \simeq \lim_{n\to\infty} h^*[x_{\mathbb{C}^n}]/(x_{\mathbb{C}^n}^n) \simeq h^*[[x_{\mathbb{C}^2}]] = h^*[[x_1]]$$

The general case follows from the Künneth formula since cohomology of factors are free h^*-modules.

□

Proof of Thm. 10.1. Consider the universal U_n–space EU_n and the subgroup of the diagonal matrices $T^n \subset U_n$. Choose model of $EU_n = V_n(\mathbb{C}^\infty)$. The flag bundle described in Thm. 9.2 of the universal bundle is homeomorphic to projection $BT^n = EU_n/T^n \to EU_n/U_n = BU_n$. We can choose a different model for ET^n - a space $V_1(\mathbb{C}^\infty) \times \dots \times V_1(\mathbb{C}^\infty)$ with the diagonal T^n-action. There is a T^n-equivariant embedding $V_1(\mathbb{C}^\infty) \times \dots \times V_1(\mathbb{C}^\infty) \subset V_n(\mathbb{C}^\infty \times \dots \times \mathbb{C}^\infty) = V_n(\mathbb{C}^\infty)$[1] which is a homotopy equivalence, thus embedding $(ET^n)/T^n \subset EU_n/T^n$ is a homotopy equivalence. It is now easy to see that pull-back of the universal vector bundle to EU_n/T^n restricted to ET^n/T^n splits into cartesian product of the Hopf bundle over factors ($\mathbb{C}P(\infty)$). According to the Splitting Principle 9.2 the induced homomorphism $\pi^*\colon h^*(BU_n) \to h^*(BT^n)$ is injective .

Recall that the Weyl group of U_n is isomorphic to the symmetric group: $NT^n/T^n \simeq \Sigma_n$, which acts on the torus T^n permuting factors. For any permutation $\sigma \in \Sigma_n$ let $c_\sigma\colon U_n \to U_n$ denote the corresponding inner automorphism, preserving the subgroup $T^n \subset U_n$. The induced map $Bc_\sigma\colon BU_n \to BU_n$ is homotopic to identity and the diagram

$$\begin{tikzcd} BT^n \arrow[r, "\pi"] \arrow[d, "Bc_\sigma"] & BU_n \arrow[d, "Bc_\sigma"] \\ BT^n \arrow[r, "\pi"] & BU_n \end{tikzcd} \tag{13}$$

is homotopy commutative. The Weyl group acts on $BT^n = (BU_1)^n$ by permuting factors thus $\operatorname{im}\pi^* \subset h^*(BT^n)^{\Sigma_n}$. The subring $h^*(BT^n)^{\Sigma_n} = h^*[[x_1, \dots x_n]]^{\Sigma_n}$ consists of the symmetric power series, thus $h^*[[x_1, \dots x_n]]^{\Sigma_n} \simeq h^*[[s_1, \dots s_n]]$ where s_i denote elementary symmetric polynomials in $x_1, \dots x_n$. From construction of generators we have that $x_i = c_1(L_i)$ and
$s_k(x_1, \dots x_n) = c_k(\pi^!(EU_n \times \mathbb{C}^n)) = \pi^! c_k(EU_n \times \mathbb{C}^n) \in \operatorname{im}\pi^*$, hence

$$h^*(BU_n) \simeq \operatorname{im}\pi^* = h^*[[c_1, \dots c_n]].$$

□

Zad. 24. Formulate and prove the corresponding result for real grassmannians and cohomology theories in which all bundles are oriented.

[1] Note that the embedding is defined on finite dimensional Stiefel manifolds: $V_1(\mathbb{C}^m) \times \dots \times V_1(\mathbb{C}^m) \subset V_n(\mathbb{C}^{nm})$ which is $(m-1)$-equivalence.

11 Comparing Chern classes and SW-classes

Let $E \to X$ be a complex vector bundle. We'll calculate the SW-classes of its realification $E_{\mathbb{R}} \to X$ in terms of the Chern classes of $E \to X$.

Theorem 11.1. *Let $E \to X$ be a n-dimensional complex vector bundle. Then for every $k \geq 0$ we have: $w_{2k+1}(E) = 0$ and $w_{2k}(E) = c_k(E) \mod 2$.*

Proof. Characteristic classes are completely described by multiplicative structure in cohomology of projectivization. Thus we have to consider map $\pi\colon \mathbb{P}(E_{\mathbb{R}}) \to \mathbb{P}(E)$ which is a S^1-sphere bundle. The map fits into a commutative diagram:

$$\begin{tikzcd} \mathbb{P}(E_{\mathbb{R}}) \arrow[rr, "\pi"] \arrow[dr] & & \mathbb{P}(E) \arrow[dl] \\ & X & \end{tikzcd} \tag{14}$$

From the Leray-Hirsch theorem 8.1 we know that $\mathbb{Z}_2$-cohomology of projectivization is a free module over $H^*(X)$ generated by the Euler class of appropriate Hopf bundle.

$$H^*(\mathbb{P}(E)) = H^*(X)\{1, x_{E^*} \dots x_{E^*}^{n-1}\} \xrightarrow{\pi^*} H^*(X)\{1, y_{E^*_{\mathbb{R}}} \dots y_{E^*_{\mathbb{R}}}^{2n-1}\} = H^*(\mathbb{P}(E_{\mathbb{R}}))$$

where $x_{E^*} = e(H_{E^*}) \in H^2(\mathbb{P}(E))$ and $y_{E^*_R} = e(H_{E^*_{\mathbb{R}}}) \in H^1(\mathbb{P}(E_{\mathbb{R}}))$. Thus it is enough to calculate

$$\pi^*(x_{E^*}) = \pi^*(e(H_{E^*})) = e(\pi^! H_{E^*}) = e(H_{E^*_{\mathbb{R}}} \oplus H_{E^*_{\mathbb{R}}}) = e(H_{E^*_{\mathbb{R}}})^2 = y^2_{E_{\mathbb{R}}}$$

(cf. Zad.10). Applying π^* to the equation which defines Chern classes we obtain

$$0 = \pi^*(x^n_{E^*} + \sum_{i=0}^{n-1} c_i(E) x_{E^*}^{n-i}) = y_{E^*}^{2n} + \sum_{i=0}^{n-1} c_i(E) y_{E^*}^{2(n-i)}.$$

Conclusion follows immediately from definition of the SW-classes. □

Zad. 25. For any real vector bundle $E \to X$, and $i \geq 0$ the equality $c_i(E^{\mathbb{C}}) \mod 2 = w_i(E)^2$.

12 The Pontryagin classes

Let $E \to X$ be a real vector bundle and $E^{\mathbb{C}} \to X$ its complexification. We can define integral charactersitic classes of $E \to X$ as the Chern classes $c_i(E^{\mathbb{C}}) \in H^{2i}(X;\mathbb{Z})$. Since $\overline{E^{\mathbb{C}}} \simeq E^{\mathbb{C}}$ by Zad. 23 we have equality of Chern classes $c_i(E^{\mathbb{C}}) = (-1)^i c_i(\overline{E^{\mathbb{C}}}) = (-1)^i c_i(E^{\mathbb{C}})$ and therefore odd Chern classes are 2-torsion. Thus the odd Chern classes are in the kernel of the natural homomorphism of cohomology $H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Q})$ induced by $\mathbb{Z} \subset \mathbb{Q}$.

Definition 12.1. *For a n-dimensional real vector bundle $E \to X$ we define its Pontryagin classes*

$$p_i(E) := (-1)^i c_{2i}(E^{\mathbb{C}}) \in H^{4i}(X;\mathbb{Q}) \quad \textit{and the total Pontryagin class} \quad p(E) := \sum_{i=0}^{[\frac{n}{2}]} p_i(E).$$

Proposition 12.1. *The assignment $Vect_{\mathbb{R}}(X) \ni E \mapsto p(E) \to H^*(X;\mathbb{Q})$ has the following properties:*

1. *It is a natural transformation i.e. for any map $f\colon Y \to X$, $p(f^! E) = f^* p(E)$,*
2. $p(E_1 \oplus E_2) = p(E_1)p(E_2)$,
3. *For any $2n$-dimensional orientable real vector bundle $p_n(E) = e(E)^2$.*

Proof. It follows immediately from properties of Chern classes thm. 9.1 and Zad.23. □

Theorem 12.1.

$$H^*(BSO_n;\mathbb{Q}) \simeq \begin{cases} \mathbb{Q}[p_1,\dots p_m] & dla \quad n = 2m+1 \\ \mathbb{Q}[p_1,\dots p_{m-1}, e] & dla \quad n = 2m \end{cases}$$

where p_i and e denote the Pontryagin classes and the Euler class of the universal bundle over BSO_n.

Proof. BĂŹdzie uzupeÅĆniony. Na razie p. Milnor-Stasheff §15. □

13 Obstruction theory

13.1 Cohomology with local coeffcients

We need to define cellular cochains with twisted coefficients.

Definition 13.1. *A local system on a space X is a covariant functor from the fundamental groupoid $\mathcal{P}(X)$ to some category (e.g. category of groups).*

Note that if the space is path-connected then isomorphism classes of local systems are in bijective correspondence with objects equipped with $\pi_1(X, x_0)$-action.

Let $p\colon E \to B$ be a fibration; for any subset $A \subset B$ we denote $E_A := p^{-1}(A)$. Every homotopy class of paths $[\omega] \in \mathcal{P}(B)$ defines a homotopy class of maps $h_{[\omega]}\colon E_{\omega(0)} \to E_{\omega(1)}$. If we assume that fibers E_b are n–simple i.e. their fundamental groups act trivially on $\pi_n(E_b, e_b)$ then the maps define homomorphisms $(h_{[\omega]})_\#\colon \pi_n(E_{\omega(0)}, e_{\omega(0)}) \to \pi_n(E_{\omega(1)}, e_{\omega(1)})$ and the assignment $B \ni b \mapsto \pi_n(E_{\omega(0)}, e_b)$ is a well-defined local system on the space B. If B is path connected we can identify the system with $\pi_1(B, b_0)$ action on $\pi_n(E_{b_0}, e_{b_0})$ by group automorphisms. If B has the universal covering $\tilde{B} \to B$ then we can define a bundle of groups: $\tilde{B} \times_{\pi_1} \pi_n(F, e) \to B$. If the bundle $E \to B$ is associated with a principal G-bundle then the local coefficients bundle can be defined as $P \times_G \pi_n(F, e) \to B$.

For a connected CW-complex X let $p\colon \tilde{X} \to X$ is the universal covering equipped with $\pi_1(X, x_0)$ action. For a local system M on X we define a chain complex $C^*(X; A) := \mathrm{Hom}_{\pi_1(X,x_0)}(S_*(\tilde{X}), A)$ where $S_*(\tilde{X})$ is the singular complex. For the constant coefficient system A we have $C^*(X; A)$ is just the ordinary singular cochain complex of X with coefficients in the group A.

For our purpose it will be convenient to identify the group of cycles with appropriate relative homotopy group. The Hurewicz isomorphism theorem (or a consequence of the addition theorem) implies the following:

Corollary 13.1. *For any simply-connected CW-complex Y the Hurewicz homomorphism $\pi_n(Y^{(n)}, Y^{(n-1)}) \xrightarrow{\chi} H_n(Y^{(n)}, Y^{(n-1)})$ is an isomorphism. For the triple $(Y^{(n+1)}, Y^{(n)}, Y^{(n-1)})$ we have the following commutative diagram:*

$$\begin{tikzcd}
\pi_{n+1}(Y^{(n+1)}, Y^{(n)}) \arrow[r, "\chi", "\simeq"'] \arrow[d, "\partial"] & H_{n+1}(Y^{(n+1)}, Y^{(n)}) \arrow[d, "\partial"] \\
\pi_n(Y^{(n)}) \arrow[r, "\chi"] \arrow[d] & H_n(Y^{(n)}) \arrow[d] \\
\pi_n(Y^{(n)}, Y^{(n-1)}) \arrow[r, "\chi", "\simeq"'] & H_n(Y^{(n)}, Y^{(n-1)})
\end{tikzcd} \tag{15}$$

Proposition 13.1. *Let $p_u\colon \tilde{X} \to X$ be the universal covering of a connected CW-complex X. Then CW-decomposition of X can be lifted to CW-decomposition on $\tilde{X}$ in such way that p_u maps cells of $\tilde{X}$ to cells of X, $\tilde{X}^{(k)} = p_u^{-1}(X^{(k)})$, and the fundamental group $\pi_i(X, x_0)$-action on $\tilde{X}$ permutes the cells.*

Proof. Assume that X is a connected CW-complex. Let $p_u\colon \tilde{X} \to X$ be the universal covering of X. Let $\alpha\colon (D^k, S^{k-1}, 1) \to (X^{(k)}, X^{(k-1)}, x_0)$ represents an element of $\pi_n(X^{(k)}, X^{(k-1)})$. It can be lifted to maps $(D^k, S^{k-1}, 1) \xrightarrow{\tilde{\alpha}_g} (p^{-1}(X^{(k)}), p^{-1}(X^{(k-1)}), \tilde{x}_g)$ indexed by elements $g \in \pi_1(X, x_0)$ which define CW-decomposition of $\tilde{X}$. The group $\pi_1(X, x_0)$ obviously acts freely on cells of $\tilde{X}$. □

Thus in definition of cohomology with local coefficients we can substitute homology with homotopy groups: $C^k(X; A) := \mathrm{Hom}_{\pi_1(X,x_0)}(\pi_k(\tilde{X}^{(k)}, \tilde{X}^{(k-1)}), A)$.

13.2 Obstruction to extending a section and the difference cochain

Theorem 13.1. *Let $p\colon E \to X$ be a fibration with simple fibers. Suppose there is a section on the k-skeleton $s_k\colon X^{(k)} \to E$. Then the obstruction to the existence of a section to the $(k+1)$-skeleton, $s_{(k+1)}\colon X^{(k+1)} \to E$ that extends s_k, is a cellular cochain $c(s_k) \in C^{k+1}(X; \{\pi_k(E_x, e_x)\})$. That is, $c(s_k) = 0$ if and only if such a lifting s_{k+1} exists. Construction of $c(-)$ is natural with respect to pull-back via a cellular map i.e. for the pull-back diagram*

$$\begin{tikzcd}
 & f^!E \arrow[r, "\tilde{f}"] \arrow[d, "p_f"] & E \arrow[d, "p"] & \\
Y^{(k)} \arrow[ur, "s'_k"] \arrow[r, "\subset"] & Y \arrow[r, "f"] & X & X^{(k)} \arrow[l, "\supset"'] \arrow[ul, "s_k"']
\end{tikzcd} \tag{16}$$

where s'_k is a section corresponding to s_k: $c(s'_k) = f^*(c(s_k))$. Moreover, $c(s_k)$ is a cocycle.

Proof. $p_u\colon \tilde{X} \to X$. In order to define a cochain we assign to each homotopy class of maps

$\alpha\colon (D^{k+1}, S^k, 1) \to (\tilde{X}^{(k+1)}, \tilde{X}^{(k)}, x_0)$ an element defined in the following way:

$$\begin{tikzcd}
 & (\alpha^! \tilde{E})_1 \arrow[r, "="] \arrow[d, "i"] & \tilde{E}_{\tilde{x}} \arrow[r, "="] \arrow[d, "i"] & E_{x_0} \arrow[d, "i"] & \\
 & \alpha^! \tilde{E}_{k+1} \arrow[r, "\tilde{\alpha}"] \arrow[d] & \tilde{E}_{k+1} \arrow[r, "\tilde{p}_u"] \arrow[d] & E_{k+1} \arrow[d] & \\
(S^k, 1, 1) \arrow[r] \arrow[ur, "s_\alpha"] & (D^{k+1}, S^k, 1) \arrow[r, "\alpha"] & (\tilde{X}^{(k+1)}, \tilde{X}^{(k)}, \tilde{x}) \arrow[r, "p_u"] & (X^{(k+1)}, X^{(k)}, x_0) & (X^{(k)}, X^{(k)}, x_0) \arrow[l, "\supset"'] \arrow[ul, "s_k"']
\end{tikzcd} \tag{17}$$

where $E_k := p^{-1}(X^{(k)})$. The top vertical arrows are inclusions of fibers. Note that inclusion $i\colon (\alpha^! E)_1 \subset \alpha^! E$ induces isomorphism on homotopy groups, since D^{k+1} is contractible. To the element $[\alpha] \in \pi_{k+1}(\tilde{X}^{(k+1)}, \tilde{X}^{(k)})$ we associate element of $\pi_k(E_{x_0}, e_0)$ corresponding to the map $S^k \xrightarrow{s_\alpha} (\alpha^! \tilde{E})_1$. Thus $c(s_k)(\alpha) := i_\#^{-1}[s_\alpha]$. From definition it follows immediately that if $c(s_k) = 0$ then s_k can be extended to the skeleton X^{k+1}. From definition of $c(-)$ follows easily that it is a group homomorphism and is natural with respect to pull-back.

Observe also that it is a cocycle i.e. $\delta(c(s_k))(\beta) = c(s_k)(\partial\beta) = 0$ since for any
$\beta\colon (D^{k+2}, S^{k+1}, 1) \to (X^{(k+2)}, X^{(k+1)}, x_0)$ the map $\partial\beta | S^k \sim *$ is null-homotopic. $\square$

For two sections defined $s_k, t_k\colon X^{(k)} \to E$ defined on the k-skeleton, such that $s_k|X^{(k-1)} \sim t_k|X^{(k-1)}$ we define a *difference cochain* $d_k(s_k, t_k) \in C^k(X; \{\pi_k(E_b, e_b)\})$ which is an obstruction to extension of the homotopy on $X^{(k)}$. Construction is analogous to the construction of the obstruction $c(s_k)$.

Theorem 13.2. *Let $p\colon E \to X$ be a fibration. Suppose there are two sections on the k-skeleton $s_k, t_k\colon X^{(k)} \to E$ and a fiberwise homotopy $s_k|X^{(k-1)} \sim t_k|X^{(k-1)}$. Then the obstruction to the existence of its extension a fiberwise homotopy on $X^{(k)}$ is a cellular cochain $d(s_k, t_k) \in C^k(X; \{\pi_k(E_b), e_b)\})$, called* the difference cochain. *That is, $d(s_k, t_k) = 0$ if and only if such homotopy exists. Construction of the difference cocycle is natural with respect to pull-back via a cellular map i.e. for the pull-back diagram Moreover, $\delta d(s_k, t_k) = c(s_k) - c(t_k)$.*

Proof. We define the cochain $d(s_k, t_k)$ by the following diagram:

$$\begin{tikzcd}
 & (\alpha^! \tilde{E})_1 \arrow[r, "="] \arrow[d, "i"] & \tilde{E}_{\tilde{x}} \arrow[r, "="] \arrow[d, "i"] & E_{x_0} \arrow[d, "i"] & \\
 & \alpha^! \tilde{E}_k \arrow[r, "\tilde{\alpha}"] \arrow[d] & \tilde{E}_k \arrow[r, "\tilde{p}_u"] \arrow[d] & E_k \arrow[d] & \\
D_S \arrow[r] \arrow[ur, "s_\alpha \cup H_\alpha \cup t_\alpha"] & (D^k, S^{k-1}) \times I \arrow[r, "\alpha"] & (\tilde{X}^{(k)}, \tilde{X}^{(k-1)}) \arrow[r, "p_u"] & (X^{(k)}, X^{(k-1)}) & (X^{(k)}, X^{(k-1)}) \arrow[l, "\supset"'] \arrow[ul, "{s_k, t_k}"']
\end{tikzcd} \tag{18}$$

where $D_S := D^k \times \{0, 1\} \cup S^{k-1} \times I$. Thus $d(s_k, t_k)([\alpha]) := [i_\#^{-1}(s_\alpha \cup H_\alpha \cup t_\alpha)]$. $\square$

13.3 Characteristic classes as obstructions

Lemma 13.1. *Let $E \to X$ be a fibration over a CW-complex such that homotopy groups of its fiber $\pi_i(F, e_0) = 0$ for $i < k$. Then there is a section $s_k \colon X^{(k)} \to p^{-1}(X^{(k)})$ and any two such sections are fibrewise homotopic on $X^{(k-1)}$.*

Proof. Induction on skeleta.(ZAD) $\square$

Proposition 13.2.

$$\pi_i(V_k(\mathbb{R}^n)) = \begin{cases} 0 & \text{if} \quad i \leq n-k-1 \\ \mathbb{Z} & \text{if} \quad i = n-k \quad \text{is even,} \quad \text{or} \quad k = 1 \\ \mathbb{Z}_2 & \text{if} \quad i = n-k \quad \text{is odd,} \quad \text{and} \quad k > 1 \end{cases}$$

Homotopy class of inclusion $S^{n-k} = V_1(\mathbb{R}^{n-k+1}) \to V_2(\mathbb{R}^{n-k+1}) \subset \ldots \subset V_k(\mathbb{R}^n)$ is a generator of the $(n-k)$-th group.

Proof. Apply homotopy exact sequence of a fibration (ZAD) $\square$

Let $E \to X$ be a n-dimensional real vector bundle with a metric. For every $k \leq n$ we define a bundle $p_k \colon V_k(E) \to X$ as an associated bundle whose fiber is the Stiefel manifold $V_k(\mathbb{R}^n)$. From Lemma 13.1 and Prop.13.2 it follows that that the bundle p_k has a section s_{n-k} on the $(n-k)$-skeleton and any two such sections are fiberwise homotopic on the $(n-k-1)$-skeleton. Thus Thm. 13.1 implies that the obstruction $[c(s_{n-k})] \in H^{n-k+1}(X; \{\pi_{n-k}(V_k(E_x))\})$ is well defined and it doesn't depend on choice of section s_{n-k}. Seting $j := n-k+1$, following Milnor we introduce notation:

$$\mathfrak{o}_j(E) := [c(s_{n-k})] \in H^j(X; \{\pi_{j-1}(V_{n-j+1}(E_b))\}$$

According to Prop. 13.2 $\pi_{j-1}(V_{n-j+1}(E_x))$ is isomorphic either to $\mathbb{Z}$ or $\mathbb{Z}_2$. In any case, there is a unique nontrivial homomorphism of local systems $\pi_{j-1}(V_{n-j+1}(E_b)) \to \mathbb{Z}_2$ which induces reduction mod 2 homomorphism of cohomology groups:

$$H^j(X; \{\pi_{j-1}(V_{n-j+1}(E_b))\}) \longrightarrow H^j(X; \mathbb{Z}_2)$$

Theorem 13.3. *For every real vector bundle $E \to X$, $w_j(E) = \mathfrak{o}_j(E) \mod 2$.*

We can state the theorem in the following form:

> *The SW class $w_j(E)$ is a mod 2-reduction of the obstruction to extension of $(n-j+1)$-sections defined on $(j-1)$-skeleton which are linearly independent in each point, to such sections on the j-skeleton.*

Proof. (after Milnor) We consider the universal bundle $E(\gamma^n) \to G_n(\mathbb{R}^\infty) = BO_n$. Recall that according to Thm. 10.1 there is an algebra isomorphism $H^*(G_n; \mathbb{Z}_2) \simeq \mathbb{Z}_2[w_1, \ldots w_n]$ where $w_i = w_i(\gamma^n)$ are SW-classes of the universal bundle. Thus there is a polynomial f_j of $\deg f_j \leq j$ such that $\mathfrak{o}_j(\gamma^n)_2 = f_j(w_1, \ldots w_n) = f'_j(w_1, \cdots w_{j-1}) + \lambda_{n,j} w_j$. Since both SW-classes and obstructions are natural with respect to pull-backs, for any n-dimensional vector bundle:

$$\mathfrak{o}_j(E)_2 = f_j(w_1(E), \ldots w_n(E)) = f'_j(w_1(E), \cdots w_{j-1}(E)) + \lambda_{n,j} w_j(E)$$

Consider the canonical inclusion $\iota\colon O_{j-1} \subset O_n$, the induced map $B\iota\colon BO_{j-1} \to BO_n$ and the pull-back bundle $(B\iota)^!\gamma_n = \gamma^{j-1} \oplus \theta^{n-j+1} =: \eta$. The bundle admits $n-j+1$ linearly independent sections, thus the bundle $V_{n-j+1}(\eta)$ admits a section hence

$$0 = \mathfrak{o}_j(\eta) = f_j'(w_1, \ldots w_{j-1}) + \lambda_j w_j(\eta) = f_j'(w_1, \ldots w_{j-1}).$$

It forces $f_j' = 0$, because the SW-classes of the universal bundle are algebraically independent. Thus for every n-dimensional vector bundle $\mathfrak{o}_j(E)_2 = \lambda_{n,j} w_j(E)$ where $\lambda_{n,j} = 0, 1$. In order to prove that $\lambda_{n,j} = 1$ it is enough to find a single n-dimensional bundle for which it is non-zero.

First consider the case $j = n$ and pick up a bundle $E_n \to \mathbb{R}P(n)$ which is a complement of the Hopf bundle i.e. $H_{n+1} \oplus E_n \simeq \theta^{n+1}$. The space $E_n = \{(L, \mathbf{v}) \,|\, L \in \mathbb{R}P(n),\, \mathbf{v} \in L^\perp\}$. Let L_0 be a fixed line generated by a unit vector $\mathbf{u}_0 \in L_0$. For any unit vector $\mathbf{u} \in \mathbb{R}^{n+1}$ we define a section of $s\colon \mathbb{R}P(n) \to E_n$ by the formula: $s(Lin\{\mathbf{u}\}) = \mathbf{u}_0 - \langle \mathbf{u}_0, \mathbf{u}\rangle \mathbf{u}$. Clearly $s(L) \neq 0$ for all $L \neq L_0$. Expressing s in local trivialization around L_0 we easily check that the relevant cocycle assigns to the top cell containing L_0 generator of the group $\pi_{n-1}(V_1(F)) = \pi_{n-1}(F \setminus 0) \simeq \mathbb{Z}$.

For $j < n$ one uses the n-dimensional vector bundle $E_j \oplus \theta^{n-j}$ over $\mathbb{R}P(j)$ and description of the generator of the group $\pi_{j-1}(V_{n-j+1}(\mathbb{R}^n))$ given in Prop. 13.2. $\square$

Zad. 26. Complete details of the last assertions.

Zad. 27. Prove that for an oriented n-dimensional real vector bundle $E \to X$ the top obstruction is equal to the Euler class: $\mathfrak{o}_n(E) = e(E)$.

Zad. 28. Describe Chern classes as appropriate obstructions.

14 Stiefel-Whitney classes *via* Steenrod squares

Our approach follows Mike Hopkins [4]

14.1 Equivariant cohomology a la Borel

Construction of the Steenrod squares becomes more transparent if one looks at it from a general viewpoint involving symmetries i.e. group actions on spaces. In the present section we introduce an important invariant of G-spaces – equivariant cohomology theory, developed by A. Borel[2]. Let G be a compact Lie group and $EG \to BG$ a universal principal G-bundle. Recall that universal G-bundle is unique up to G-homotopy i.e. if $E'G$ and $E''G$ are two universal principal G-spaces then there exists G-homotopy equivalence $E'G \to E''G$, unique up to G-homotopy. For any space X we consider the associated bundle $EG \times_G X \to BG$.

The total space $EG \times_G X$ is called the *Borel construction* on G-space X or the homotopy orbit space of X. Note that projection $EG \times X \to X$ induces a map $EG \times_G X \to X/G$ which in general is not a homotopy equivalence.

For arbitrary coefficient group (or ring) we define a functor $H^*_G\colon \mathcal{T}op_{G,2} \to \mathcal{A}b^*$ from the category of pairs G-spaces to the category of graded abelian groups (or rings, if we consider coefficients in a ring). Let X be a G-space and $f\colon (X, A) \to (Y, B)$ a G-map:

$$H^*_G(X, A) := H^*(EG \times_G (X, A)) = H^*(EG \times_G X, EG \times_G A)$$

$$f^* = (id \times_G f)\colon H^*(EG \times_G (Y, B)) \to H^*(EG \times_G (X, A)).$$

[2]Armand Borel La Chaux-de-Fonds, CH 1923 – 2003 Princeton, NJ, USA

Cross-product, and hence cup-product extends to equivariant cohomology with coefficients in a ring. For any two pairs of G-spaces $(X,A),(Y,B)$ we have multiplication, defined by composition of arrows in the following diagram:

$$\begin{tikzcd} H^*_G(X,A)\otimes H^*_G(Y,B) \arrow[r, dotted, "\times"] \arrow[d, "\times"] & H^*_G((X,A)\times(Y,B)) \\ H^*((EG\times_G(X,A))\times(EG\times_G(Y,B))) \arrow[r, "\simeq"] & H^*((EG\times EG)\times_G(X,A)\times(Y,B)) \arrow[u, "\Delta^*"'] \end{tikzcd} \tag{19}$$

We'll state several properties of equivariant cohomology:

Proposition 14.1.

1. *If (X,A) is a pair with trivial G-action, for cohomology with coefficients in a field there is an isomorphism $H^*_G(X,A)\simeq H^*(BG)\otimes H^*(X,A)$,*

2. *If (X,A) is a pair of free G-spaces (no base-point!), then projection $EG\times_G(X,A)\to(X/G,A/G)$ induces an isomorphism $p^*\colon H^*(X/G,A/G)\to H^*_G(X,A)$,*

3. *If (X,A) is a pair of G-spaces then embedding of fiber $j\colon(X,A)\to EG\times_G(X,A)$, $j(x):=[e_0,x]$ induces a homomorphism $j^*\colon H^*_G(X,A)\to H^*(X,A)^G$.*

4. *Consider a pair (X,A) such that $A\subset X^G$. Then inclusion $A\subset X$ induces a map $BG\times A=EG\times_G A\to EG\times_G X$ thus $H^*_G(X,A)=H^*(EG\times_G X, BG\times A)$.*

If we consider G-pairs (X,A) such that inclusion $A\to X$ is a cofibration we can recover values of equivariant cohomology on arbitrary pair from its values on pointed G-spaces (X,x_0). Note that the point x_0 defines a section $s_0\colon BG\to EG\times_G X$ and

$$H^*_G(X,x_0):=H^*(EG\times_G(X,x_0)):=H^*(EG\times_G X, EG\times_G x_0)=H^*(EG\times_G X, BG\times\{x_0\})$$

14.2 Construction of the Steenrod squares

Coefficients of all cohomology groups are assumed to be $\mathbb{Z}_2$.

Theorem 14.1. *There exist (additive) natural transformations*

$$Sq^i\colon H^n(X,A)\to H^{n+i}(X,A),\quad Sq:=\sum Sq^i$$

such that:

1. *Naturality. If $f\colon(X,A)\to(Y,B)$ then $Sq\circ f^*=f^*\circ Sq$.*

2. *If $a\in H^n(X,A)$, then $Sq^0(a)=a$, $Sq^n(a)=a\cup a$, and $Sq^i(a)=0$ for $i>n$.*

3. *The Cartan formula. $Sq(a\times b)=Sq(a)\times Sq(b)$.*

The rest of the present section will be devoted to construction of the Steenrod squares. First we collect some facts which will be used for it. Crucial role is played by the Eilenberg-MacLane spaces (EM-spaces) $K(M,n)$ defined for every abelian group and every $n\geq 0$, uniquely up to weak homotopy type.

Proposition 14.2.
$$H^n(K(M,m),N) = \begin{cases} 0 \quad \textit{if} \quad n < m \\ \mathrm{Hom}(M,N) \quad \textit{if} \quad n = m \end{cases}$$

Proof. $H^n(K(M,n);N) \simeq \mathrm{Hom}(H_n(K(M,n);\mathbb{Z}),N) \simeq \mathrm{Hom}(\pi_n(K(M,n)),N) = \mathrm{Hom}(M,N)$. $\square$

The cohomology functor $H^n(X,A;M)$ is represented on the homotopy category of CW-complexes by the Eilenberg-Maclane space $K(M,n)$:

$$H^n(X,A;M) \simeq [X/A, K(M,n)]_*$$

The natural isomorphism $[X,K(M,n)]_* \xrightarrow{\simeq} \tilde{H}^n(X;M)$ is defined by the *fundamental class* $\iota_{M,n} \in H^n(K(M,n);M)) = \mathrm{Hom}(M,M)$ corresponding to $id_M \in \mathrm{Hom}(M,M)$. Recall the Yoneda lemma in the form we'll need it later. We will consider contravariant functors.

Lemma 14.1. *Let $\mathcal{C}$ be a category which has an object $* \in \mathrm{ob}\,\mathcal{C}$ which is both initial and final. Then any representable functor $R^C\colon \mathcal{C} \to \mathcal{S}et$ has unique lifting to the category of pointed sets. If $F\colon \mathcal{C} \to \mathcal{S}et_*$ is a functor to the category of pointed sets, then there is a natural bijection of pointed sets: $\mathrm{Hom}(R^C,F) \simeq F(C)$ where $\mathrm{Hom}(R^C,F)$ is set of natural transformations of functors into $\mathcal{S}et_*$.*

Proof. The base-point in $R^C(X) = \mathrm{Mor}_{\mathcal{C}}(X,C)$ is defined by the unique morphism $X \to *$: $R^C(*) = \mathrm{Mor}_{\mathcal{C}}(*,C) \to \mathrm{Mor}_{\mathcal{C}}(X,C) = R^C(X)$. Clearly for any $\varphi\colon X \to Y$ the induced map preserves the base-point. It is equally easy to see that the Yoneda equivalence preserves the base-point. $\square$

The Steenrod squares can be considered as a kind of "filtration" of the squaring operation:

$$\tilde{H}^n(X/A) = H^n(X,A) \ni x \mapsto x \times x \in H^{2n}(X \times X, X \times A \cup A \times X) = \tilde{H}^{2n}(X/A \wedge X/A).$$

Note that because we are using $\mathbb{Z}_2$ coefficient, the squaring map is an additive homomorphism. The "filtration" comes from action of the cyclic group $\mathbb{Z}_2$ on $X \times X$ by permuting factors. Instead of general pairs we will consider pointed spaces and equivariant cohomology of pointed $\mathbb{Z}_2$-spaces.

For an arbitrary pointed space (X,x_0) consider its smash-square $(X \wedge X, \hat{x}_0)$ - to shorten notation we'll denote its base-point $[x_0,x_0] =: \hat{x}_0$. The cyclic group $\mathbb{Z}_2$ acts on $(X \wedge X, \hat{x}_0)$ by permutation of coordinates. For a universal principal $\mathbb{Z}_2$-bundle $E\mathbb{Z}_2 \to B\mathbb{Z}_2$ consider an associated fiber bundle pair $E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (X \wedge X, x_0) \to B\mathbb{Z}_2$.The diagonal map $\delta\colon (X,x_0) \subset (X \wedge X, \hat{x}_0)$ is precisely inclusion of the $\mathbb{Z}_2$-fixed-point set and it induces an inclusion of the trivial bundle:

$$\begin{tikzcd}
(X,x_0) \arrow[rr, "\Delta"] \arrow[d] & & (X \wedge X, \hat{x}_0) \arrow[d] \\
B\mathbb{Z}_2 \times (X,x_0) \arrow[rr, "\Delta_{\mathbb{Z}_2}"] \arrow[dr] & & E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (X \wedge X, \hat{x}_0) \arrow[dl] \\
& B\mathbb{Z}_2 &
\end{tikzcd} \tag{20}$$

The top square defines the following diagram of cohomology groups in which we'll be looking for the natural lifting of the squaring map (dotted arrow), and show that it is unique:

$$\begin{tikzcd} & H^{2n}_{\mathbb{Z}_2}(X\wedge X,\hat{x}_0) \arrow[r, "\Delta^*_{\mathbb{Z}_2}"] \arrow[d, "j^*"] & H^{2n}_{\mathbb{Z}_2}(X,x_0) \arrow[d, "j^*"] \\ H^n(X,x_0) \arrow[r, "x\times x"] \arrow[ur, dotted, "P^e_X"] & H^{2n}(X\wedge X,\hat{x}_0) \arrow[r, "\Delta^*"] & H^{2n}(X,x_0) \end{tikzcd} \tag{21}$$

Since cohomology functor H^n on the category of pairs (or pointed spaces) is represented by the pointed EM-space, according to Lemma 14.1 it is enough to consider $(X, x_0) = (K(\mathbb{Z}_2, n), x_0) =: (K_n, x_0)$ and find an element $P^e_{K_n}(\iota_n) \in H^{2n}_{\mathbb{Z}_2}(K_n \wedge K_n, \hat{x}_0)$ such that

$$P^e_{K_n}(\iota_n)|K_n \wedge K_n = \iota_n \times \iota_n = \iota_n \otimes \iota_n \in H^n(K_n) \otimes H^n(K_n) \simeq \tilde{H}^{2n}(K_n \wedge K_n) \simeq \mathbb{Z}_2.$$

We'll prove that restriction to fiber $H^{2n}_{\mathbb{Z}_2}(K_n \wedge K_n) \xrightarrow{j^*} H^{2n}(K_n \wedge K_n)$ is in fact an isomorphism, thus the lifting is unique.

Lemma 14.2. *Suppose (Y, y_0) is a $(m-1)$-connected pointed space equipped with $\mathbb{Z}_2$-action. Then the inclusion of fiber $j\colon (Y, y_0) \subset E\mathbb{Z}_2 \times_{Z_2} (Y, y_0)$ induces, for $i \le m$ an isomorphism*

$$j^*\colon H^i_{\mathbb{Z}_2}(Y, y_0) \to H^i(Y, y_0)^{\mathbb{Z}_2}.$$

Proof. We choose S^∞ with antipodal $\mathbb{Z}_2$-action as a model of $E\mathbb{Z}_2$. Note that S^∞ has a filtration by finite dimensional spheres $S^0 \subset S^1 \subset \cdots \subset S^\infty$, which are $\mathbb{Z}_2$-invariant. First we'll prove that for $k \ge 1$ and $i \le n$ the inclusion $S^k \subset S^{k+1}$ induces an isomorphism

$$H^i(S^{k+1} \times_{\mathbb{Z}_2} (Y, y_0)) \to H^i(S^k \times_{\mathbb{Z}_2} (Y, y_0)).$$

For $k \ge 0$ there is a push-out diagram of $\mathbb{Z}_2$-spaces where $\mathbb{Z}_2$ acts as antipodal map on the spheres, and interchanges copies of disks (left diagram). The horizontal maps are inclusions. To make the vertical maps $\mathbb{Z}_2$-equivariant we define $q(1, v) = v$, $q(-1, v) = -v$ which remains push-out after applying $- \times_{\mathbb{Z}_2} (Y, y_0)$ functor to it (right diagram). Hence we have a map of cofibred sequences

$$\begin{tikzcd} \mathbb{Z}_2 \times S^k \arrow[r, "\subset"] \arrow[d, "q"] & \mathbb{Z}_2 \times S^{k+1}_+ \arrow[d, "q"] \\ S^k \arrow[r, "\subset"] & S^{k+1} \end{tikzcd} \tag{22}$$

$$\begin{tikzcd} S^k \times (Y, y_0) \arrow[r, "\subset"] \arrow[d] & S^{k+1}_+ \times (Y, y_0) \arrow[d] \\ S^k \times_{\mathbb{Z}_2} (Y, y_0) \arrow[r, "\subset"] & S^{k+1} \times_{\mathbb{Z}_2} (Y, y_0) \end{tikzcd} \tag{23}$$

and resulting homeomorphism of cofibers $(S^{k+1}, S^k) \times_{\mathbb{Z}_2} (Y, y_0) \simeq (S^{k+1}_+, S^k) \times (Y, y_0)$ and projection $(S^{k+1}_+, S^k) \times (Y, y_0) \to \Sigma^{k+1}(Y, y_0)$ which induces isomorphism in cohomology. The last space is $(m+k)$-connected. Now consider the exact sequence of the pair (we introduce shorter notation)

$$(S^{k+1} \times_{\mathbb{Z}_2} Y/S^{k+1} \times_{\mathbb{Z}_2} \{y_0\}, S^k \times_{\mathbb{Z}_2} Y/S^k \times_{\mathbb{Z}_2} \{y_0\}) =: (S^{k+1} \wedge_{\mathbb{Z}_2} Y, S^k \wedge_{\mathbb{Z}_2} Y).$$

$$... \to H^i(S^{k+1}\wedge_{\mathbb{Z}_2}Y, S^k\wedge_{\mathbb{Z}_2}Y) \to H^i(S^{k+1}\wedge_{\mathbb{Z}_2}Y) \to H^i(S^k\wedge_{\mathbb{Z}_2}Y) \xrightarrow{\delta} H^{i+1}(S^{k+1}\wedge_{\mathbb{Z}_2}Y, S^k\wedge_{\mathbb{Z}_2}Y) \to ...$$

The relative groups can be easily calculated in low dimensions:

$$H^i(S^{k+1} \wedge_{\mathbb{Z}_2} Y, S^k \wedge_{\mathbb{Z}_2} Y) \simeq H^i(\Sigma^{k+1}(Y, y_0)) = 0 \quad \text{for} \quad i \leq m+k.$$

hence $H^i(S^{k+1} \wedge_{\mathbb{Z}_2} Y) \to H^i(S^k \wedge_{\mathbb{Z}_2} Y)$ is an isomorphism for $i < m+k$ and consequently $H^i(S^{k+1} \wedge_{\mathbb{Z}_2} Y) \to H^i(S^1 \wedge_{\mathbb{Z}_2} Y)$ is an isomorphism for $k \geq 1$ and $i \leq m$. Note that we can also take $k = \infty$, since for every fixed i, $H^i(S^\infty \wedge_{\mathbb{Z}_2} Y) \to H^i(S^{i+1} \wedge_{\mathbb{Z}_2} Y)$.

We'll analize the restriction homomorphism:

$$H^i(S^1 \times_{\mathbb{Z}_2} (Y, y_0)) \to H^i(Y, y_0) \quad \text{for} \quad i \leq m.$$

The space $S^1 \times_{\mathbb{Z}_2} (Y, y_0)$ is a homotopy push-out of the following diagram:

$$\begin{tikzcd}
(Y, y_0) \vee (Y, y_0) \arrow[r, "id \vee id"] \arrow[d, "id \vee \tau"] & (Y, y_0) \arrow[d, "\subset"] \\
(Y, y_0) \arrow[r, "\subset"] & S^1 \times_{\mathbb{Z}_2} (Y, y_0)
\end{tikzcd} \tag{24}$$

where $\tau\colon (Y, y_0) \to (Y, y_0)$ denotes action of the $\mathbb{Z}_2$-generator. The diagram 24 leads to a long Mayer-Vietoris exact sequence (we write $\tilde{H}^i(Y) := H^i(Y, y_0)$ for short) :

$$\tilde{H}^{i-1}(Y) \oplus \tilde{H}^{i-1}(Y) \xrightarrow{\delta} H^i(S^1 \times_{\mathbb{Z}_2} (Y, y_0)) \xrightarrow{(i^*, i^*)} \tilde{H}^i(Y) \oplus \tilde{H}^i(Y) \xrightarrow{(id,id)\oplus(id,\tau^*)} \tilde{H}^i(Y) \oplus \tilde{H}^i(Y)$$

It follows easily that for $i \leq m$ the restriction $H^i(S^1 \times_{\mathbb{Z}_2} (Y, y_0)) \xrightarrow{i^*} H^i(Y, y_0)^{\mathbb{Z}_2}$ is an isomorphism. □

Proposition 14.3. *There exists unique natural transformation P^e for which the diagram:*

$$\begin{tikzcd}
 & H^{2n}_{\mathbb{Z}_2}(X \wedge X, \hat{x}_0) \arrow[r, "\Delta^*_{\mathbb{Z}_2}"] \arrow[d, "j^*"] & H^{2n}_{\mathbb{Z}_2}(X, x_0) \arrow[d, "j^*"] \\
H^n(X, x_0) \arrow[ur, dotted, "P^e_X"] \arrow[r, "x \times x"] & H^{2n}(X \wedge X, \hat{x}_0) \arrow[r, "\Delta^*"] & H^{2n}(X, x_0)
\end{tikzcd} \tag{25}$$

commutes.

Proof. The cohomology functors H^i on pointed spaces are represented by the pointed EM-spaces $(K_i, x_0) := (K(\mathbb{Z}_2, i), x_0)$ thus according to the Yoneda-type lemma 14.1 it is enough to find a (unique) element

$$\iota_n \tilde{\times} \iota_n \in H^{2n}_{\mathbb{Z}_2}(K_n \wedge K_n, \hat{x}_0) \quad \text{such that} \quad i^*(\iota_n \tilde{\times} \iota_n) = \iota_n \times \iota_n.$$

The Hurewicz theorem implies that the space $K_n \wedge K_n$ is $(2n-1)$-connected Lemma 14.2 implies that restriction to fiber

$$H^{2n}_{\mathbb{Z}_2}(K_n \wedge K_n, \hat{x}_0) \to H^{2n}(K_n \wedge K_n, \hat{x}_0) \simeq \tilde{H}^n(K) \otimes \tilde{H}^n(K) \simeq \mathbb{Z}_2$$

is an isomorphism. The element $x \times x \in H^{2n}(K_n \wedge K_n, \hat{x}_0)$ is a generator $x \times x = \operatorname{im} i^*$. □

Remark 1. Note that P^e is natural transformation of functors valued in the category of pointed sets, not groups i.e. *a priori* is not a homomorphism.

We're ready to define the Steenrod squaring operations. Consider composition of maps in diagram 21 with the Künneth isomorphism and denote it P, and it i-th component by P_i.

$$\begin{tikzcd} H^n(X,x_0) \arrow[r, dotted, "P"] \arrow[d, "P^e"] & (H^*(X,x_0)[t])^{2n} \arrow[r, "\simeq"] & \bigoplus_{i=0}^{2n} H^{2n-i}(X,x_0)t^i \\ H^{2n}_{\mathbb{Z}_2}(X \wedge X, \hat{x}_0) \arrow[r, "\Delta^*_{\mathbb{Z}_2}"] & H^{2n}(B\mathbb{Z}_2 \times (X,x_0)) \arrow[u, "\simeq"'] & \end{tikzcd} \tag{26}$$

where $\deg t = 1$. Thus $P(x) = \sum_{i=0}^{2n} P_i(x)t^i$ where $P_i(x) \in H^{2n-i}(X,x_0)$. Note that in the diagram 26 two arrows $\Delta^*_{\mathbb{Z}_2}$ and Künneth isomorphism $\simeq$ are additive and multiplicative. The lifting P^e is only a transformation of pointed set valued functors. In next section we'll prove that in fact it preserves both addition and cross product.

Definition 14.1. *We define operations $Sq^i \colon H^n(X,x_0) \to H^{n+i}(X,x_0)$ by the formula*

$$Sq^i(x) := P_{n-i}(x), \quad Sq_t(x) := \sum_{i=0}^{2n} Sq^{n-i}(x)t^i$$

We'll prove that $Sq^i(x) = 0$ for $i < 0$ thus $Sq_t(x) := \sum_{i=0}^{n} Sq^{n-i}(x)t^i$.

14.3 Checking axioms

We'll prove that natural transformations of the $\mathcal{S}et_*$-valued functors $Sq^i \colon \tilde{H}^n(X) \to \tilde{H}^{n+i}(X)$, defined in Def. 14.1 satisfy the following properties (in different order then in Thm. 14.1) :

1. If $a \in H^n(X,A)$, then $Sq^i(a) = 0$ for $i > n$, $Sq^n(a) = a \cup a$,
2. $Sq^k(a \times b) = \sum_{i+j=k} Sq^i(a) \times Sq^j(b)$,
3. $Sq^0(a) = a$,
4. They commute with suspension isomorphism, thus are additive.

Proposition 14.4. *If $a \in \tilde{H}^n(X)$, then $Sq^i(a) = 0$ for $i < 0$ and $Sq^n(a) = a \cup a$*

Proof. If $i < 0$ then $Sq^i \colon \tilde{H}^n(X) \to \tilde{H}^{n-|i|}(X)$. The Yoneda lemma together with Prop. 14.2 imply that it must be zero. The second equality follows immediately from commutativity of the right square in diagram 21 and definition of the cup product. . □

Proposition 14.5 (The Cartan formula)**.** $Sq^k(a \times b) = \sum_{i+j=k} Sq^i(a) \times Sq^j(b)$

Proof. The Cartan formula is equivalent to the claim that transformation

$$P^e_X\colon H^n(X,x_0) \to H^{2n}(E\mathbb{Z}_2 \wedge_{\mathbb{Z}_2} X \wedge X)$$

is multiplicative (cf. remark before Def. 14.1). i.e. that the following diagram of natural transformations commutes:

$$\begin{tikzcd}
H^n(X,x_0) \times H^m(Y,y_0) \arrow[r, "\times"] \arrow[d, "P^e_X \times P^e_Y"] & H^{n+m}(X \wedge Y, [x_0,y_0]) \arrow[d, "P^e_{X\wedge Y}"] \\
H^{2n}_{\mathbb{Z}_2}(X \wedge X, [x_0,x_0]) \times H^{2m}_{\mathbb{Z}_2}(Y \wedge Y, [y_0,y_0]) \arrow[r, "\times"] & H^{2(n+m)}_{\mathbb{Z}_2}((X\wedge Y)^2, [x_0,y_0]^2)
\end{tikzcd} \tag{27}$$

Again, we use the Yoneda lemma: it is enough to prove that the diagram commutes for the Eilenberg-MacLane spaces $X = K_n$, $Y = K_m$ and element $(\iota_n, \iota_m) \in H^n(K_n) \times H^m(K_m)$. We need to calculate $H^{2(n+m)}(E\mathbb{Z}_2 \wedge_{\mathbb{Z}_2} (K_n \wedge K_m)^2)$. Note that the space $K_n \wedge K_m$ is $(n+m-1)$-connected, thus $(K_n \wedge K_m)^2$ is $(2(n+m)-1)$-connected. Hence according to Lemma 14.2,

$$H^{2(n+m)}(E\mathbb{Z}_2^+ \wedge_{\mathbb{Z}_2} (K_n \wedge K_m)^2) \simeq H^{2(n+m)}((K_n \wedge K_m)^2)^{\mathbb{Z}_2} \simeq \mathbb{Z}_2$$

Since the last group has only one non-trivial element, to prove that two transformations are equal it is enough to show that they are non-trivial for some spaces X, Y. Consider $X = Y = \mathbb{R}P(\infty)$ and elements $x^n \in H^n(\mathbb{R}P(\infty))$ and $x^m \in H^m(\mathbb{R}P(\infty))$. Then

$$Sq^{n+m}(t^n \times t^m) = (t^n \times t^m)^2 = t^{2n} \times t^{2m} = Sq^n(t^n) \times Sq^m(t^m) \neq 0.$$

$\square$

Proposition 14.6. *The operation Sq^0 is the identity operation i.e. $Sq^0(a) = a$.*

Proof. Again, we'll compare two transformations (in fact endomorphisms) of a representable functor: $Sq^0, id\colon \tilde{H}^n(X) \to \tilde{H}^n(X)$. According to the Yoneda lemma such transformations correspond to elements of $H^n(K_n) = \mathbb{Z}_2$. We have to show that Sq^0 corresponds to the nontrivial element i.e. it is non-zero on some space. Take $X = S^n = S^1 \wedge \cdots \wedge S^1$ and calculate $Sq^0(\iota_n) = Sq^0(\iota_1) \times \cdots \times Sq^0(\iota_1)$. By the Cartan formula 14.5 it is enough to show that $Sq^0(\iota_1) \neq 0$ for $\iota_1 \in H^1(S^1)$. We'll show that the map $H^2(E\mathbb{Z}_2 \wedge_{\mathbb{Z}_2} (S^1 \wedge S^1) \to H^2(B\mathbb{Z}_2 \times S^1)$ is nontrivial. We begin with observation that the space $S^1 \wedge S^1$ with $\mathbb{Z}_2$-action which permutes coordinates is $\mathbb{Z}_2$-homeomorphic to the sphere S^2 with $\mathbb{Z}_2$-action by reflection in a plane. There is a push-out diagram of $\mathbb{Z}_2$-spaces:

$$\begin{tikzcd}
\mathbb{Z}_2 \times (S^1,1) \arrow[r, "\subset"] \arrow[d, "q"] & \mathbb{Z}_2 \times (S^2_+,1) \arrow[d, "q"] \\
(S^1,1) \arrow[r, "\subset"] & (S^2,1)
\end{tikzcd} \tag{28}$$

$$\begin{tikzcd}
E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (\mathbb{Z}_2 \times (S^1,1)) \arrow[r, "\subset"] \arrow[d, "q"] & E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (\mathbb{Z}_2 \times (S^2_+,1)) \arrow[d, "q"] \\
E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (S^1,1) \arrow[r, "\subset"] & E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (S^2,1)
\end{tikzcd} \tag{29}$$

The right push-out diagram is homotopy equivalent to the right square in the diagram:

$$\begin{tikzcd}
\mathbb{Z}_2 \times (S^1,1) \arrow[r, "\subset"] \arrow[d, "q"] & \mathbb{Z}_2 \times (S^2_+,1) \arrow[r] \arrow[d, "q"] & (S^2,1) \arrow[d, "\simeq"] \\
B\mathbb{Z}_2 \times (S^1,1) \arrow[r, "\subset"] & E\mathbb{Z}_2 \times_{\mathbb{Z}_2} (S^2,1) \arrow[r] & E\mathbb{Z}_2 \times_{\mathbb{Z}_2} S^2/B\mathbb{Z}_2 \times S^1
\end{tikzcd} \tag{30}$$

hence respective cofibers are equivalent (right top arrow). where $\mathbb{Z}_2$ acts on top spaces by interchanging coordinates and trivially of the bottom S^1; the map q is just projection. Consider equivariant cohomology exact sequence of the triple of $\mathbb{Z}_2$-spaces $(S^2, S^1, 1)$:

$$H^2_{\mathbb{Z}_2}(S^2, S^1) \to H^2_{\mathbb{Z}_2}(S^2, 1) \to H^2_{\mathbb{Z}_2}(S^1, 1) \to H^3_{\mathbb{Z}_2}(S^2, S^1) = 0$$

The Künneth formula implies that $H^2_{\mathbb{Z}_2}(S^1, 1) \simeq \mathbb{Z}_2$ thus restriction map is non-zero. $\square$

Corollary 14.1. *The operations Sq^i are stable i.e. it commutes with the suspension isomorphism:*

$$\begin{tikzcd} \tilde{H}^n(X) \arrow[r, "Sq^i"] \arrow[d, "\simeq"', "\sigma"] & \tilde{H}^{n+i}(X) \arrow[d, "\simeq"', "\sigma"] \\ \tilde{H}^{n+1}(\Sigma X) \arrow[r, "Sq^i"] & \tilde{H}^{n+i+1}(\Sigma X) \end{tikzcd} \tag{31}$$

Proof. The suspension isomorphism σ is given by cross product with $\iota_1 \in \tilde{H}^1(S^1)$: $\sigma(x) = \iota_1 \times x$ hence:

$$Sq^i\sigma(x) = Sq^i(\iota_1 \times x) = Sq^0(\iota_1) \times Sq^i(x) = \iota_1 \times Sq^i(x) = \sigma(Sq^i(x)).$$

$\square$

Proposition 14.7. *The operation Sq^i is additive i.e. $Sq^i(x_1 + x_2) = Sq^i(x_1) + Sq^i(x_2)$.*

Proof. Cor. 14.1 implies that it is enough to prove that $Sq^i\colon H^n(\Sigma X) \to H^{n+i}(\Sigma X)$ is a homomorphism. Note that group structure on $H^m(\Sigma X) = [\Sigma X, K(\mathbb{Z}_2, m)]_*$ can be defined using co-H-group structure on suspension ΣX i.e. the map $\nu\colon \Sigma X \to \Sigma X \vee \Sigma X$. Since Sq^i is a natural transformation the following diagram commutes, what proves that Sq^i is a group homomorphism.

$$\begin{tikzcd} \tilde{H}^n(\Sigma X) \times \tilde{H}^n(\Sigma X) \arrow[r, "{(Sq^i, Sq^i)}"] \arrow[d, "\nu^*"] & \tilde{H}^{n+i}(\Sigma X) \times \tilde{H}^{n+i}(\Sigma X) \arrow[d, "\nu^*"] \\ H^n(\Sigma X) \arrow[r, "Sq^i"] & H^{n+i}(\Sigma X) \end{tikzcd} \tag{32}$$

$\square$

14.4 The SW-classes and the Steenrod squares

Definition 14.2. *Let $E \to X$ be a vector bundle, $U_E \in H^n(\mathrm{Th}(E); \mathbb{Z}_2)$ its (unique) $\mathbb{Z}_2$-Thom class and $\Phi_E\colon H^i(X; \mathbb{Z}_2) \to \tilde{H}^{i+n}(\mathrm{Th}(E); \mathbb{Z}_2)$ the corresponding Thom isomorphism. Then we define*

$$w_i(E) := \Phi_E^{-1}(Sq^i(U_E)).$$

Theorem 14.2. *The classes defined above satisfy properties 9.3, thus they coincide with the SW-classes defined in Sec. 9.*

Proof. See Milnor-Stasheff §8. Properties 1)-3) of Thm. 9.3 are straightforward. We have to check that $w_1(H_{\mathbb{R}}) \neq 0$ in $H^1(S^1)$. For that it is enough to show that $Sq^1(U_{H_{\mathbb{R}}}) \neq 0$ in $\tilde{H}^1(\mathrm{Th}(H_{\mathbb{R}})$. According to Prop. 7.2 $\mathrm{Th}(H_{\mathbb{R}}) \simeq \mathbb{R}P(2)$ and according to the proof of Thm. 8.3 the Thom class is a generator $x \in H^1(\mathbb{R}P(2))$, thus $Sq^1 x = x^2 \neq 0$. $\square$

14.5 The Steenrod algebra and Wu formulas

Consider graded polynomial algebra $\mathbb{Z}_2[Sq^0, Sq^1, \dots]$ where $\deg Sq^i = i$. The algebra obviously acts on cohomology of any space $H^*(X)$ and cohomology becomes a graded module over the algebra. It turns out that some compositions of the Steenrod squares are related by an equation.

Theorem 14.3 (Adem relation)**.** *For any space X Steenrod squares on $H^*(X)$ satisfy equation*

$$Sq^i Sq^j - \sum_{k=0}^{[\frac{i}{2}]} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k = 0$$

whenever $i, j > 0$ and $i < 2j$.[3]

The quotient algebra of $\mathbb{Z}_2[Sq^0, Sq^1, \dots]$ by the ideal generated by Adem relations is called *the Steenrod algebra*, and denotes $\mathcal{A}_2$. Cohomology is a functor form topological spaces to the category of graded $\mathcal{A}_2$-modules.

Corollary 14.2. *The Steenrod algebra is generated by elements Sq^{2^r}, $r \geq 0$.*

The following theorem describes how Steenrod operation act on the Stiefel-Whitney classes.

Theorem 14.4 (Wu formula 1)**.** *For any vector bundle $E \to X$ and $j \geq i$ the following formula holds (we denote $w_i := w_i(E)$):*

$$Sq^i(w_j) = \sum_{t=0}^{i} \binom{j-i-1+t}{t} w_{i-t} w_{j+t} = w_i w_j + \binom{j-i}{1} w_{i-1} w_{j+1} + \dots .$$

Proof. The formula is clearly natural with respect to pullbacks. If it holds for a bundle then it holds for its pullback. Thus it is enough to prove it for the universal bundle. By splitting principle applied to the universal bundle it is enough to prove it for product of Hopf bundles over projective spaces; $H_\mathbb{R} \times \dots \times H_\mathbb{R}$. For $H_\mathbb{R}$ we have $Sq^1(w_1) = w_1^2$ thus the formula holds. The multiplicative properties of the Steenrod squares and SW-classes imply that if the formula holds for a bundle E then it holds for $E \times H_\mathbb{R}$. [4] ☐

To apply Wu formula we need to know when the binomial coefficients are odd, thus nonzero mod 2.

Theorem 14.5. *[E. Lucas] For every prime p and two integers $a \geq b$ a binomial coefficient $\binom{a}{b} = 0 \mod 2$ if and only if at least one of the base 2 digits of b is greater than the corresponding digit of a.*

Corollary 14.3. *For every $k, i > 0$, $\binom{2^k-1}{i} = 1 \mod 2$.*

An important corollary of the Thm. 14.4 says that over the Steenrod algebra SW-classes in different dimensions are not independent.

Proposition 14.8. *The SW-class w_k of an arbitrary vector bundle can be expressed over the Steenrod $\mathcal{A}_2$ in terms of classes w_{2^r} where $2^r \leq n$.* [5]

[3]For proof of the Adem relation cf. Mike Hopkins [4]
[4]For detailed calculation cf. Robert R. Bruner, Michael Catanzaro, J. Peter May *Characteristic classes* Thm. 6.3.
[5]For detailed explanation in next paragraph we quote Tyler Lawson

Proof. For an arbitrary decomposition $k = i + j$ where $0 < i \leq j$ the Wu formula 14.4 implies that:

$$\binom{j-1}{i} w_k = Sq^i(w_j) + \sum_{t=0}^{i-1} \binom{j+i-t-1}{i} w_{i-t} w_{j+t}.$$

If the binomial coefficient on the left is nonzero, the class w_k can be expressed as a sum of terms generated by lower SW-classes. So for fixed k this becomes a question as to whether exist i, j such that $k = i + j$, $0 < i \leq j$ and $\binom{j-1}{i} \neq 0 \mod 2$.

If k is not a power of 2, we can write it as $k = i + j$ where $j = 2^k > i$. Thus $w_k = Sq^i(w_j) + \ldots$. Iterating such procedure we express w_n in terms of w_{2^r}.

If $k = i + j = 2^r$, then $i + (j-1)$ has no zeros in its binary expansion, which means that i and $j-1$ can't share any nonzero binary digits. Hence we can not find a suitable decomposition $k = i + j$. $\square$

Corollary 14.4. [6] *If $E \to X$ is a vector bundle such that $w(E) \neq 1$, then the smallest n for which $w_m(E) \neq 0$ is a power of 2.*

14.6 The SW-classes of manifolds

We define SW-classes of a smooth manifold M as SW-classes of its tangent bundle: $w_k(M) := w_k(TM)$. We proceed towards proof of a surprising fact that the SW-classes of tangent bundle of a closed smooth manifold can be recovered just from the $\mathcal{A}_2$-module structure on $H^*(M)$. The key role is played by the Poincaré duality.

Theorem 14.6 (Poincaré duality)**.** *Let M be a closed smooth n-dimensional manifold and $\mu_M \in H_n(M; \mathbb{Z}_2)$ be its fundamental class. Then for every k the bilinear form $PD\colon H^k(M) \otimes H^{n-k}(M) \to \mathbb{Z}_2$, defined as $PD(a, b) := \langle a \cup b, \mu_M \rangle$ is nondegenerate i.e. it defines an isomorphism*

$$H^k(M) \simeq \operatorname{Hom}(H^{n-k}(M), \mathbb{Z}_2).$$

The Steenrod operation $Sq^k\colon H^{n-k}(M) \to H^n(M)$ composed with evaluation $H^n(M) \xrightarrow{\langle -, \mu_M \rangle} \mathbb{Z}_2$ is an element of $\operatorname{Hom}(H^{n-k}(M), \mathbb{Z}_2)$. Corresponding element $v_k \in H^k(M)$ we call *k-th Wu class* of M. Similarly to the total Stiefel-Whitney class we define the total Wu class: $v := 1 + v_1 + \cdots + v_n$.

Theorem 14.7 (Wu Wenjun)**.** *Let M be a closed smooth n-dimensional manifold. Then*

$$w(M) = Sq(v) \quad \textit{i.e.} \quad w_k(M) = \sum_{i+j=k} Sq^i(v_j).$$

Since Poincaré duality holds for topological manifolds, the Wu theorem allows us to define the Stiefel-Whitney classes for closed topological manifolds (without smooth structure!). The first step of proof of the Wu theorem is showing that homological properties of the Thom space of the tangent bundle can be idepend only on topolology of the manifold, not the smooth structure. We need some prerequisites from differential topology; namely the tubular neighborhood theorem.

Let $j\colon M \to N$ be an immersion i.e. a smooth map such that for each $x \in M$ differential $Dj_x\colon TM_x \to TN_{j(x)}$ is injective. The quotient bundle $j^! TM/TN =: \nu(j)$ over M is called *the normal bundle* to j. If $j\colon M \subset N$ is a inclusion of a submanifold then the normal bundle $\nu(j)$ is called the normal bundle of M in N and denoted $\nu(M, N)$. Note that normal bundle is a local invariant of $M \subset N$ i.e. if U is an open subset such that $M \subset U \subset N$ then $\nu(M, N) = \nu(M, U)$. the normal bundle is a summand of the tangent bundle to N restricted to M. Introducing a Riemannian metric on N we have a decomposition $\nu(M, N) \simeq TN^{\perp} \subset TM|N$.

[6] Milnor-Stasheff Problem 8-B

Zad. 29. If $E \to M$ is a smooth vector bundle, then for every smooth section $s\colon M \to E$, $\nu(s) \simeq E$.

Zad. 30. For every smooth manifold normal bundle to the diagonal embedding $M \subset M \times M$ is isomorphic to the tangent bundle TM.

Theorem 14.8 (Tubular Neighborhood Theorem)**.** *For any submanifold $M \subset N$ there exists a tubular neighbourhood that is an open subset $U \supset M$, an open neighbourhood V of the zero-section of $\nu(M, N)$ such that $M \subset V$ is a deformation retract and a diffeomorphism $h\colon U \to V$ which maps zero-section to N, so that the following diagram commutes:*

$$\begin{tikzcd} U \arrow[rr, "h"] & & V \\ & M \arrow[ul, "\subset"] \arrow[ur, "s_0"'] & \end{tikzcd} \tag{33}$$

In particular h defines a diffeomorphism of pairs $h\colon (U, U \setminus s_0(M)) \xrightarrow{\simeq} (V, V \setminus M)$.

Note that excision axiom implies that every tubular neighbourhood defines isomorphism of (co-)homology with arbitrary coefficients:

$$\tilde{H}^*(\mathrm{Th}(\nu(M, N)) \simeq H^*(\nu(M, N), \nu(M, N)_0) \simeq H^*(N, N \setminus M).$$

Consider again diagonal embedding $\Delta\colon M \subset M \times M$. Zad. 30 implies that we have an isomorphism:

$$\tilde{H}^*(\mathrm{Th}(TM)) = \tilde{H}^*(\mathrm{Th}(\nu(\Delta, M \times M)) \simeq H^*(M \times M, M \times M \setminus \Delta).$$

and the Thom class $U_M \in \tilde{H}^*(\mathrm{Th}(TM))$ corresponds to a class $U'_M \in H^*(M \times M, M \times M \setminus \Delta)$ such that for each $x \in M$ its restriction $U'_M|x \in H^*(M, M \setminus \{x\})$ is a generator. Restriction of the class U'_M to $H^n(M \times M)$ is called the *diagonal cohomology class of* M and is denoted U''_M.

Lemma 14.3. *Let $p_1, p_2\colon M \times M \to M$ be projections on factors. Then for any class $a \in H^*(M)$:*

$$p_1^*(a) \cup U'_M = p_2^*(a) \cup U'_M \in H^*(M \times M, M \times M \setminus \Delta) \quad \textit{and} \quad p_1^*(a) \cup U''_M = p_2^*(a) \cup U''_M \in H^*(M \times M).$$

Proof. Let $U \supset \Delta$ be a tubular neighbourhood of the diagonal $M = \Delta \subset M \times M$. By excision property $H^*(M \times M, M \times M \setminus \Delta) = H^*(U, U \setminus \Delta)$. The key observation is the following: two projections are equal on the diagonal. Since embedding $\Delta \subset U$ is a homotopy equivalence, the restrictions $p_1, p_2\colon U \to M$ are homotopic, thus $p_1^*(a)|U = p_2^*(a)|U$. Now we have to apply some diagram chasing. Naturallity of the cup product implies commutativity of the diagram which immediately implies conclusion of the theorem:

$$\begin{tikzcd} H^*(M \times M) \otimes H^*(M \times M, M \times M \setminus \Delta) \arrow[r, "\cup"] \arrow[d] & H^*(M \times M, M \times M \setminus \Delta) \arrow[d, "\simeq"] \\ H^*(U) \otimes H^*(U, U \setminus \Delta) \arrow[r, "\cup"] & H^*(U, U \setminus \Delta) \end{tikzcd} \tag{34}$$

□

We need to recall the slant product, pairing cohomology of a product of two pairs with homology of factor (coefficients in a ring).

$$H^{p+q}((X, A) \times (Y, B)) \otimes H_q(Y, B) \xrightarrow{/} H^p(X, A)$$

If ring of coefficients is a field , we have the Künneth isomorphism

$$H^*(X,A) \otimes H^*(Y,B) \xrightarrow{\times} H^p((X,A) \times (Y,B))$$

and the slant product composed with its inverse:

$$H^*(X,A) \otimes H^*(Y,B) \otimes H_*(Y,B) \xrightarrow{/} H^*(X,A)$$

is given by the formula: $a \otimes b \otimes z \mapsto a\langle b, z\rangle$.

Lemma 14.4. *If $\mu_M \in H_n(M)$ is the fundamental homology class of the compact manifold M then $U''_M/\mu = 1 \in H^0(M)$.*

Proof. Let $x \in M$ and $j\colon M \to M \times M$ be inclusion $j(y) := (x, y)$. Conclusion follows from the following commutative diagram:

$$\begin{tikzcd}
H^n(M \times M, M \times M \setminus \Delta) \otimes H_n(M) \arrow[r] \arrow[d, "j^* \otimes id"] & H^n(M \times M) \otimes H_n(M) \arrow[r, "/"] \arrow[d, "j^* \otimes id"] & H^0(M) \arrow[d, "\simeq"] \\
H^n(M, M \setminus x) \otimes H_n(M) \arrow[r] & H^n(M) \otimes H_n(M) \arrow[r, "/"] & H^0(pt)
\end{tikzcd} \tag{35}$$

□

The Poincaré duality implies that for any homogeneous basis $b_1, \dots b_r \in H^*(M)$ there exists a dual basis $b_1^\#, \dots b_r^\# \in H^*(M)$ such that $PD(b_i, b_j^\#) = \delta_{ij}$.

Theorem 14.9. $U''_M = \sum (-1)^{\dim b_i} b_i \times b_i^\#$.

Proof. We use the Künneth isomorphism $H^*(M) \otimes H^*(M) \xrightarrow{\times} H^*(M \times M)$ and express the class U''_M in terms of the basis b_i. There exist elements $c_i \in H^{n - \dim b_i}(M)$ such that

$$U''_M = b_1 \times c_1 + \cdots + b_r \times c_r.$$

We need to identify coefficients c_i. According to Lemmas 14.4 and 14.3 for any $a \in H^*(M)$

$$a = ((a \times 1) \cup U''_M)/\mu_M = ((1 \times a) \cup U''_M)/\mu_M.$$

On the right side, substituting $U''_M = \sum b_i \times c_i$ we obtain ($\deg a =: |a|$)

$$a = \sum (-1)^{|a||b_j|} (b_j \times (a \cup c_j))/\mu_M = \sum (-1)^{|a||b_j|} b_j \langle a \cup c_j), \mu_M\rangle.$$

Since $b_1, \dots b_r$ form a basis, substituting b_i for a it follows that:

$$(-1)^{|b_i||b_j|} \langle b_i \cup c_j, \mu_M\rangle = \delta_{ij} \quad \text{and} \quad c_i = (-1)^{|b_i|} b_i^\#.$$

□

Proposition 14.9. $w_i(M) = Sq^i(U''_M)/\mu_M$.

Proof. By definition of the SW-class

$$(w_i(M) \times 1) \cup U_M = p_1^*(w_i(M)) \cup U_M = Sq^i(U_M)$$

where $p_1 \colon M \times M \to M$. Calculate

$$Sq^i(U_M)/\mu_m = ((w_i(M) \times 1) \cup U_M)/\mu_m = w_i(M) \cup U_M/\mu_m = w_i(M).$$

□

Proof of Thm. 14.7. Let $b_1, \dots b_r \in H^*(M)$ be a homogeneous basis and $b_1^\#, \dots b_r^\# \in H^*(M)$ be its dual with respect to PD. Then any element $x \in H^*(M)$ can be expressed as

$$x = \sum_{i=1}^{r} \langle x \cup b_i^\#, \mu_M \rangle b_i.$$

Let $v(M) \in H^*(M)$ be the total Wu class, then from its definition it follows that:

$$v(M) = \sum_{i=1}^{r} \langle v \cup b_i^\#, \mu_M \rangle b_i = \sum_{i=1}^{r} \langle Sq^i(b_i^\#, \mu_M \rangle b_i$$

and from Thm. 14.9 and Prop. 14.9 follows that

$$Sq(v(M)) = \sum_{i=1}^{r} \langle Sq(b_i^\#), \mu_M \rangle Sq(b_i) = \sum_{i=1}^{r} \langle Sq^i(b_i^\#) \times Sq(b_i)/\mu_M \rangle = Sq(U_M'')/\mu_M = w(M).$$

□

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